NEW TRENDS IN SYZYGIES: 18w5133

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1 Overview of the Field

Since the pioneering work of David Hilbert in the 1890s, syzygies have been a central area of research in commutative algebra. The idea is to approximate arbitrary modules by free modules. Let R be a Noetherian, commutative ring and let M be a finitely generated R-module. A free resolution is an exact sequence of free R-modules $\cdots \to F_{i+1} \to F_i \to \cdots F_0$ such that $H_0(F_\bullet) \cong M$. Elements of F_i are called ith syzygies of M. When R is local or graded, we can choose a minimal free resolution of M, meaning that a basis for F_i is mapped onto a minimal set of generators of $\mathrm{Ker}(F_{i-1} \to F_{i-2})$. In this case, we get uniqueness of minimal free resolutions up to isomorphism of complexes. Therefore, invariants defined by minimal free resolutions yield invariants of the module being resolved. In most cases, minimal resolutions are infinite, but over regular rings, like polynomial and power series rings, all finitely generated modules have finite minimal free resolutions.

Beyond the study of invariants of modules, syzygies have connections to algebraic geometry, algebraic topology, and combinatorics. Statements like the Total Rank Conjecture connect algebraic topology with free resolutions. Bounds on Castelnuovo-Mumford regularity and projective dimension, as with the Eisenbud-Goto Conjecture and Stillman's Conjecture, have implications for the computational complexity of Gröbner bases and computational algebra systems. Green's Conjecture provides a link between graded free resolutions and the geometry of canonical curves. These are just some of the important problems that have seen great recent progress and were discussed at this BIRS meeting. In the following we focus on these four problems in detail. Then we summarize some other meeting highlights and positive feedback from meeting participants.

2 Recent Developments and Open Problems

The theory of free resolutions and syzygies has seen many spectacular results in the last few years with several long-standing conjectures being resolved. Here we briefly summarize the biggest results and their connections with this workshop.

2.1 Buchsbaum-Eisenbud-Horrocks Conjecture

Let k be any field, let R be a regular local ring of dimension n. The Buchsbaum-Eisenbud-Horrocks rank conjecture roughly says that the Koszul complex is the smallest possible minimal free resolution. The conjecture was formulated by David Buchsbaum and David Eisenbud [BE]. The conjecture is also implicit in the work of Horrocks. See [Har]. The precise statement is as follows.

Conjecture 2.1 (BEH Conjecture)**.** *Let M be an R module of finite length. Then*

$$\beta_i(M) \geq \binom{n}{i}.$$

Note that if $M = k$, then the minimal free resolution is a Koszul complex and the ith total Betti number is exactly $\binom{n}{i}$. The conjecture has been settled for $n \leq 4$ by Graham Evans and Phillip Griffith [EG2]. A similar statement can be made in the graded case. A special case of this question was proved by Daniel Erman [E].

One can ask the weaker question as to whether the sum of all the Betti numbers is at least 2^n. This is sometimes known as the Weak Horrocks Conjecture or Total Rank Conjecture. In a recent breakthrough, Mark Walker proved the Total Rank Conjecture for all characteristics except 2. Walker gave a talk on his work at the meeting. The proof relies on techniques from K-theory and fits nicely into a one-hour talk. In fact, his result applies more generally to finitely generated modules of finite projective dimension over local complete intersections not of characteristic two or over local rings containing a field of odd, prime characteristic.

Srikanth Iyengar gave a related talk he presented joint work with Walker constructing counterexamples to various generalizations of the Total Rank Conjecture. (See description below.)

2.2 Eisenbud-Goto Conjecture

Let $S = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field and let M be a finitely generated graded S-module. We define the (Castelnuovo-Mumford) regularity of M to be

$$\text{reg}(S/I) = \max\{j \mid \beta_{i,i+j}(S/I) \neq 0.$$

Thus the regularity of S/I is the index of the last nonzero row in the Betti table of S/I. David Eisenbud and Shiro Goto showed [EG] that this agrees with the more classical definition of regularity given in terms of vanishing of twists of sheaf cohomology modules. In particular, note that the regularity of an ideal is an upper bound on the degrees of the minimal generators. Thus finding effective bounds on regularity has attracted a lot of attention.

David Bayer and Michael Stillman [BS] gave another characterization of regularity: $\text{reg}(I)$ is equal to the maximal degree of a Gröber basis element of I in the revlex monomial order if we first take a generic change of coordinates. Since Gröbner bases are required for many computational tasks, this means that finding upper bounds on regularity equates to finding bounds on the computational complexity of an ideal. Unfortunately, in the most general setting possible, such upper bounds are quite large. Set $\text{maxdeg}(I)$ to be the maximal degree of a minimal generator of I. A well-known result of David Bayer and David Mumford [BM] and later Giulio Caviglia and Enrico Sbarra [CS] shows that $\text{reg}(I) \leq (2\text{maxdeg}(I))^{2^{n-2}}$, where n is the number of variables.

This doubly exponential bound grows quickly with respect to n. Unfortunately, this bound is nearly optimal. In one construction derived from the so-called Mayr-Meyer [MM3] ideals, Koh [Koh] proved that for any integer $r \geq 1$ there exists an ideal I_r in $S_r = k[x_1, \ldots, x_{22r-1}]$ such that

$$\begin{aligned} \text{maxdeg}(I_r) &= 2 \\ \text{reg}(S_r/I_r) &\geq 2^{2^{r-1}}. \end{aligned}$$

Thus doubly exponential behavior cannot be avoided. We note for later reference that I_r is generated by $22r - 3$ quadrics and 1 linear form while the regularity is realized at the first syzygies of I (second syzygies of S/I).

The ideals I_r have many associated primes and embedded primes; in particular, they are far from prime. Better bounds were expected for ideals with some geometric content. This expectation was expressed in the following conjecture:

Conjecture 2.2 (Eisenbud-Goto (1984))**.** *Let $S = k[x_1, \ldots, x_n]$ with $k = \bar{k}$ and suppose $\mathfrak{p} \subset (x_1, \ldots, x_n)^2$ is a homogeneous prime ideal. Then*

$$\mathrm{reg}(S/\mathfrak{p}) \leq e(S/\mathfrak{p}) - \mathrm{ht}(\mathfrak{p}).$$

The condition $\mathfrak{p} \subset (x_1, \ldots, x_n)^2$ is equivalent to saying that the projective variety corresponding to $\mathfrak{p}$ is not contained in a hyperplane and thus is optimally embedded. Such ideals are called nondegenerate. It is worth noting that the right-hand side of the inequality above is always positive for nondegenerate prime ideals.

The Eisenbud-Goto Conjecture can be viewed as an expectation that for ideals with more geometric content, regularity is better behaved. Via the Bayer-Stillman result, this would then ensure that computations involving prime ideals are much better behaved than for arbitrary ideals. Castelnuovo essentially showed in 1893 that smooth curves in $\mathbb{P}^3_{\mathbb{C}}$ satisfy the conjecture. In 1983 Gruson-Lazarsfeld-Peskine [GLP] proved the bound for all curves (smooth and singular) in any projective space. (Remember that $\mathfrak{p}$ defines a projective curve when $\dim(S/\mathfrak{p}) = 2$.) Pinkham [Pi] and Lazarsfeld [La] proved the bound for smooth projective surfaces over $\mathbb{C}$. Ran [Ra] proved the bound for most smooth projective 3-folds over $\mathbb{C}$. Eisenbud and Goto also proved the Cohen-Macaulay case. Numerous other special cases have also been proved.

In 2016, Jason McCullough and Irena Peeva [MP] constructed counterexamples to the Eisenbud-Goto Conjecture. The construction involved two new concepts: Rees-like algebras and Step-by-step homogenization. Given an ideal $I \subset S$, the Rees algebra of I is $S[It]$, where t is a new indeterminate. It defines the blow up of projective space along the subvariety defined by I and thus is important in the resolution of singularities. McCullough and Peeva defined the Rees-like algebra as $S[It, t^2]$. Contrary to the case of the Rees algebra, where defining equations are difficult to find, the defining equations of the Rees-like algebra are easy. Moreover, McCullough and Peeva constructed the entire minimal graded free resolution of the defining prime ideal of any Rees-like algebra. Thus they were able to compute the degree and regularity of these ideals and show that they yield counterexamples to the Eisenbud-Goto Conjecture 2.2. Moreover, they showed that there was no polynomial bound on the regularity of nondegenerate prime ideals purely in terms of the degree. A new homogenization was constructed to form standard graded analogs of the non-standard graded prime ideals above.

In more recent work, Giulio Caviglia, Marc Chardin, Jason McCullough, Irena Peeva, and Matteo Varbaro [CCMPV] showed that there is a (necessarily large) bound on the regularity or projective dimension of nondegenerate primes in terms of the degree alone. They also gave new counterexamples to the Eisenbud-Goto Conjecture 2.2 that use Rees algebras instead of Rees-like algebras and ones that do not rely on the Mayr-Meyer ideals. Work of Craig Huneke, Paolo Mantero, Jason McCullough, and Alexandra Seceleanu [HMMS4] shows that no similar bound is possible more generally for primary ideals.

While these counterexamples indicate that the Eisenbud-Goto Conjecture 2.2 is false in general, it remains open in the smooth case. Suppose $\mathrm{char}(k) = 0$ and $X \subset \mathbb{P}^{p-1}_k$ is a smooth variety. In this case Sijong Kwak and Jinhyung Park [KP] and Atsushi Noma [No] reduced it to the Castelnuovo's Normality Conjecture that X is r-normal for all $r \geq \deg(X) - \mathrm{codim}(X)$. They do so by bounding the regularity of the structure sheaf of X. Sijong Kwak attended the meeting and gave a talk on "$\mathcal{O}_X$ regularity bound for smooth varieties with classification of extremal and next to extremal examples." The proof uses geometric properties of double point divisors from inner projections.

2.3 Green's Conjecture

Let $S = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field K. Let I be a homogeneous ideal of S. We denote by $\beta_{ij}(S/I) = \dim_K \operatorname{Tor}_i^K(S/I, K)$ the graded Betti numbers of S/I. Typically one displays the Betti numbers of a module in what is called the Betti table. We place $\beta_{ij}(S/I)$ in column i and row $j - i$. The length of the 2-linear strand of the Betti table is $2\mathrm{LP}(S/I) = \max\{i \mid \beta_{i,i+1}(S/I) \neq 0\}$, and therefore measures the index of the last nonzero entry in the second row in the Betti table.

Green's conjecture concerns the shape of the Betti tables of canonical curves. A projective curve may be embedded into projective space in many ways, but for non-hyperelliptic cruves there is a canonical embedding into projective space. Resolving the associated homogeneous ideal I, one finds that S/I is Gorenstein and has Betti table of the form

	0	1	2	$\cdots$	g-4	g-3	g-2
0:	1	-	-	$\cdots$	-	-	-
1:	-	a_1	a_2	$\cdots$	a_{g-4}	a_{g-3}	-
2:	-	a_{g-3}	a_{g-4}	$\cdots$	a_2	a_1	-
3:	-	-	-	$\cdots$	-	-	1,

where the a_i are the only possibly nonzero Betti numbers and g denotes the genus of the curve. We recall the definition of the clifford index. For a given line bundle $\mathcal{L}$ on C, we set

$$\mathrm{Cliff}(\mathcal{L}) = g + 1 - \left(h^0(\omega \otimes \mathcal{L}^{-1})\right) = \deg \mathcal{L} - 2(h^0(\mathcal{L}) - 1).$$

The Clifford index of C (in the case when the genus of C is at least 3) is the minimum of all Clifford indices of bundles $\mathcal{L}$ with $h^0(\mathcal{L}) \geq 2$ and $h^0(\omega \otimes \mathcal{L}^{-1}) \geq 2$. We can now state Green's Conjecture:

Conjecture 2.3 (Green)**.** *The length of the 2-linear strand of the resolution of the canonical ring of a curve of genus g and clifford index c is*

$$2\mathrm{LP}(S/I) = g - c - 2.$$

Equivalently, $a_{g-3} = a_{g-4} = \cdots a_{g-c-1} = 0$ and $a_{g-c-2} \neq 0$ in the Betti table above.

The generic curve of genus g is known to have Clifford index $\lfloor (g-1)/2 \rfloor$. Since the Betti number vanishing condition above defines a Zariski open set, the generic case of Green's Conjecture asserts that there exists a smooth curve whose canonical ring satisfies the conclusion. Special cases of the Conjecture were known, such as for low genus curve, but even the generic case was open until Clair Voisin resolved the generic case in characteristic 0 in 2005 [V1, V2].

At this meeting Claudiu Raicu presented a new proof (joint work with Marian Aprodu, Gavril Farkas, Stefan Papadima, and Jerzy Wayman) of the generic Green's Conjecture that goes further and simplifies Voisin's proof. Their proof uses a vanishing theorem for Koszul modules and connects to other exciting work in [AFPRW]

2.4 Stillman's Conjecture

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. View S as a standard graded ring $S = \bigoplus_{i \geq 0} S_i$, where S_i denotes the K-vector space of degree i homogeneous polynomials. Let $I = (f_1, \ldots, f_m)$ denote a homogeneous ideal of S. Around 2000, Michael Stillman posed the following:

Conjecture 2.4 ([PS, Problem 3.14])**.** *Fix a sequence of natural numbers $d_1, \ldots, d_g$. Does there exist a number p such that $\mathrm{pd}_S(S/I) \leq p$, where I is a homogeneous ideal in a polynomial ring S with a minimal system of generators of degrees $d_1, \ldots, d_g$? Note that the number of variables in S is not fixed.*

It follows from Hilbert's Syzygy Theorem that $\mathrm{pd}_S(S/I) \leq n$, but the conjecture does not reference the number of variables. It follows from work of Buch [Bu], Kohn [Ko], and Bruns [Br] that there is no bound on the projective dimension of three-generated ideals. Thus to find any such bound we must at least take into account the degrees of the generators. More recent work of Caviglia (see [MS]) that Conjecture 2.4 is equivalent to one where we replace projective dimension by regularity.

Previously, only special cases of the Conjecture had been proved, e.g. [En1, En2, HMMS1, HMMS3, MM2, MM3, AH1]. In 2016, Ananyan and Hocster [AH2] gave a proof of Stillman's Conjecture 2.4 in full generality. The bound they prove is generally non-constructive and it remains open as to what an asymptotically tight bound would look like. But they introduced many new ideas that have already had a significant impact on the field. Notably, their result led to the result of Caviglia, Chardin, McCullough, Peeva, and Varbaro [CCMPV] mentioned previously. They also introduced the notion of the strength of a form which we now discuss.

Let $f \in K[x_1, \ldots, x_n]$ with $\deg(f) > 0$. We say that F has a k-collapse for $k \in \mathbb{N}$, if f is in an ideal generated by k elements of strictly smaller positive degree. We say that f has strength k if it has a $(k+1)$-collapse but no k-collapse. For instance, all linear forms have infinite strength. If f is a nonzero form of positive degree then f has strength at least 1 if and only if f is irreducible.

Another new concept was that of a prime (resp. R_i) sequence. A sequence of elements $f_1, \ldots, f_m \in S$ is a prime sequence (respectively R_i -sequence, where $i \in \mathbb{N}$), if $f_j \notin (f_1, \ldots, f_{j-1})$ and $S/(f_1, \ldots, f_j)$ is a domain (respectively, satisfies (R_i)) for $j = 1, \ldots, m$. One of the key observations of Ananyan-Hochster was that F has a small collapse if and only if the singular locus of f is large (i.e. has small codimension). Their proof is a massive yet subtle inductive argument that relies on a technical theorem of Grothendieck. However, their paper also provides other bounds on the number of associated primes, degrees and number of generators of primary components, and more for ideals with a fixed number of generators of fixed degree.

More recently, Daniel Erman, Steven Sam and Andrew Snowden have given two brand new proofs of Stillman's Conjecture. Both rely on a new characterization of infinitely generated polynomial rings via derivations. One of their proofs uses the inverse limit of polynomial rings in an increasing finite number of variables. The other proof uses ultra products. The meeting was fortunate to have Steven Sam give a talk on this paper.

Yet another new proof of Stillman's Conjecture has been given by Jan Draisma, Michael Lason and Anton Leykin [DLL]. Their proof relies on an initial term and Gröbner basis argument. Despite all the progress on Stillman's Conjecture, it remains open as to what the optimal bounds should be in general. Ananyan and Hochster have announced a forthcoming paper giving explicit bounds in the case of ideals generated by quadrics, cubics, and quartics but even these bounds are large and far from optimal.

3 Presentation Highlights

Including the talks discussed above, there were 22 talks given at the workshop. Here we survey the highlights among those talks.

Hai Long Dao spoke about joint work [DDM] with Allesandro de Stefani and Linquan Ma. Inspired by a question raised by Eisenbud-Mustaţă-Stillman regarding the injectivity of maps from Ext modules to local cohomology modules, they introduce a class of rings which called cohomologically full rings. In positive characteristic, this notion coincides with that of F-full rings, while in characteristic 0, they include Du Bois singularities. They prove many basic properties of cohomologically full rings, including their behavior under flat base change. Ideals defining these rings satisfy many desirable properties, in particular they have small cohomological and projective dimension. Furthermore, they obtain Kodaira-type vanishing and strong bounds on the regularity of cohomologically full graded algebras.

Takayuki Hibi spoke on joint work [HM] with Kazunori Matsuda entitled "Regularity and h-polynomials of monomial ideals." Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$ and $I \subset S$ a homogeneous ideal of S with $\dim S/I = d$. The Hilbert series of S/I is of the form $h_{S/I}(\lambda)/(1-\lambda)^d$, where $h_{S/I}(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s$ with $h_s \neq 0$ is the h-polynomial of S/I. It is known that, when S/I is Cohen–Macaulay, one has $\operatorname{reg}(S/I) = \deg h_{S/I}(\lambda)$, where $\operatorname{reg}(S/I)$ is the (Castelnuovo–Mumford) regularity of S/I. Hibi showed how, given arbitrary integers r and s with $r \geq 1$ and $s \geq 1$, a monomial ideal I of $S = K[x_1, \ldots, x_n]$ with $n \gg 0$ for which $\operatorname{reg}(S/I) = r$ and $\deg h_{S/I}(\lambda) = s$ could be constructed.

Srikanth Iyengar spoke on joint work with Mark Walker [IW] on "Examples of finite free complexes of small rank and small homology." Given Walker's resolution of the Total Rank Conjecture, one could imagine that there are lower bounds on the total Betti numbers of certain complexes more generally. To the contrary, Iyengar and Walker construct finite free complexes over commutative noetherian rings such that the total rank of their underlying free modules, or the total length of their homology, is less than predicted by various conjectures in the theory of transformation groups and in local algebra.

Satoshi Murai spoke on "h-vectors and the number of generators of fundamental groups," representing joint work [MN] with Isabella Novik. In his talk, he showed how to resolve a conjecture of Kalai asserting that the g_2-number of any (finite) simplicial complex Δ that represents a normal pseudomanifold of dimension $d \geq 3$ is at least as large as $\binom{d+2}{2} m(\Delta)$, where $m(\Delta)$ denotes the minimum number of generators of the fundamental group of Δ. He also showed how to prove that a weaker bound, $h_2(\Delta) \geq \binom{d+1}{2} m(\Delta)$, applies to any d-dimensional pure simplicial poset Δ all of whose faces of co-dimension at least 2 have connected links.

Maria Rossi gave a talk on "A generalization of Macaulay's correspondence for Gorenstein k-algebras and applications." Macaulay's inverse systems give a one-to-one correspondence between Artinian, Gorenstein ideals in $R = k[x_1, \ldots, x_n]$ and finitely-generated R-submodules of the injective envelope of R. Rossi gave a talk on how this picture generalizes to Gorenstein ideals of arbitrary dimension. This is joint work [ER] with Juan Elias.

Hal Schenck spoke on joint work with Matt Mastroeni and Mike Stillman. In his talk entitled "Theta characteristics, Koszul algebras, and the Gorenstein property," Schenck addressed a question of Conca who asked if every quadratic, Gorenstein algebra R of regularity at most 3 is Koszul. They construct a family of counterexamples using the idealization process.

Matteo Varbaro reported on joint work [CV] with Aldo Conca on "Square-free Gröbner Degenerations." They address a conjecture of Jürgen Herzog regarding square-free initial ideals. It is well known that the graded Betti numbers, projective dimension, regularity of the initial ideal of an ideal I are upper bounds for those of I itself. A celebrated result of Bayer and Stillman says that in the degree revlex order and in generic coordinates, these upper bounds become equalities. It had been conjectured by Herzog that this was also the case under any monomial order when the initial ideal was a square-free monomial ideal. Varbaro gave a detailed talk in which he and Conca prove Herzog's conjecture.

4 Outcome of the Meeting

This meeting brought together 41 researchers in commutative algebra and algebraic geometry. The meeting allowed for participants from far away a chance to talk and collaborate. The participants hailed from Canada, France, Germany, Italy, Japan, South Korea, Spain, and USA.

The meeting was very received very positively with many excellent talks. Several participants reported starting new collaborations as a result of attending the meeting. We include here some of the feedback and testimonials the organizers received regarding the benefits and outcomes of their attendance.

David Eisenbud wrote: "It really was a great workshop. Hightlights for me were the talks reporting on 4 spectacular results:

1. New, simpler proof of Voisin's theorem on Green's conjecture (Raicu)
2. New, very clever and simpler proof of Ananyan-Hochster's result on Stillman's problem (Sam)
3. Wonderful and surprising results on square free initial ideals (Varbaro)
4. Proof of Kalai's conjecture (Murai)

...and lots more.

For my own research, I was particularly happy to interact with Mats Boij, who showed me some things related to inverse systems that I will now publish as part of the next Macaulay2 release."

Takayuki Hibi wrote: "My participation in the BIRS workshop created a motive to write a new paper as well as a new research project. In fact, (a) During the coffee break just after my talk, Marc Chardin kindly made a frank comment about the argument in my talk. His suggestion, in fact, can make our proof much elegant and, as a result, yields a motive to write a new paper. (b) Talking to Adam Van Tuyl over lunch about a conjecture on edge ideals, he agreed with me about making a new joint research project on regularity of edge ideals. Even though each of us have written a lot of papers on edge ideals, I have never had an opportunity to have a conversation with him. This is one of the most productive benefits gotten from participating in the workshop."

Srikanth Iyengar wrote: "Besides giving a chance to catch up and pursue "old" collaborators and collaborations (among whom I count Mark Walker, Claudia Miller and Liana Sega), the conference gave me a chance to talk to Lukas Katthaen. I am not sure this will lead to any theorems right away, but we certainly started a conversation that will be continued. Another benefit of the meeting was that I learnt some new things from Bernd Ulrich's talks that are pertinent to a ongoing project I have with Ryo Takahashi."

Thomas Kahle wrote: "I really enjoyed the workshop and how it helped me to stay up to date with some very recent developments in commutative algebra such as the result by Varbaro and Conca that extremal Betti numbers are preserved under square-free Grbner degenerations.

Attending the workshop also gave me the chance to renew some long-running collaborations. Together with Lukas Katthn we tested some ideas on random structures in commutative algebra. The wonderful atmosphere at BIRS and Banff makes all this much more enjoyable and effective than collaboration via e-mail."

Paolo Mantero wrote: "This workshop also gave me the opportunity to make progress on two joint projects, one with J. McCullough and one with A. Sammartano; most likely the projects will be done around the end of the Fall semester. I also had the opportunity to discuss research ideas with Matteo Varbaro and Maria Evelina Rossi, which I am very grateful for, since both researchers reside and work in Europe, so without the workshop it is unlikely I could have had this great interactions with them.

Additionally, the workshop helped me strengthen the relation with Greg Smith and Adam Van Tuyl; as a likely outcome I will give talks at their institutions (Queen's University and McMaster University) in the near future. It opens the door to potential future collaborations."

Irena Swanson wrote: "I enjoyed the conference, the talks, and the people very much. Thank you for inviting me. I will be teaching a course on homological algebra on my Fulbright at University of Graz this fall, so connecting with very recent research in the area is very beneficial. I learned much."

Adam Van Tuyl wrote: "I just wanted to write and thank you again for inviting me to conference at BIRS. I really enjoyed the talks, and I came away with a couple of new ideas to play with."

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