Geometric Quantization

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1 Overview

Quantization is an important theme in many areas of mathematics and physics. *Geometric quantization* aims to associate to any classical phase space, modeled by a symplectic manifold, a corresponding quantum space, modeled by a Hilbert space. Classical observables, given by smooth functions (Hamiltonians), should be quantized to quantum observables.

The original concept of quantization, nowadays usually referred to as *canonical quantization*, goes back to the work of H. Weyl, J. von Neumann, P. Dirac and I.E. Segal. Nevertheless, canonical quantization encountered several problems, and among other things failed to provide a unified framework for the Schrödinger and Bargmann-Fock representation.

In the 1960s, a theory called *geometric quantization* emerged. Its main goal is to set a relation between classical and quantum mechanics from a geometrical point of view, taking as a model the canonical quantization method, but removing some ambiguities involved in the canonical quantization procedure.

The pioneering works on geometric quantization are due to J.M. Souriau [24], B. Kostant [12], although many of their ideas were based on previous works by A.A. Kirillov [13]. In this theory the Hilbert space of quantum states of a mechanical system is constructed of some sections of a complex line bundle over a symplectic manifold. The foundation of geometric quantization is the fact, discovered independently by Kirillov, Souriau and Kostant, that every coadjoint orbit of a Lie group is endowed with a symplectic structure, and for the *integrable* one the symplectic structure is *pre-quantized* by a line bundle.

A relevant feature of geometric quantization is its close relationship with the theory of irreducible unitary representations of Lie groups. Kirillov [13] proposed a program of realizing irreducible unitary representations of a Lie group G by geometric quantization of its coadjoint orbits. This program, which became known as the *orbit method*, has been highly successful, with contributions from many mathematicians such as Auslander-Kostant, Duflo, Rossmann, Schmid, Vogan, and others.

In the 1980s, Guillemin and Sternberg [10] studied the geometric quantization of more general G-equivariant Kähler manifolds M. The resulting G-representations are reducible in general. Guillemin and Sternberg proved the ground-breaking result that the multiplicities of this G-representation are calculated in terms of geometric quantizations of the symplectic quotients of M. They conjectures that the phenomenon of “quantization commutes with reduction” (in short, “[Q,R]=0”) holds in much greater generality, for the quantization of arbitrary prequantized symplectic manifolds by Dirac operators.

In the 1990s, following Witten’s idea of non-abelian localization, there was considerable progress on the “[Q,R]=0” problem, through work of Meinrenken, Vergne, Jeffrey-Kirwan, Guillemin, Duistermaat and Wu. The conjecture was finally solved by Meinrenken (1996) [18] using Lerman’s technique of symplectic cutting. Subsequently, Tian and Zhang (1997) gave an analytic proof of the Guillemin-Sternberg conjecture using a deformation of the Dirac operator, and Paradan developed a K-theoretic approach using the theory of transversally elliptic operators (2001).

2 Recent Developments

Today, geometric quantization continues to be a highly active area, with links to physics, symplectic geometry, representation theory, index theory, differential geometry and geometric analysis. One purpose of the workshop was to present different facets of the “geometric quantization” theme, including topics such as

– quantization of Hamiltonian manifolds,
– group valued momentum maps,
– moduli space problems,
– branching rules in representation theory,
– semiclassical asymptotics,
– index theory (Atiyah-Singer and Connes-Kasparov).

There are many open problems and active developments surrounding geometric quantization, many of which were addressed during the workshop. For example, the recent generalization of the Verlinde formulas to *equivariant* Verlinde formula opens a wide range of question. The application of simplicial techniques to the geometric realization of higher cohomology groups is a new direction; for instance, there still is no good finite-dimensional realization of the multiplicative gerbe on compact Lie groups. Also quite recently, there have been exciting new developments on the application of C^*-algebras to problems in representation theory. These and other directions were all presented in workshop talks.

3 Presentation Highlights

The workshop featured researchers with distinct backgrounds and research interests, as well as PhD students and postdoctoral fellows. We made an effort to organize each day of the workshop around a different theme, starting with one or two one-hour survey lectures. This was followed by more specialized 45-minute talks in the afternoon.

Below, we describe some of the highlights in each of these themes.

3.1 Foundational aspects of geometric quantization

Yael Karshon: Geometric quantization with metaplectic-c structures

The classical geometric quantization procedure with the ”half-form correction”, as developed in the 1960s and 1970s, has a number of deficiencies. For example, this approach does not provide a quantization of complex projective space of even complex dimension, due to the absence of a ”half form bundle”, and one cannot equivariantly quantize any symplectic toric manifold, due to the absence of an ”equivariant half form bundle”. In her talk, Yael Karshon reported on work with Jennifer Vaughan (expanding on earlier work of Harald Hess), of a new type of geometric quantization procedure. The new approach uses metaplectic-c structures to incorporate the ”half form correction” into the prequantization stage, and it does apply to the examples mentioned above.

Michèle Vergne: Semi-classical limits of geometric quantization and the graded equivariant Todd class

Let T be a torus with Lie algebra $\mathfrak{t}$ and let M be a compact Hamiltonian T-manifold with Kostant pre-quantum line bundle L, and with a proper moment map. The quantization $RR(M, L)$ of these data is a virtual representation of T, defined as the index of the T-equivariant Spin-c Dirac operator with coefficients in L. Replacing L with its tensor powers, one obtains a sequence $RR(M, L^k)$ of such representations. Let $Q \subset \mathfrak{t}^*$ be the weight lattice, and

$$m(\lambda, k)$$

the multiplicity of the irreducible representation of highest weight $\lambda \in Q$ in $RR(M, L^k)$. The distribution

$$\mathfrak{m} = k^{-\dim M/2} \sum_{\lambda \in Q} m(\lambda, k) \delta_{\frac{1}{k}\lambda}$$

has been much-studied in the literature; its limit for $k \to \infty$ is the Duistermaat-Heckman measure of the Hamiltonian T-space. In her lecture, Michèle Vergne explained that this distribution has a complete asymptotic expansion

$$\mathfrak{m} \sim \sum_{i=0}^{\infty} k^{-i} \operatorname{Td}_i$$

where the coefficients are the twisted DuistermaatHeckman distributions associated with the *graded equivariant Todd class* of M. When f polynomial, the resulting asymptotic series for the pairing $\langle \mathfrak{m},\ f \rangle$ is finite and exact.

Mathai Varghese: Equivariant index theory in the noncompact context, and the relation to quantisation, reduction and PSC metrics

Mathai Varghese gave a survey lecture on aspects of index theory and quantization, in situations where the group or the manifold (or both) are non-compact. For compact spin manifolds, the non-vanishing of the index gives a topological obstruction to the existence of positive scalar-curvature metrics (PSC's). Mathai explained his lecture how to generalize Lichnerowicz' theorem, to the setting where M is a noncompact spin manifold with a proper cocompact action of a Lie group G. To define the equivariant index, he works with a group C^*-algebra, whose K-theory $K_0(C^*(G))$ is a recipient of the equivariant index map. In recent joint work with Hao Guo and Hang Wang, he thus obtains an obstruction to the existence of G-invariant positive scalar-curvature metrics as an element of this K_0-group. Mathai also described some existence results, for the case of 'almost connected Lie groups' whose maximal compact subgroup has nonabelian identity component. For instance, if the G-action is cocompact and proper, and if the action has a global slice with a proper K-action, then a G-invariant PSC metric exists. He discussed many applications of these results, as well as further generalizations. He furthermore explained various results of Mathai-Zhang and Mathai-Hochs on the 'quantization commutes with reduction' conjecture, again for situations where the group or the manifold are non-compact.

3.2 Applications of geometric quantization

Martin Puchol: G-invariant holomorphic Morse inequalities

Let G be a connected compact Lie group, acting on a manifold M, and let $L \to M$ be a holomorphic line bundle. Suppose $E \to M$ is a G-equivariant holomorphic vector bundle, and consider the sequence of vector bundles $E \otimes L^k$. In his talk, Martin Puchol described an extension of Demailly's holomorphic Morse inequalities to this equivariant context, computing the invariant part of the Dolbeault cohomology of $E \otimes L^k$. The formula is an instance of the "quantization commutes with reduction", relating the invariant part of the Dolbeault cohomology of M to the Dolbeault cohomology of the symplectic quotient $M /\!\!/ G = \mu^{-1}(0)/G$, where $\mu \colon M \to \mathfrak{g}^*$ is Kostant's momentum map for the line bundle L, and using a regularity assumption to guarantee that the quotient is smooth. The inequalities are expressed in terms of the curvature of the bundle induced by $E \otimes L^k$ on this reduction.

Laurent Charles: Toeplitz operators and entanglement entropy

The first part of Laurent Charles' talk was an expository introduction to the Berezin-Toeplitz quantization of Khler manifolds. The second part described his work with B. Estienne, focussing on a particular class of Berezin-Toeplitz operators whose symbols are characteristic functions. He described their spectral distribution, and presented a two-terms Weyl law. As an application, he obtained the so-called area law for the entanglement entropy in the Quantum Hall Effect.

George Marinescu: Berezin-Toeplitz quantization for eigenstates of the Bochner-Laplacian on symplectic manifolds

George Marinescu's talk continued the theme of Berezin-Toeplitz quantization, describing joint work with L. Ioos, W. Lu and X. Ma. In this context, he regarded as the *quantum space* the space of eigenstates of the renormalized Bochner Laplacian on a symplectic manifold, corresponding to eigenvalues localized near the origin. He showed that this quantization has the correct semiclassical behavior, and constructed the corresponding star-product.

3.3 Geometric quantization and mathematical physics

Sergei Gukov: Geometric quantization and the equivariant Verlinde formula

Sergei Gukov reported on his work, with Andersen and Pei, on the Verlinde formula for the quantization of the Higgs bundle moduli spaces and stacks for any simple and simply-connected reductive group G. This was generalized to a Verlinde formula for the quantization of parabolic Higgs bundle moduli spaces and stacks. These authors also proved that these dimensions form a one-parameter family of 1 + 1-dimensional TQFT, uniquely classified by the complex Verlinde algebra. It is a one-parameter family of Frobenius algebras obtained as a deformation of the classical Verlinde algebra for G.

Daniel Freed: Eta-invariants on pin manifolds and time-reversal symmetry

Daniel Freed described some recent joint work with Mike Hopkins. As he explained, this work may be seen as a consistency check for M-theory, specifically its invariance property under time reversal. This consistency check leads to concrete calculations, involving eta-invariants for odd-dimensional pin manifolds.

Stephan Stolz: From factorization algebras to functorial field theories

The concept of functorial field theories was introduced by Atiyah-Segal in the 1980s, while factorization algebras were pioneered by Beilinson and Drinfeld (2004) and further developed in the recent work by Costello and Guilliam. Stefan Stolz decribed a further understanding of the relationship between these two approaches, based on his joint work with Gwyer and Teichner. In broad terms, their result states that any $\mathcal{G}$-factorization algebra determines a twisted $\mathcal{G}$-field theory. Here $\mathcal{G}$ is additional structure, and a $\mathcal{G}$-factorization algebra is a functor from manifolds M with a $\mathcal{G}$-structure, with morphisms the embeddings, into the category of cochain complexes, the latter playing the role of observables. One advantage of this approach is that concrete examples factorization algebra are rather easy to describe in practice.

Nikhil Savale: A Gutzwiller type trace formula for the magnetic Dirac operator

The usual Gutzwiller trace formula gives a semi-classical approximation to the spectrum of a Schrödinger operator, in terms of data (periods, Maslov indices and so on) of the periodic orbits of the associated classical mechanical system. For a suitable class of manifolds, including metric contact manifolds with non-resonant Reeb flow, Nikhil Savale proved a Gutzwiller type trace formula for the associated magnetic Dirac operator involving contributions from Reeb orbits on the base. As an application, he obtained a semiclassical limit formula for the eta invariant.

3.4 Geometric quantization in representation theory

Nigel Higson: Discrete series representations, the Dirac operator and C^*-algebra K-theory

Nigel Higson gave an expository talk on the subject of C^*-algebra K-theory for reductive groups. Following an explanation of the foundations of operator K-theory, he explained how some of the known results of the representation theory of Lie groups can be formulated in K-theoretic, and described some of the new results suggested by this viewpoint. A central theme in this story is Harish-Chandras parametrization of the discrete series representations, and the realization of discrete series representations using the Dirac operator. Higson also touched on other parts of Harish-Chandras theory of tempered representations that are relevant from the K-theoretic point of view, as well as the the philosophy of *Mackey's machine*.

Yanli Song: Fourier transform, orbital integral and character of representations

In the 1980s, Connes and Moscovici studied the equivariant index theory of G-invariant elliptic pseudo-differential operators, acting on non-compact homogeneous spaces. They proved an L^2-index formula using the heat kernel method, which is related to the discrete series representation of Lie groups. In his talk, Yanli Song, following his joint work with Xiang Tang, discussed the orbital integral of the heat kernel, and its relation with the Plancherel formula. One result in this context is a generalization of Connes-Moscovici's analytic index to the limit of discrete series case. Song's talk also touched on recent work by Hochs and Wang, who had obtained a fixed point theorem for the topological side of the index.

Hang Wang: K-theory, fixed point theorem and representation of semisimple Lie groups

The approach to representation theory of Lie groups through the K-theory of C^*-algebras was continued in the talk of Hang Wang, who reported on her work with Peter Hochs. More specifically, their work used the K-theory of reduced group C^*-algebras and their trace maps, in order to study tempered representations of semisimple Lie groups from the viewpoint of index theory. She explained how for a semisimple Lie group G, every K-theory generator can be viewed not only as the equivariant index of some Dirac operator, but also as a *family* of representations, parametrised by the A-factor of the Levi component of a cuspidal parabolic subgroup. In particular, if the Lie group G admits discrete series representations, then the corresponding K-theory classes are realized as equivariant geometric quantizations of the associated coadjoint orbits. Applying orbital traces to the K-theory group, Hochs and Wang obtained a fixed point formula; its application to the realization of discrete series representations, recovered the Harish-Chandra character formula. The result may be regarded as a noncompact analogue of the Atiyah-Segal-Singer fixed point theorem, in relation to the Weyl character formula.

Peter Hochs: K-types of tempered representations

Peter Hochs desribed his joint work with Song and Yu, as well as with Higson and Song. Let G be a real semisimple Lie group, and $K \subset G$ a maximal compact subgroup. A tempered representation π of G is an irreducible representation that occurs in the Plancherel decomposition of $L^2(G)$. Similar to the fact that an irreducible representation of a compact Lie group is determined by its restriction to the maximal torus, a substantial amount of information about π is captured by the restriction $\pi|_K$ of π to the maximal compact subgroup K. By realizing this restriction as the geometric quantization of a suitable space, which (under a regularity assumption on π) is a coadjoint orbit, a suitable version of the *quantization commutes with reduction* principle leads to geometric expressions for the multiplicities of the irreducible representations of K in $\pi|_K$, that is, the K-types of π. This was done for the discrete series representations in work of Paradan in 2003. Hochs' joint work with Song and Yu extended this result to arbitrary tempered representation. The resulting multiplicity formula was obtained in a different way, for tempered representations with regular parameters, by Duflo and Vergne in 2011. In joint work with Higson and Song, Hochs also gave a new proof of Blattner's formula for multiplicities of K-types of discrete series representations, using the ideas of geometric quantization. This formula was first proved by Hecht and Schmid in 1975, and later by Duflo, Heckman and Vergne in 1984.

3.5 Loop groups, Higher structures

Konrad Waldorf: Fusion in loop spaces

In his lecture, Konrad Waldorf gave a general overview of fusion operations in loop spaces, based on his work since around 2010. The notion of a *fusion structure* on objects over the loop space was emphasized by Stolz and Teichner in their 2005 article *"The spinor bundle on loop spaces"*. Specifically, these authors had developed the notion of an orientation on loop spaces, giving precise definitions and proofs for the fact that orientations of the loop space LM are in canonical 1-1 correspondence with spin structures on M. Waldorf proved that the category of abelian gerbes with connection over a smooth manifold is equivalent to a category of principal bundles over the free loop space, equipped with a connection and with a fusion product with respect to triples of paths.

Chris Kottke: A new theory of higher gerbes

Complex line bundles on a manifold M are classified naturally up to isomorphism by degree two integer cohomology $H^2(M,\mathbb{Z})$, and it is of interest to find finite-dimensional geometric objects which are similarly associated to higher degree cohomology, and which would allow for a similar theory of connections, curvature and so on. For degree 3 cohomology $H^3(M,\mathbb{Z})$, such geometric realizations are known as *gerbes*. There are various models for gerbes, due respectively to Giraud, Brylinski, Hitchin and Chatterjee, and Murray. Various notions of"higher gerbes" have also been defined, though these tend to run into technicalities and complicted bookkeeping associated with higher categories.

Chris Kottke's talk, based on his joint work with Melrose, proposed a new geometric version of higher gerbes in the form of "multi simplicial line bundles". The basic idea is to replace the simplicial manifolds, as used for example in Murray's notion of bundle gerbes, by multi-simplicial manifolds, and to study line bundles with connections on such spaces. The theory avoids many of the higher categorical difficulties, yet still captures key examples including the string obstruction associated to the first Pontrjagin class

$$\frac{1}{2}p_1(M) \in H^4(M,\mathbb{Z}).$$

The authors found that every integral cohomology class is represented by one of these objects, in the guise of a line bundle on the iterated free loop space equipped with a "fusion product" (as defined by Stolz and Teichner and further developed by Waldorf) for each loop factor.

Richard Melrose: Generalized products and quantization

The 'generalized products' in Richard Melroses lecture are simplicial manifolds, for example a sequence of manifolds

$$M,\ M\times M,\ M\times M\times M,\ldots$$

with the usual face and degeneracy maps. He explained how any simplicial manifold $M(1), M(2),\ldots$ generates an algebra of pseudo-differential operators on $M(1)$, using conormal distributions relative to the degeneracy map $M(1)\to M(2)$ as integral kernels. The fact that these form an algebra is encoded in the properties of a simplicial manifold. Instead of manifolds, one can also consider other categories, and Melrose's talk focused on the category of manifolds with corners. He then applied these ideas to various examples, including compactifications of Lie groups.

Yiannis Loizides: Quantization of Hamiltonian loop group spaces

A theorem of Freed, Hopkins and Teleman relates the representation theory of the loop group LG of a compact Lie group G to the equivariant twisted K-theory of G. In the special case of a connected, simply connected and simple Lie group, the theorem states that there is an isomorphism of rings

$$R_k(G)\cong K_0^G(G,\mathcal{A}^{k+h^\vee})$$

where the left hand side is the level k Verlinde ring, and the right hand side the equivariant K-homology of G with twisting (Dixmier-Douady class) at the shifted level $k+h^\vee$, where $h^\vee$ is the dual Coxeter number.

(Freed-Hopkins-Teleman work with twisted equivariant K-cohomology, which is related by a Poincare duality isomorphism.) In his talk, explained how to realize classes in $K_0^G(G, \mathcal{A}^{k+h^\vee})$ in terms of D-cycles, thereby defining an explicit inverse to the Freed-Hopkins-Teleman map on the chain level. One idea in this construction is an abelianization procedure, working with a tubular neighborhood of the maximal torus T in G. One application of this result, relating it to earlier work of Loizides with Meinrenken and Song, provides the equivalence of two methods for quantizing Hamiltonian loop group spaces.

3.6 Index theory

Jean-Michel Bismut: Hypoelliptic Laplacian and the trace formula

The hypoelliptic Laplacian gives a natural interpolation between the Laplacian and the geodesic flow. This interpolation preserves important spectral quantities. In his talk, Jean-Michel Bismut explained the construction of the hypoelliptic Laplacian in the context of compact Lie groups. In this case, the hypoelliptic Laplacian is the analytic counterpart to localization in equivariant cohomology on the coadjoint orbits of loop groups. The construction for noncompact reductive groups ultimately produces a geometric formula for the semisimple orbital integrals, which are the key ingredient in Selberg trace formula. In both cases, the construction of the hypoelliptic Laplacian involves the Dirac operator of Kostant.

Xiang Tang: Hochschild Homology of Proper Lie Groupoids

Xiang Tang reported on his ongoing work with Pflaum and Posthuma. For a compact Lie group acting on a smooth manifold, this work studies the complex of basic relative forms on the inertia space of a proper Lie groupoid, as originally constructed by Brylinski. In his talk, Xiang Tang explained how basic relative forms can be used to study the Hochschild homology of the convolution algebra.

Rudy Rodspohn: Diff-equivariant index theory

Since the early 1980s, Alain Connes had developed his *Noncommutative Geometry* program. One of the primary goals of this program was to extend index theory to singular situations, where the usual tools of differential geometry are no longer available. A typical instance of such a setting are regulations foliations on manifolds for which there does not exist a transverse measure invariant under the action of the holonomy group. In the late 1990s, Connes and Moscovici proved an equivariant index theorem for this these context, and formulated a conjecture concerning the calculation of this index in terms of characteristic classes. In his talk, Rudy Rodspohn gave an account of the history and motivation of the index problem. And explained his solution, together with Denis Perrot, of the Connes-Moscovici conjecture, with special emphasis on aspects relating to quantization.

Frédéric Rochon: Torsion on hyperbolic manifolds of finite volume

Frédéric Rochon described joint work with Werner Mueller. Let X be any finite volume hyperbolic manifold of dimension d. Given a finite dimensional irreducible complex representation of the group $G = \mathrm{SO}_0(d, 1)$, one can associate a canonical flat vector bundle E over X, together with a canonical bundle metric h. For d odd and under some mild hypotheses on the manifold X, Rochon explained how to obtain a formula relating the analytic torsion of (X, E, h) with the Reidemeister torsion of an associated manifold with boundary. The argument involves a family of compact manifolds degenerating to X in a suitable sense. In the arithmetic setting, this formula can be used to derive exponential growth of torsion in cohomology for various sequences of congruence subgroups.

Thomas Schick: Index theory and secondary spectral invariants to understand moduli spaces of Riemannian metrics

Thomas Schick presented an overview of applications of index theory to the existence of positive scalar curvature metrics. Given a manifold X, his talk considered the moduli space of positive scalar curvature metrics by the action of the diffeomorphism group of X, fixing some base point x_0 and also having differential at x_0 equal to the identity:

$$\mathrm{Riem}^+(X)/\,\mathrm{Diff}_{x_0}(X)$$

There are many questions about this space – for example, whether it is empty, if non-empty what the number of connected components would be. Another interesting quotient to consider is the the set of concordance equivalence classes

$$\mathrm{Riem}^{+}(X)/\,\mathrm{concordance}\,.$$

To investigate the relationship, one considers more general bordisms, and introduces a space $\mathrm{Pos}(X)$ of cycles (M, g, f) (where g is a PSC metric and $f\colon M \to X$ is a map) modulo an equivalence relation. A deep theorem of Stolz fits $\mathrm{Pos}_m(X)$ into an exact sequence gives an exact sequence involving the spin bordism group $\Omega^m(X)$. Schick's lecture described the many unknown questions about this exact sequence, and some recent progress. The second part of the lecture explained the relationship of these ideas to the Lichnerowicz-Schrödinger formula for the square of a Dirac operator.

4 Outcome of the Meeting

The workshop at BIRS brought together mathematicians whose work involves quantization in rather different forms and with many different techniques: loop groups, topological K-theory, analytic estimates, C^*-algebras, representation theory. The philosophy of geometric quantization served as a focal point for the interaction between all of these areas. The workshop provided an opportunity for experts working on different aspects of the theory to exchange ideas, leading to fresh insights and new developments.

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