

New Directions in Applied Linear Algebra

John Pearson (University of Edinburgh),
Jennifer Pestana (University of Strathclyde),
David Silvester (University of Manchester),
Valeria Simoncini (Università di Bologna)

27 August–1 September, 2023

1 Overview of the Field

Linear algebra is a fundamental component of pure mathematics. It also lies at the heart of many scientific, engineering, and industrial applications. Research and development in numerical linear algebra includes theoretical studies, algorithmic implementations on advanced computer architectures, and applications to various disciplines. The research landscape is constantly changing with an array of new theoretical problems continually arising as a result of the rapidly developing application areas of data science and uncertainty quantification.

2 Recent Developments and Open Problems

The distinctive aspect of the workshop was the focus on general theoretical problems, that is, on the development of rigorous numerical analysis. It is well established that bespoke numerical linear algebra techniques, including the design of direct and iterative methods, are a vital component of solvers for huge-scale problems arising in optimization constrained by deterministic physical models (partial differential equations). What is much less well established is the design of efficient linear algebra components in the contexts of machine learning (where the underlying model is associated with large data) and in nonlocal network dynamics (where deterministic physical models are defined on lattices), making optimal control more challenging.

Uncertainty quantification (UQ) has emerged as an important area of scientific computing within the last decade. This development is motivated by engineering practice. In a UQ setting one would like to model uncertainty in the input data (for example, material properties) in a mathematically rigorous way and then propagate the effect of the uncertainty in a quantity of interest (for example, the reliability of a component under loading). Efficient numerical linear algebra plays a crucial role in the design of efficient algorithms that accomplish this task (so-called *forward* UQ) and has led to significant interest and research developments in low-rank approximation and tensor decompositions.

3 Programme Structure

The hybrid workshop had 26 in-person participants with an additional 16 participants attending online. The in-person participants included 8 females and 7 early-career researchers. The list of in-person participants included experts from several EU countries (Belgium, Croatia, Germany, Italy, Ireland, Norway) as well as

from the UK, USA, and Canada. All in-person participants were given the opportunity to give a talk. In addition to the technical program we organised three informal after-dinner social events and a hike/ramble on the Tuesday afternoon.

The programme included four discussion sessions. Two of these sessions were on Wednesday afternoon. The first of these was organised by Jon Cockayne and focused on the development of probabilistic linear solvers, where solution accuracy is guaranteed in a statistical setting. The second discussion session on Wednesday focused on EDIA aspects. This session also featured specific issues raised by early-career researchers. It was noted that the requirement to be mobile is clearly challenging for any postdoctoral researcher in a two-person relationship. The other two discussion sessions were held on Thursday afternoon. The first of these was organised by Michael Mahoney and focused on the development of library software for randomized linear algebra. The final discussion session returned to the issues raised by the early-career researchers. This was a productive session in the sense that strategies were identified by more senior participants that might alleviate issues raised by the early-career researchers on the previous day.

4 Programme Specifics

The general themes that provided a structure for the discussions were

- Optimization & Machine Learning
- UQ & Related Linear Algebra Challenges
- Low-Rank & Randomized Linear Algebra
- Applications to Physics & Network Science

All the talks that were given can be categorised into one of these 4 topics. We built in considerable flexibility, however, and several talks spanned multiple themes.

A key driver of state-of-the-art applied linear algebra techniques arises from the fast and robust solution of problems from high-dimensional optimization and machine learning. There has been much cutting-edge research on randomization for interior point methods [1, 2], preconditioning for double-saddle point systems [3], numerical linear algebra for four-dimensional variational data assimilation (4D-Var) [4], low-rank methods for data-driven optimization [5, 6], convergence of Anderson acceleration [7], stabilization for reduced-order models [8] with application to PDE-constrained optimization, interior point methods for optimal transport [9], preconditioners for such problems [10], and scalable preconditioned iterative solvers for PDE-constrained optimization [11]. The main focus of the workshop on Monday and Wednesday was on recent developments in these subject areas, and we received presentations about these topics. Abstracts for the talks as well as additional references are provided below.

Petros Drineas: Randomized linear algebra for interior point methods

Linear programming is a central problem in computer science and applied mathematics with numerous applications across a wide range of domains, including machine learning and data science. Interior point methods (IPMs) are a common approach to solving linear programs with strong theoretical guarantees and solid empirical performance. The time complexity of these methods is dominated by the cost of solving a linear system of equations at each iteration. In common applications of linear programming, particularly in data science and scientific computing, the size of this linear system can become prohibitively large, requiring the use of iterative solvers which provide an approximate solution to the linear system. Approximately solving the linear system at each iteration of an IPM invalidates common analyses of IPMs and the theoretical guarantees they provide. In this talk we will discuss how randomized linear algebra can be used to design and analyze theoretically and practically efficient IPMs when using approximate linear solvers [1, 2].

Chen Greif: Numerical solution of double saddle-point systems

Double saddle-point systems are drawing increasing attention in the past few years, due to the importance of associated applications and the challenge in developing efficient numerical solvers. In this talk we describe some of their numerical properties and make some distinctions between this family and the more well-established class of classical 2-by-2 block-structured saddle-point systems. We derive eigenvalue bounds,

expressed in terms of extremal eigenvalues and singular values of block sub-matrices. The analysis includes bounds for preconditioned matrices based on block diagonal preconditioners using Schur complements, and it is shown that in this case the eigenvalues are clustered within a few intervals bounded away from zero. Analysis for approximations of Schur complements is included [3].

Jemima Tabcart: Block alpha-circulant preconditioners for all-at-once diffusion-based correlation operators

I will present practical methods to apply block α -circulant preconditioners approximately to an all-at-once system from ocean data assimilation. We apply the preconditioner via a nested iterative method within an outer Chebyshev semi-iteration. We present theory on the rate of convergence and use this to design an adapted approach which ensures each sub-problem is solved to the same tolerance. A panel of numerical experiments reveal that using an appropriate choice of α our new approaches are competitive in terms of outer iterations and matrix-vector products compared to current choices of preconditioner.

Madeleine Udell: Low rank approximation for faster optimization

Low rank structure is pervasive in real-world datasets. This talk shows how to accelerate the solution of fundamental computational problems, including eigenvalue decomposition, linear system solves, composite convex optimization, and stochastic optimization (including deep learning), by exploiting this low rank structure. We present a simple method based on randomized numerical linear algebra for efficiently computing approximate top eigendecompositions, which can be used to replace large matrices (such as Hessians and constraint matrices) with low rank surrogates that are faster to apply and invert. The resulting solvers for linear systems (NystromPCG), composite convex optimization (NysADMM), and deep learning (SketchySGD) demonstrate strong theoretical and numerical support, outperforming state-of-the-art methods in terms of speed and robustness to hyperparameters [5, 6].

Hans De Sterck: Asymptotic convergence speed of windowed Anderson acceleration: an overview of results and open problems

Anderson acceleration with window size m is a nonlinear convergence acceleration mechanism for fixed-point iterative methods that is widely used in scientific computing, optimization and machine learning. For many applications $AA(m)$ dramatically improves the convergence speed, both when iteration counts are small, and asymptotically for large iteration counts. Nevertheless, there are still no known results yet that can bound or quantify the improvement in asymptotic convergence speed provided by windowed $AA(m)$. In this talk I will give an overview of what is known about the asymptotic convergence speed of windowed $AA(m)$ and highlight some recent results and open problems [7].

Howard Elman: A new method of model order reduction for parametrized constrained partial differential equations

Model order reduction techniques are effective for solving parametrized models involving PDEs, including models of incompressible flow, where the constraint is the incompressibility constraint, and in optimal control, where the constraints themselves are PDEs. However, reduced models may fail to be inf-sup stable. We present a new approach for generating reduced bases in this scenario, using a so-called stacked reduced basis, which avoids some of the difficulties associated with inf-sup stability. We show that this approach is effective although in some circumstances it also requires stabilization, which can be done using either classic methods of penalization or through Petrov-Galerkin methods. Computational tests are presented for models based on PDE-constrained optimization and incompressible flow. This is joint work with Kayla D. Davie (University of Maryland, College Park).

Jacek Gondzio: Applications of interior point methods: From sparse approximations to discrete optimal transport

A variety of problems in modern applications of optimization require a selection of a ‘sparse’ solution, a vector with preferably few nonzero entries. Such problems may originate from very different applications in computational statistics, signal or image processing or compressed sensing, finance, machine learning and discrete optimal transport, to mention just a few. Sparse approximation problems are often solved with dedicated and highly specialised first-order methods of optimization. In this talk I will argue that these problems may be very efficiently solved by interior point methods. The key to their success is a design of suitable preconditioners [9].

Enrico Facca: Linear algebra challenges in optimal transport problems

Optimal Transport (OT) is a branch of optimization theory that has prompted a number of successful applica-

tions in different fields in the last few years. However, the wide-spread application of OT ideas in science and engineering is still fragmentary, due to the lack of accurate and efficient numerical solvers. We will explore some recent advances and open issues in the construction of efficient preconditioners involved in the solution of PDE-based formulations of OT [10].

Tucker Hartland: A scalable interior-point method for PDE and bound-constraint optimization

We present a scalable method for large-scale elliptic PDE and bound constrained optimization. Such problems are a means to learn unknown aspects of PDE-based models and are particularly challenging as their optimality systems are coupled PDEs and complementarity conditions. We utilize an interior-point Newton method with a filter-line search strategy and solve the Newton linear systems with a block Gauss-Seidel preconditioned Krylov-subspace method. We show that, in certain problem regimes, the number of Krylov-subspace iterations is independent of discretization and ill-conditioning from the bound constraints. We conclude with scaling results on an example problem using the MFEM and hypre packages [11].

The main focus of the workshop on Tuesday was on uncertainty quantification and associated linear algebra challenges. Abstracts for the talks as well as additional references are provided below.

Elisabeth Ullmann: Uncertainty quantification with PDE-based models

In this talk I will outline recent trends in uncertainty quantification (UQ) with PDE-based models and connect the trends to linear algebra. The topics in UQ are: (1) advanced random field models such as hierarchical random fields, (2) dynamical systems with random coefficients and their sensitivity analysis, (3) reliability analysis and rare events. The associated linear algebra topics are: matrix factorization, matrix eigenvalue problems, numerical continuation, and dimension reduction [12].

Christopher Miller: Iterative solvers for stochastic Galerkin discretizations of Stokes flow with random data

We study the iterative solution of linear systems of equations arising from stochastic Galerkin finite element discretizations of symmetric saddle point problems with stochastic data. As preconditioners we consider block matrices with building blocks that have Kronecker product structure. We derive bounds for the eigenvalues of the preconditioned system matrix and show how the existence requirements of a Bramble-Pasciak-type conjugate gradient method can be met. Based on numerical experiments for Stokes flow with random viscosity, we compare a MINRES solver with block diagonal preconditioner to a conjugate gradient method with block triangular preconditioner [13].

Aretha Teckentrup: Smoothed circulant embedding and applications in multilevel Monte-Carlo methods

Parameters in mathematical models for physical processes are often impossible to determine fully or accurately, and are hence subject to uncertainty. In this talk, we employ the multilevel Monte Carlo (MLMC) method to compute expected values of quantities of interest related to partial differential equations with random coefficients. We make use of the circulant embedding method for sampling from the coefficient, and devise a smoothing technique to further improve the computational complexity of the MLMC estimator [14].

Giang Tran: Sparse random features and applications in time series data

Random feature methods have been successful in various machine learning tasks, are easy to compute, and come with theoretical accuracy bounds. They serve as an alternative approach to standard neural networks since they can represent similar function spaces without a costly training phase. However, for accuracy, random feature methods require more measurements than trainable parameters, limiting their use for data-scarce applications or problems in scientific machine learning. In this talk, we will introduce the sparse random feature expansion to obtain parsimonious random feature models. Specifically, we leverage ideas from compressive sensing to generate random feature expansions with theoretical guarantees even in the data-scarce setting. We also present a sparse random mode decomposition to extract intrinsic modes from challenging time-series data and propose a new method to predict the infectious using sparse random feature models and time-delay equations. Applications of our methods on identifying important variables in high-dimensional settings as well as on decomposing music pieces, visualizing black-hole mergers, and epidemiologic forecasting will be addressed [15].

Jon Cockayne: BayesCG: A probabilistic linear solve

Probabilistic linear solvers are iterative solvers for linear systems that return probability measures. These measures are designed to capture uncertainty in the solution due to incomplete computation. In this talk we

review the probabilistic linear solver BayesCG, and discuss ongoing research into applications of the solver. Specifically, we introduce a new application of the solver to the problem of multiple related linear systems, and discuss its role as a means of providing both warm starting and subspace recycling [16].

The availability of huge amounts of data, especially in data science, has created new great challenges for numerical linear algebra: the need for compact though flexible data representations and reduced (towards dimension-independent) computational costs. Migration from deterministic to probabilistic regime has led to the successful development of *randomized* linear algebra algorithms, which allow the approximate solution of problems with dramatic memory savings [17, 18]. These approaches are combined with strategic data allocations, such as low rank matrices, tensor representations, and structure-aware (hierarchical) formulations. Throughout the workshop many presentations have addressed these challenges, by focusing both on the theoretical foundations underlying randomized strategies, as well as on the impact of this methodology in various problems in data science and scientific computing. Abstracts for the talks as well as additional references are provided below.

Misha Kilmer: Structured matrix approximations via tensor decompositions

We provide a computational framework for approximating a class of structured matrices; here, the term structure is very general. Our matrix-to-tensor invertible mapping allows us to pose the matrix-approximation problem as a tensor-approximation problem. Mapping the tensor approximant back to matrix space, we obtain a structured matrix approximation that can be expressed as a sum of structured Kronecker products or written in block low-rank form. We illustrate the ability of our method to uncover block-low-rank format from applications such as system identification and show we can uncover sum of structured Kronecker products structure on several matrices from the SuiteSparse collection [19].

Martina Iannacito: An algebraic algorithm for blind source separation and tensor decomposition

The blind source separation problem is a classical signal processing problem. Mathematically it translates into the factorization of S , an observed signal matrix, into a mixing and a source signal matrix, i.e., $X = MS^T$. In [Domanov, 2016], the authors propose a criteria checking list to guarantee the decomposition generic uniqueness for a deterministic method class. From this and [De Lathauwer, 2006; Domanov, 2014] results, we design an algebraic algorithm to retrieve the unique factorization of X . Moreover, this algorithm finds application in the canonical polyadic tensor decomposition. The structure of the objects involved in the algorithm allows an efficient implementation. This is joint work with Ignat Domanov and Lieven De Lathauwer.

Zvonimir Bujanović: Randomized algorithms with rank-one random vectors

Randomized algorithms in numerical linear algebra typically generate random vectors from standard distributions, such as Gaussian. However, in certain applications it may be advantageous to run computations with vectors compatible with the underlying structure of the problem. In this talk we discuss algorithms that use rank-one vectors, i.e., Kronecker products of two random vectors, which may allow for faster operations with matrices represented as short sums of Kronecker products. Possible applications include trace estimation and large-scale eigenvalue computation; we provide theoretical and numerical evidence that the use of rank-one instead of unstructured random vectors still leads to good estimates [20].

Davide Palitta: Sketched and truncated polynomial Krylov subspace methods

Sketching can be seen as a random dimensionality reduction able to preserving the main features of the original problem with probabilistic confidence. Sketching is one of the most promising tools to boost numerical computations although its understanding is still limited. In this talk we present the main features of sketching and how this can be combined with Krylov methods for the solution of large-scale linear systems. However, thanks to the novel sketched Arnoldi relation we will illustrate, the results discussed in this talk can be extended to the numerical evaluation of matrix functions and the solution of matrix equations [21].

Lars Grasedyck: Challenges in Hierarchical Low Rank Tensor Approximation

The first part of the talk will provide a general introduction to low rank tensor formats for the data sparse representation of higher order tensors and multivariate functions (CP, Tucker, TT, HT). The second half of the talk is geared towards more recent results on multiparametric multigrid methods and challenges in approximation with regard to special distance measures (e.g. KL) and constraints on the set of admissible tensors (e.g. positivity) [22].

The focus of the workshop on Thursday afternoon and Friday morning was on applications to physics and network science. Abstracts for the talks as well as additional references are provided below.

David Silvester: Fast solution of incompressible flow problems with two-level pressure approximation

Reliable and efficient iterative solvers for models of steady incompressible flow emerged in the early 1990s. Strategies based on block preconditioning of the underlying matrix operators using (algebraic or geometric) multigrid components have proved to be the key to realising mesh independent convergence (and optimal complexity) without the need for tuning parameters, particularly in the context of classical mixed finite element approximation. The focus of this contribution is on efficient solver strategies in cases where (an inf-sup) stable Taylor-Hood mixed approximation is augmented by a piecewise constant pressure in order to guarantee local conservation of mass. The augmentation leads to over-specification of the pressure solution requiring a redesign of the established solver technology. This enrichment process causes over-specification of the pressure, which complicates the design and implementation of efficient solvers for the resulting linear systems. We first describe the impact of this choice of pressure space on the matrices involved. Next, we show how to recover effective solvers for Stokes problems, using a preconditioner based on the singular pressure mass matrix, and for Oseen systems arising from linearising the Navier-Stokes equations, by using a two-stage pressure convection-diffusion strategy [23]. This is joint work with Jennifer Pestana.

David Bindel: Linear algebra, invariant circles, and fusion plasmas

Stellarators are a class of non-axisymmetric toroidal magnetic confinement systems for fusion relevant plasmas. Confinement depends on nested magnetic flux surfaces that act as transport barriers. Flow along magnetic field lines has a Hamiltonian structure, and we generically expect such invariant surfaces to exist in stellarators by KAM theory. However, many devices exhibit not only nested flux surfaces, but also magnetic islands and chaotic regions. Physicists often explore this visually via Poincaré plots that show the structure of the monodromy map for magnetic field line flow around a device. However, to optimize devices we want fast and automatic methods to identify such structure. In this talk, we describe several such analysis methods, with an emphasis on new methods with roots in linear algebra, vector sequence extrapolation, and system identification. We illustrate our method with a variety of examples [24].

Igor Simunec: Computation of the von Neumann entropy of large matrices via trace estimators and rational Krylov method

We consider the problem of approximating the von Neumann entropy of a large and sparse symmetric positive semidefinite matrix A , defined as $\text{trace}(f(A))$ where $f(z) = -z \log(z)$, which is an important task in several fields, such as quantum information theory and network science. We consider the use of polynomial and rational Krylov subspace algorithms [25] within both randomized trace estimators [26] and probing methods based on graph colorings [27]. We develop error bounds and heuristics that are employed in the implementation of the algorithms. The performance of the methods is illustrated with several numerical experiments. The activities mentioned herein are performed in the framework of the project MUR_PNRR_PE_NQSTI_BELTRAM funded by the Italian Ministry for University and Research (MUR). A preprint is available on arXiv [28]. This is a joint work with Michele Benzi and Michele Rinelli.

Niall Madden: Linear solvers for singularly perturbed problems

We consider the solution of linear systems arising from *parameter robust* discretizations of singularly perturbed convection-diffusion problems, whose accuracy is proven to be independent of the perturbation parameter. There are many studies of such methods, but few of the robust solution of resulting linear systems. I'll discuss challenges for both direct and iterative solvers for such systems, and propose a multigrid-based preconditioning strategy tuned to the matrix structure induced by using layer-adapted meshes. Supporting theoretical and numerical results will be presented [29]. This is joint work with Anh Thi Nhan (SCU) and Scott MacLachlan (MUN).

Felix Kwok: Multigrid interpretation of a three-level Parareal algorithm

In this talk, we show how a recently introduced three-level variant of Parareal can be interpreted as a multigrid reduction in time (MGRIT) method for a particular choice of restriction/prolongation operators and relaxation scheme. Specifically, we show that when suitably initialized, the MGRIT V-cycle with F-relaxation and injection as the prolongation operator generates iterates that are identical to three-level Parareal. This equivalence allows one to analyze the convergence of three-level Parareal from two different perspectives, using either ODE or multigrid techniques [30]. This is joint work with Stephanie Friedhoff and Martin J. Gander.

5 Conclusion

The primary achievement of the workshop was that it enabled the goal of connecting expert researchers who would not interact otherwise. It also successfully enabled the informal interaction of early-career researchers with more senior experts: specifically by providing an opportunity for them to showcase their opinions and their research work in a non-intimidating atmosphere. This resulted in active collaboration and exchange of knowledge between the participants. New perspectives were found on a range of challenging problems and new collaborations were initiated. Follow-on workshops and activities are being planned and are likely to take place over the next two years.

References

- [1] R. Bhattacharjee, G. Dexter, P. Drineas, C. Musco, and A. Ray, Sublinear time eigenvalue approximation via random sampling. In *50th International Colloquium on Automata, Languages, and Programming*, Leibniz International Proceedings in Informatics, **261**, 21:1–21:18, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023.
- [2] G. Dexter, A. Chowdhury, H. Avron, and P. Drineas, On the convergence of inexact predictor-corrector methods for linear programming, In *Proceedings of the 39th International Conference on Machine Learning*, Proceedings of Machine Learning Research, **162**, 5007–5038, 2022.
- [3] S. Bradley and C. Greif, Eigenvalue bounds for double saddle-point systems, *IMA Journal of Numerical Analysis* (2022), drac077.
- [4] J. M. Tabart and J.W. Pearson, Saddle point preconditioners for weak-constraint 4D-Var, *arXiv preprint arXiv:2105.06975*, 2023.
- [5] S. Zhao, Z. Frangella, and M. Udell, NysADMM: Faster composite convex optimization via low-rank approximation, In *Proceedings of the 39th International Conference on Machine Learning*, Proceedings of Machine Learning Research, **162**, 26824–26840, 2022.
- [6] Z. Frangella, P. Rathore, S. Zhao, and M. Udell, SketchySGD: Reliable stochastic optimization via randomized curvature estimates, *arXiv preprint arXiv:2211.08597*, 2022.
- [7] D. Wang, Y. He, and H. De Sterck, On the asymptotic linear convergence speed of Anderson acceleration applied to ADMM, *Journal of Scientific Computing* **88** (2021).
- [8] K.D. Davie and H.C. Elman, An Assessment of the supremizer and aggregation methods of stabilization for reduced-order models, *arXiv preprint arXiv:2303.06101*, 2023.
- [9] F. Zanetti and J. Gondzio, An interior-point-inspired algorithm for linear programs arising in discrete optimal transport, *INFORMS Journal on Computing*, 2023.
- [10] E. Facca, G. Todeschi, A. Natale, and M. Benzi, Efficient preconditioners for solving dynamical optimal transport via interior point methods, *arXiv preprint arXiv:2209.00315*, 2022.
- [11] T.A. Hartland, *Hierarchical Approaches for Efficient and Scalable Solution of Inverse Problems Governed by Partial Differential Equations*, Ph.D. thesis, University of California, Merced, CA, 2022.
- [12] D. Kressner, J. Latz, S. Massei and E. Ullmann, Certified and fast computations with shallow covariance kernels, *Foundations of Data Science* **2** (2020), 487–512.
- [13] C. Miller, S. Ullmann, and J. Lang, A Bramble-Pasciak conjugate gradient method for discrete Stokes equations with random viscosity. *SIAM/ASA Journal on Uncertainty Quantification* **7** (2019), 787–805.
- [14] A. Istratuca and A. Teckentrup, Smoothed circulant embedding with applications to multilevel Monte Carlo methods for PDEs with random coefficients, *arXiv preprint arXiv:2306.13493*, 2023.

- [15] A. Hashemi, H. Schaeffer, R. Shi, U. Topcu, G. Tran, and R. Ward, Generalization bounds for sparse random feature expansions, *Applied and Computational Harmonic Analysis* **62** (2023) 310–330.
- [16] J. Cockayne, C.J. Oates, I.C.F. Ipsen, and M. Girolami, A Bayesian conjugate gradient method (with discussion), *Bayesian Analysis* **14** (2019), 937–1012.
- [17] P. Martinsson, J. Tropp, Randomized numerical linear algebra: Foundations and algorithms. *Acta Numerica*, **29** (2020), 403–572.
- [18] D.P. Woodruff, Sketching as a tool for numerical linear algebra. *Foundations and Trends in Theoretical Computer Science*, **10** (2014), 1–157.
- [19] M.E. Kilmer and A.K. Saibaba, Structured matrix approximations via tensor decompositions, *SIAM Journal on Matrix Analysis and Applications* **43** (2022), 1599–1626.
- [20] Z. Bujanović and D. Kressner, Norm and trace estimation with random rank-one vectors, *SIAM Journal on Matrix Analysis and Applications* **42** (2021), 202–223.
- [21] D. Palitta, M. Schweitzer, and V. Simoncini, Sketched and truncated polynomial Krylov methods: Evaluation of matrix functions *arXiv preprint arXiv:2306.06481*, 2023.
- [22] L. Grasedyck, M. Klever, C. Löbbert, and T.A. Werthmann, A parameter-dependent smoother for the multigrid method, *arXiv preprint arXiv:2008.00927*, 2020.
- [23] J. Pestana and D.J. Silvester, Fast solution of incompressible flow problems with two-level pressure approximation, *arXiv preprint arXiv:2303.10233*, 2023.
- [24] M. Ruth, Symplectic Map Tools, *GitHub Repository*, <https://github.com/maxeruth/SymplecticMapTools.jl>, 2023.
- [25] S. Gttel, Rational Krylov approximation of matrix functions: Numerical methods and optimal pole selection, *GAMM-Mitteilungen* **36** (2013), 8–31.
- [26] R. A. Meyer, C. Musco, C. Musco, and D. P. Woodruff, Hutch++: Optimal stochastic trace estimation, In *SIAM Symposium on Simplicity in Algorithms (SOSA)*, 142–155, 2021.
- [27] A. Frommer, C. Schimmel, and M. Schweitzer, Analysis of probing techniques for sparse approximation and trace estimation of decaying matrix functions, *SIAM Journal on Matrix Analysis and Applications* **42** (2021), 1290–1318.
- [28] M. Benzi, M. Rinelli, and I. Simunec, Computation of the von Neumann entropy of large matrices via trace estimators and rational Krylov methods, *arXiv preprint arXiv:2212.09642*, 2022.
- [29] S.P. MacLachlan, N. Madden, and T.A. Nhan, A boundary-layer preconditioner for singularly perturbed convection diffusion. *SIAM Journal on Matrix Analysis and Applications* **43** (2022), 561–583.
- [30] S. Friedhoff, M.J. Gander, and F. Kwok, Multigrid interpretation of a three-level Parareal algorithm, In *Domain Decomposition Methods in Science and Engineering XXVII*, Lecture Notes in Computational Science and Engineering, Springer-Verlag, in print, 2023.