Motives and Invariants: Theory and Applications to Algebraic Groups and Their Torsors

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1 Overview of the Field

The algebraic theory of quadratic forms was founded by Witt in 1937, when he introduced the Witt ring of a general field, thereby providing a rich new viewpoint on a topic that had been studied in number-theoretic contexts for many decades prior. After undergoing a major expansion throughout the second half of the 20th century, the subject reached a high point in the 1990s with Voevodsky's celebrated proof of the Milnor conjecture relating the graded Witt ring of a field to its mod-2 Galois cohomology and Milnor K-theory rings. This landmark result established a long-expected weak classification theorem for quadratic forms over fields in terms of (partially-defined) Galois cohomological invariants. At the same time, its proof, based on pioneering developments in motivic cohomology and motivic homotopy theory, has gone on to inspire a broad new algebraic-geometric approach to the subject. The central theme here is the study of algebraic cycles on (products of) quadrics and isotropic Grassmannians, which are homogeneous varieties for orthogonal groups and are naturally viewed in this context as objects of various motivic categories. The resulting theory has complemented existing approaches greatly, leading to the solution of old problems that seemed beyond the reach of classical methods, and enabling us to gain deeper insight into the classification of quadratic forms.

As has long been understood, the picture painted above finds its most natural generalization within the study of (linear) algebraic groups over general fields. Beyond the orthogonal groups defined by quadratic forms, it was shown by Weil in the 1960s that all reductive algebraic groups of classical type over a field can be described in terms of a larger class of algebraic objects, namely algebras with involution (or, equivalently, hermitian forms over involutive division algebras). From this perspective, the theory of quadratic forms appears as a key example, which despite its specificities, has been a constant source of inspiration for the wider study of algebraic groups and their torsors. This includes the study of groups of exceptional type, though more specialized tools and arguments are often needed here in the absence of a good general description of these groups in terms of concrete algebraic structures (still, many realizations of these groups also involve quadratic or hermitian forms in one way or another).

In this broader context, a cohomological invariant of an algebraic group G over a field F is understood as a natural transformation from the functor $H^1_{ét}(-, G)$ of isomorphism classes of G-torsors on the category of extensions of F to a suitable abelian cohomological functor, typically taken to be Galois cohomology with coefficients in a torsion Galois module (although it is also natural to consider other cycle modules in the sense of Rost, as well as Witt groups of quadratic forms). Echoing the situation for quadratic forms (i.e., torsors for split orthogonal groups), the problem of determining the cohomological invariants of a given group G plays a central role in studying the classification of its torsors. The existence of non-trivial invariants already imposes lower bounds on the so-called essential dimension of G, a numerical invariant that informally measures the number of parameters needed to describe a generic G-torsor, and which has been the subject of much investigation in recent years. In the same spirit, the theory of cohomological invariants offers means by which to attack from the negative side the generalized Noether problem asking whether the classifying space of a connected algebraic group over an algebraically closed field is stably rational. The last decade has seen continued progress in these directions, not only for connected algebraic groups, but finite groups also. In the former case, a significant development came in work of Merkurjev that establishes an exact sequence describing the degree-3 (Galois) cohomological invariants of an arbitrary semisimple group, extending foundational work of Rost from the 1990s on the simply connected case. This goes some way towards establishing a complete understanding of the low-degree invariants of semisimple groups. While a general understanding of higher-degree cohomological invariants remains a very challenging goal, important progress has also been made here in special cases. For instance, Semenov exploited ideas in the proof of the Milnor conjecture to construct a (partially defined) degree-5 invariant for the split exceptional group E_8, answering in the process an open question of Serre on groups of type E_8 over the field of rational numbers.

Despite playing a central role in the general theory, the cohomological invariants discussed above are insufficient for many purposes. To this end, the motivic algebraic-geometric developments emerging from the proof of the Milnor conjecture have proved extremely fruitful, particularly in their application to index-reduction problems that feature prominently throughout the subject as a whole. A basic unifying question here is the following: Given projective homogeneous varieties X and Y under actions of semisimple algebraic groups, when does X admit a point over the function field of Y? In the case where X is a generalized Severi-Brauer variety, this was fully answered in the 1990s by Panin, Merkurjev, Wadsworth and others using tools from algebraic K-theory. Beyond this result, precise answers are only known in special cases. Over the past two decades, however, the advent of new tools with which to investigate rationality problems for algebraic cycles on projective homogeneous varieties has led to significant progress on cases where X is an isotropic Grassmannian attached to a quadratic or hermitian form (we note here the work of Karpenko, Merkurjev and Vishik, in particular). This progress has seen the introduction and systematic investigation of new discrete invariants of semisimple algebraic groups that capture important information of a motivic nature. For example, a significant advance made by Vishik on an old problem of Kaplansky concerning the possible values of the u-invariant of a field was founded on the introduction and study of the so-called elementary discrete invariant for special orthogonal groups. A certain component of this invariant, known as the J-invariant, has subsequently been adapted to the study of arbitrary semisimple groups (initially in work of Petrov, Semenov and Zainoulline), and has been applied with great effect to the study of exceptional groups in particular. While many aspects of the motivic approach have so far been somewhat isolated from the theory of cohomological invariants, other recent developments suggest a more unified picture. In particular, for a general algebraic group G, Smirnov and Vishik have proposed the investigation of refined characteristic-class invariants of G-torsors taking values in motivic cohomology groups of certain associated simplicial schemes. Implementing this for split orthogonal groups, they have constructed new invariants for quadratic forms – the subtle Stiefel-Whitney classes – that not only refine the classical Galois cohomological invariants (the ordinary Stiefel-Whitney classes), but also determine the J-invariant, among other things. This opens the door to a broader motivic-homotopic approach to the subject.

Finally, within the overall picture, the study of algebraic groups and their torsors over special fields of arithmetic or geometric interest has also occupied a position of central importance, and has served as a source of inspiration for the development of the general theory. Here, recent years have seen a great deal of progress, with a key driver on the arithmetic side being the general field patching techniques originally introduced by Harbater, Hartmann and Krashen, and further developed and applied in the work of these authors, Colliot-Thélène, Parimala, Suresh and others.

2 Recent Developments and Open Problems

- **Patching, local-global principles and field invariants.** Over the past 15 years, the development of field patching techniques originally introduced in the work of Harbater, Hartmann and Krashen has led to remarkable progress on the study of local-glocal principles for homogeneous spaces over function fields of arithmetic surfaces and other low-dimensional fields, as well as related problems concerning invariants of such fields defined in terms of torsors, e.g., the period-index problem for central simple algebras, and the u-invariant problem for quadratic forms. In a recent development, P. Gille and Parimala have exploited these methods to establish a general local-global principle for the existence of rational points on projective homogeneous varieties over semi-global fields, extending previously known results for generalized Severi-Brauer varieties and quadrics. On a more geometric side, refined tools from real and analytic geometry have been brought to bear. For instance, Benoist used methods from Hodge theory to determine the u-invariant of the function field of a real surface, and to give a positive answer to the period-index problem for function fields of real surfaces with no real points. Benoist has also applied étale cohomological techniques to obtain the best-known bounds on the Pythagoras numbers of Laurent series fields in several variables over real closed fields, thereby answering a long-standing question due to Choi, Dai, Lam and Reznick.

- **Cohomological invariants of algebraic groups.** Following Blinstein and Merkurjev's determination of the degree-2 cohomological invariants of reductive algebraic groups, Lourdeaux completed the picture in his PhD thesis by showing that the normalized degree-2 invariants of a smooth connected algebraic group G are in one-to-one correspondence with the extensions of G by the multiplicative group. A similar result for finite groups was also established by Bailey in her PhD thesis (with a more elementary and direct proof being given recently by S. Gille). A consequence of these results is the fact that these invariants arise from projective representations of the group. Building on the work of Merkurjev mentioned in §1, Baek completely determined the reductive indecomposable degree-3 invariants of all split semisimple groups of classical type. Despite recent progress, however, little is known in general about cohomological invariants of higher degree, the study of which remains a difficult and important challenge for the area.

- **Massey products in Galois cohomology.** The Milnor and Bloch-Kato conjectures, proved by Rost and Voevodsky, assert that the Galois cohomology ring of an absolute Galois group with coefficients in the integers modulo a prime p has generators in degree 1 and only one relation (the Steinberg relation) in degree 2, i.e., is a Koszul algebra. Closely related to these is the vanishing conjecture for Massey products, recently formulated by Mináč and Tân, which – if true – would give further constraints on the cohomology of absolute Galois groups. More specifically, this conjecture asserts that if the Massey product of $n \geq 3$ elements in the first Galois cohomology group of the absolute Galois group of a field F with coefficients in the integers modulo a prime p is defined, then this product (which is actually a set in the second Galois cohomology group) contains 0. This has been proven for number fields by Harpaz and Wittenberg, for $n = 3$ and arbitrary fields by Mináč and Tân as well as (independently) by Efrat and Matzri (after older results for $n = 3$ and number fields by Hopkins and Wickelgren). Recently, Merkurjev and Scavia have settled the case of four-fold Massey products at the prime 2 (after older results for $n = 4$ and number fields by Guillot, Mináč and Topaz). While the vanishing conjecture may on the surface seem tangential to the main themes of the workshop, the methods used to attack it, as well as its implications, are in fact intimately related to the theory of algebraic groups and their torsors, and the problem represents an important challenge towards establishing a better understanding of the structure of absolute Galois groups. The developments above are also in line with the recent work of De Clercq and Florence on so-called smooth profinite groups, which, as is now expected, may lead to more elementary proofs of the Milnor and Bloch-Kato conjectures.

- **Motives of projective homogenous varieties.** The focus of the research here is on understanding the possible decompositions of the motives of projective homogenous varieties under actions of semisimple algebraic groups (in various motivic categories) with a view towards applications to the classification of torsors over general fields. An important general role is played here by the aforementioned J-invariant, which governs the Chow-motivic decomposition of the variety of Borel subgroups into indecomposable motives (with coefficients in the integers modulo a prime). While this invariant was initially only defined for groups of inner type, Geldhauser and Zhykhovich have recently succeeded in extending the theory to certain groups of outer type, broadening the reach of its applicability. In other recent work of Geldhauser and Petrov, new constraints on the Chow-motivic decompositions of arbitrary projective homogeneous varieties for a given semisimple group G coming from the J-invariant have been established. This is based on a new uniform approach to the subject that exploits the Hopf algebra structure on the Chow ring of the split form of G. The results obtained have revealed new insights even in the extensively-studied case of quadrics. Another major theme here is the theory of upper indecomposable summands in the Chow motives of projective homogeneous varieties developed in the work of Karpenko. Recently, De Clercq and Quéguiner-Mathieu have revisited this theory by introducing and studying so-called Tate traces of Chow motives, i.e., maximal pure Tate summands. Using Karpenko's theory, they show that, with finite coefficients, the Chow motives of projective homogeneous varieties for semisimple groups of inner type are determined up to isomorphism by their Tate traces over all extensions of the base field. These investigations are closely related to the earlier work of De Clercq and Garibaldi on the notion of motivic equivalence for semisimple groups, and are also related to the recent work of Vishik on so-called isotropic motivic categories (the latter may provide a higher-viewpoint explanation on some of the isomorphism criteria obtained). Finally, replacing Chow groups with other oriented cohomology theories in the sense of Levine and Morel, such as Morava K-theories, provides further insight, and has been the subject of much recent work. In particular, Sechin and Semenov showed that the Morava K-theory motives of quadrics detect the vanishing of invariants of their underlying quadratic forms. As an application, they obtained strong new results on the torsion in the integral Chow groups of quadrics, something that remains poorly understood in general despite its significance for the theory of quadratic forms. These advances were, in part, made possible after the Rost nilpotence principle for Morava K-theory motives of projective homogeneous varieties was established by S. Gille and Vishik.

- **Quadratic forms in characteristic 2.** The last decade of the 20th century has seen rapid progress of the algebraic theory of quadratic forms in the wake of Voevodsky's proof of the Milnor conjecture. The methods that originated in Voevodsky's work have been further developed by Vishik, Karpenko and others (see also the previous paragraph on motives of projective homogeneous varieties) to solve many important problems related to isotropy questions, in particular isotropy and Witt indices of quadratic forms over function fields of quadrics. Much of that work was restricted to characteristic not 2, but over recent years these methods have been extended to characteristic 2. Crucial in this context was the the theory of Steenrod operations on mod-2 Chow groups of smooth varieties in characteristic 2 by Primozic that allowed Karpenko to complete the proof of the Hoffmann-Totaro conjecture on the first Witt indices of quadratic forms also in characteristic 2 and in that case also for singular forms. In fact, the theory of totally singular forms is of some independent interest as the methods to prove results on isotropy and Witt indices in the totally singular case are much more algebraic in nature. Such questions have been studied extensively by Scully. In particular, it was Scully who provided the proof of the Hoffmann-Totaro conjecture in the totally singular case.

3 Presentation Highlights and Scientific Progress Made

The programme consisted of 4 one-hour overview talks and 20 further scientific talks (mostly 50 minutes in length) on the latest developments in the field. All talks were given in person. We provide a brief summary of the presentations made, with quotes from speakers' abstracts being indicated by the use of italic letters.

We first report on a proposed talk by PAVEL SECHIN that unfortunately had to be cancelled due to visa problems. The intended topic of the talk was recent progress on *cohomological invariants of projective homogeneous varieties through Morava motives*, in particular a construction of an *injective functorial homomorphism from* $K^M_{n+1}(k)/2$ *to the group of invertible* $K(n)$*-motives over* k, which sends the class of a given nonsingular quadratic form q in the $(n+1)$st power of the fundamental ideal to an invertible summand of the $K(n)$-motive of the projective quadric associated to q. This is based on recent joint work of Sechin with Lavrenov.

Turning to the talks themselves, NIKITA KARPENKO gave an overview lecture on his recent work (partly in collaboration with Devyatov and Merkurjev) concerning the study of the Chow ring modulo torsion for generically twisted flag varieties of spin groups. As outlined in the talk, this work brings striking new applications to the study of degrees of partial splitting fields for nonsingular quadratic forms with trivial discriminant and Clifford invariant, enhancing an important 2005 paper of Totaro in which the torsion indices of the spin groups were completely determined.

In a related direction, CHARLES DE CLERCQ gave a talk *on the classification of direct summands of motives of projective homogeneous varieties, through the Tate motives they contain over field extensions*, a report on his recent work with Quéguiner-Mathieu that introduces and studies the notion of Tate traces for Chow motives. There were further talks on the Chow motives of projective homogeneous varieties by NIKITA GELDHAUSER and MAXIM ZHYKHOVICH, each presenting new results on the J-invariant, *a discrete invariant of semisimple algebraic groups which describes the motivic behaviour of the variety of Borel subgroups*. This invariant was *an important tool to solve several long-standing problems. For example, it plays an important role in the progress on the Kaplansky problem about possible values of the u-invariant of fields by Vishik*.

ALEXANDER VISHIK's talk embraced the more abstract setting of Voevodsky's motivic categories, and concerned the relation between the notions of numerical equivalence and the recently-introduced isotropic equivalence for Chow groups and Morava K-theories. Isotropic realizations *provide an algebro-geometric object with its local versions parametrized by various extensions of the base field, versions residing in the isotropic category whose* *complexity is similar to that of the topological category*. The talk culminated with a discussion of the speaker's recent proof that *over so called flexible fields, isotropic Chow groups coincide with numerical ones*.

Further talks related to motivic homotopy theory were given by BAPTISTE CALMÈS and OLIVIER HAUTION. The latter spoke on the so-called *concentration theorem for actions of linearly reductive groups on affine schemes* and a *consequence for equivariant stable motivic homotopy theory, asserting that, upon inverting appropriate elements, the equivariant cohomology of a scheme with a group action is "concentrated" on its fixed locus*. Calmès talk, based on joint work with Dotto, Harpaz, Hebestreit, Land, Moi, Nardin, Nikolaus and Steimle, concerned *a new definition of Hermitian K-theory as a universal object in the context of quadratic functors and stable infinity-categories* and how it *enables us to simplify and generalize its classical properties, study the relationship between different objects of quadratic nature such as symmetric bilinear forms or quadratic ones, and completely remove or clarify the invertibility of* 2 *assumptions scattered in the theory until now*.

DIEGO IZQUIERDO gave an overview talk on relations between some Diophantine properties and cohomological properties of fields, such as the relation between the C_i property and cohomological dimension, or Serre's conjecture II, where the cohomological dimension controls the existence of rational points for certain homogeneous spaces. This talk was followed by a presentation of GIANCARLO LUCCHINI ARTECHE on his joint work with Izquierdo on some "higher versions" of Serre's conjecture II.

In a similar fashion, ALEXANDER MERKURJEV's overview lecture on Massey products in Galois cohomology was followed by the talk of FEDERICO SCAVIA on his recent joint work with Merkurjev on the vanishing conjecture for Massey products. As already mentioned in §2, this topic is closely related to the (proven) Milnor and Bloch-Kato conjectures, which have been at the heart of MATHIEU FLORENCE's talk, where we heard about *a new approach to the norm residue isomorphism Theorem of Rost, Suslin and Voevodsky*, developed in particular with Charles De Clercq. More specifically, it was explained here that these famous conjectures follow from a lifting conjecture for Galois representations.

RAMAN PARIMALA spoke on her recent work with P. Gille that establishes a *Hasse principle for projective homogeneous spaces over semiglobal fields* (i.e., fields of transcendence degree one over a complete discretely valued field) using patching techniques. This extends previously known results on generalized Severi-Brauer varieties and quadrics, and proves for projective homogeneous spaces a conjecture of Colliot-Thélène, Parimala and Suresh initially made over function fields of p-adic curves. Parimala's talk was preceded by an overview talk by JEAN-LOUIS COLLIOT-THÉLÈNE on the degree-3 unramified cohomology of algebraic varieties over finite fields with torsion coefficients.

The theory of essential dimension for algebraic groups and related structures was addressed in the talks by DANNY OFEK and ZINOVY REICHSTEIN. Ofek reported on joint work with Reichstein in which valuation-theoretic techniques are exploited to establish a new *lower bound on the essential dimension of a Brauer class*, and a result on the *essential dimension of the Witt class of a Hermitian form, generalizing a theorem of Chernousov and Serre*. Reichstein's talk concerned recent work with Edens that establishes a new upper bound on the essential dimension of the finite symmetric group S_n over a field of odd characteristic for a certain infinite family of positive integers n (depending on the characteristic of the field). This bound runs contrary to the widely expected value of the essential dimension in characteristic 0. A fundamental concept in the study of essential dimension of algebraic groups is that of versal torsors. In his talk, URIYA FIRST spoke on recent work (partly joint with Florence and Rosengarten) that investigates the existence (or non-existence) of torsors over schemes satisfying various versality properties. Applications of these results to the symbol length problem for Azumaya algebras over semilocal rings (containing enough roots of unity) were also discussed. Continuing the theme of torsors over semilocal rings, THOMAS UNGER reported on his joint work with Astier that establishes *Pfister's local-global principle for hermitian forms over Azumaya algebras with involution over semilocal rings*, showing in particular that *the Witt group of nonsingular hermitian forms is 2-primary torsion*.

Talks on general structure of algebraic groups were given by PHILLIPE GILLE and SRIMATHY SRINIVASAN. The former spoke on joint work with Guralnick that studies *semi-continuity for the unipotent dimension of group schemes in view of application to finite groups*, and the latter on the classification of semisimple groups of classical type over a general base in terms of Azumaya algebras with involution.

ARTURO PIANZOLA reported on joint work with P. Gille that provides *a criterion for certain algebraic objects over Jacobson schemes to be forms of each other based on their behaviour at closed fibres*. An application to a question of Burban on loop algebras of simple finite-dimensional complex Lie algebras was also discussed.

Quadratic forms in characteristic 2 were addressed in two talks. ADAM CHAPMAN (based on joined work with Quéguiner-Mathieu) gave new constructions of so-called minimal quadratic forms for function fields of non-singular conics in characteristic 2. In characteristic not 2, the picture of such minimal forms is reasonably complete, but characteristic 2 poses significant problems because of the necessity to consider singular forms. DIKSHA MUKHIJA presented new results, joint with Ahmed Laghribi, on the (non)excellence of field extensions for quadratic forms in characteristic 2, including the fact that function fields of totally singular conics are generally not excellent (contrary to the case of non-singular conics considered in Chapman's talk). We also mention here CAMERON RUETHER's talk on triality, where he extended to the setting over schemes, the definition of the canonical quadratic pair of a Clifford algebra, recently defined by Dolphin and Quéguiner-Mathieu over fields of characteristic 2, and explored the consequences on triality.

4 Outcome of the Meeting

In this workshop, we brought together established specialists and young researchers working on topics related to the study of quadratic and hermitian forms, linear algebraic groups, homogeneous varieties and Galois cohomology. With a mix of expertise coming from the various sides of the overall picture outlined in §1, we hoped for a synergistic impact, stimulating new developments and collaborative projects among different groups of researchers. The group of 40 in-person participants included 8 postdoctoral researchers, 1 PhD student and 1 undergraduate student.

During the workshop, we observed a lot of informal discussions between participants, some of which were reported to us after the conference. For instance, Jean-Louis Colliot-

Thélène mentioned short discussions with six different people or groups of people, on various topics such as *an ongoing joint project on patching* and *explicit Brauer type counterexamples to the Hasse principle for biquadratic extensions as discussed in the literature*. Some discussions led to new results and new projects. The appendix by Alexander Merkurjev in Nikita Karpenko's recent preprint "Finite extensions partially splitting PGO-torsors" is an explicit outcome of the meeting. A joint project of Charles De Clercq, Nikita Karpenko, and Anne Quéguiner-Mathieu, which aims at extending the recent results of De Clercq and Quéguiner-Mathieu mentioned above to groups of outer type is work in progress, following a suggestion of several participants after De Clercq's talk.

This was the first major conference on the topics discussed in this report since the onset of the COVID-19 pandemic. As many of the participants communicated, the overall atmosphere was excellent. We had a chance to meet young colleagues, some for the first time, and several of them presented their work in talks. Zinovy Reichstein reported that *the two UBC students who attended the workshop were both happy to be invited. They enjoyed the workshop and learned a lot from it*. One of them has been following up some suggestions he got in response to his lecture, and the other has extended the result he talked about to the characteristic-2 case during the conference.