Equidistribution of quartic Gauss sums at primes arguments

Joint work with
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Alberta Number Theory Days XV
Banff, March 2024

Quadratic Gauss sums

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Let p be an odd prime and let

$$\chi_p = \left(\frac{\cdot}{p}\right) \; : \; (\mathbb{Z}/p\mathbb{Z})^* \to \{\pm 1\} \subset \mathbb{C}^*$$

$$a \mapsto \begin{cases} 1 & a \equiv \square \bmod p \\ -1 & a \not\equiv \square \bmod p \end{cases}$$

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We define the quadratic Gauss sum $g_2(p) \in \mathbb{C}^*$ by

$$g_2(p) = \sum_{a=0}^{p-1} \left(\frac{a}{p}\right) \zeta_p^a, \quad \text{where } \zeta_p = e^{2\pi i/p}.$$

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We will present a rigorous demonstration of this most elegant theorem, unsuccessfully attempted for many years in various ways, and finally successfully perfected through singular and quite subtle considerations...

Cubic Dirichlet characters

We want

$$\chi_p : (\mathbb{Z}/p\mathbb{Z})^* \to \{1, \omega, \omega^2\} \subset \mathbb{C}^*, \quad \omega = e^{2\pi i/3}.$$

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If χ_p is not trivial, then we must have

$$3 \mid p - 1 \iff p \equiv 1 \bmod 3.$$

For $p \equiv 1 \bmod 3$, and $(a, p) = 1$, let

$$\chi_p(a) = \left(\frac{a}{p}\right)_3 \equiv a^{\frac{p-1}{3}} \bmod p.$$

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This gives 2 primitive characters modulo $p \equiv 1 \bmod 3$,

$$\chi_p,\ \chi_p^2 = \overline{\chi}_p : (\mathbb{Z}/p\mathbb{Z})^* \to \{1, \omega, \omega^2\} \subset \mathbb{C}^*.$$

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Let $p \equiv 1 \bmod 3$, and

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$$|g_3(p)| = \sqrt{p} \implies \begin{aligned} g_3(p) &= e^{i\theta_p}\sqrt{p} \\ \overline{g_3(p)} &= e^{-i\theta_p}\sqrt{p} \end{aligned}$$

with a unique $\theta_p \in [0, \pi]$ such that

$$g_3(p) + \overline{g_3(p)} = 2\sqrt{p}\cos\theta_p = g_3(\chi_p) + g_3(\overline{\chi}_p).$$

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Kummer (1846) computed θ_p for $3 \leq p \leq 500$, $p \equiv 1 \bmod 3$, and how they distribute in the 3 possible intervals

$$I_1 = [0, \frac{\pi}{3}], \quad I_2 = [\frac{\pi}{3}, \frac{2\pi}{3}], \quad I_3 = [\frac{2\pi}{3}, \pi].$$

Distribution of cubic Gauss sums

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Assuming GRH, Dunn and Radziwill (2021+) proved (a smooth version of) Patterson's conjecture.

Distribution of cubic Gauss sums

p_0	n	l_1	l_2	l_3	
0	45	24	14	7	Kummer
0	611	272	201	138	von Neumann-Goldstine
0	1000	438	322	240	Lehmer
0	1259	552	416	291	Cassels
25 000	192	83	69	40	Cassels
30 000	119	49	40	30	Cassels
100 000	165	49	68	48	Cassels

Equidistribution

Let $u_1, u_2, \ldots$ be a sequence of real numbers with $u_i \in [a, b]$. The sequence is equidistributed on $[a, b]$ if for each $I = (\alpha, \beta) \subseteq [a, b]$, we have

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Theorem (Weyl's criterion, 1916)

The sequence $u_1, u_2, \ldots$ is equidistributed on $[a, b]$ iff for each $k \neq 0 \in \mathbb{Z}$,

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} e^{\frac{2\pi i u_n k}{b-a}}}{N} = 0 \iff \sum_{n=1}^{N} e^{\frac{2\pi i u_n k}{b-a}} = \sum_{n=1}^{N} e\left(\frac{u_n k}{b-a}\right) = o(N).$$

Cubic characters on $K = \mathbb{Q}(\omega)$

For each prime $\pi \in \mathbb{Z}[\omega]$, and for $a \in \mathbb{Z}[\omega]$, $(a, \pi) = 1$, we have

$$\chi_\pi(a) = \left(\frac{a}{\pi}\right)_3 \equiv a^{\frac{N(\pi)-1}{3}} \bmod \pi \subset \{1, \omega, \omega^2\}.$$

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Let $p, a \in \mathbb{Z}$, $p \equiv 1 \bmod 3$, and $p = \pi\overline{\pi}$ and $(a, p) = 1$. Then,

$$\chi_p(a) = \left(\frac{a}{\pi}\right)_3 \text{ or } \chi_p(a) = \overline{\left(\frac{a}{\pi}\right)_3} = \left(\frac{a}{\overline{\pi}}\right)_3.$$

Cubic Gauss sums modulo $c \in \mathbb{Z}[\omega]$

We define for any $c \in \mathbb{Z}[\omega], c \equiv 1 \bmod 3$

$$g_3(c) = \sum_{a \bmod c} \left(\frac{a}{c}\right)_3 \mathbf{e}\left(\frac{a}{c}\right), \quad \mathbf{e}(z) := e^{2\pi i(z+\bar{z})}$$

$$\tilde{g}_3(c) = \frac{g_3(c)}{N(c)^{\frac{1}{2}}}$$

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Gauss showed that for any $c \in \mathbb{Z}[\omega], c \equiv 1 \text{ mod } 3$

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We have for $p \equiv 1 \bmod 3, \; p = \pi\overline{\pi}$,

$$\{\tilde{g}_3(\chi_p), \tilde{g}_3(\overline{\chi}_p)\} = \{\tilde{g}_3(\pi), \tilde{g}_3(\overline{\pi})\} = \{e^{i\theta_p}, e^{-i\theta_p}\}$$

Cubic and general Gauss sums at prime arguments

By Weyl's criterion, the angles θ_p are equidistributed in $[0, \pi]$ iff for all integers $k \neq 0$

$$\sum_{\substack{N(\pi) \leq X \\ \pi \in \mathbb{Z}[\omega] \text{ prime} \\ \pi \equiv 1 \bmod 3}} \widetilde{g}_3(\pi)^k = o(\pi(X)) = o\left(\frac{X}{\log X}\right).$$

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Conjecture (Patterson, 1978)

$$\sum_{\substack{N(\pi)\leq X \\ \pi\in\mathbb{Z}[\omega] \; prime \\ \pi\equiv 1 \bmod 3}} \widetilde{g}_3(\pi) \sim \frac{2(2\pi)^{2/3}}{5\Gamma\left(\frac{2}{3}\right)} \frac{X^{5/6}}{\log X}.$$

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Patterson's conjecture (a smooth version of) was proven Dunn and Radziwill (2021+), under GRH.

Cubic Gauss sums at prime arguments

$$\sum_{\substack{N(c)\le X \\ c\in\mathbb{Z}[\omega] \\ c\equiv 1 \bmod 3}} \widetilde{g}_3(c)\,\Lambda(c) \ll X^{30/31+\varepsilon}$$ (Heath-Brown and Patterson, 1979)

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For a general number fields K such that $\zeta_n \in K$, let S be a set of places of K containing the places at ∞, and large enough such that $\mathcal{O}_K^S$, the ring of S-integers, is a PID. Then (Patterson, 1985)

$$\sum_{\substack{c\in\mathcal{O}_K^S \\ N(c)\leq X \\ c \bmod {}^{\times} U_n(S)}} \widetilde{g}_n(c)\Lambda(c) \ll_K X^{1-\theta_n(K)+\varepsilon} + X^{19/20+\varepsilon}.$$

Quartic Gauss sums at prime argument

Theorem (D-Dunn-Hamieh-Lin, 2023)

For any $c \in \mathbb{Z}[i]$, $c \equiv 1 \bmod \lambda^3$, *with* $\lambda = 1 + i$, *let*

$$g_4(c) = \sum_{a \text{ mod } c} \left(\frac{a}{c}\right)_4 \mathbf{e}\left(\frac{a}{q}\right), \quad \mathbf{e}(z) := e^{2\pi i(z+\bar{z})}$$

$$\widetilde{g}_4(c) = \frac{g_4(c)}{N(c)^{\frac{1}{2}}}$$

For quartic Gauss sums $\widetilde{g}_4(c)$, *with* $\beta \in \{1, 1+\lambda^3\}$

$$\sum_{\substack{N(c) \leq X \\ c \in \mathbb{Z}[i] \\ c \equiv \beta \text{ mod } 4}} \widetilde{g}_4(c)\Lambda(c) \ll X^{5/6+\varepsilon}.$$

Quartic Gauss sums at prime argument

Conjecture (Quartic Gauss sums at prime argument)

For $\beta \in \{1, 1+\lambda^3\} \bmod 4$*, there exists a constant* $b_\beta \neq 0$ *such that for any* $\varepsilon > 0$ *and* $\ell \in \mathbb{Z}$*,*

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ N(c) \leq X \\ c \equiv \beta \bmod 4}} \widetilde{g}_4(c) \left(\frac{\overline{c}}{|c|} \right)^{\ell} \Lambda(c) = \begin{cases} b_\beta X^{3/4} + O_\varepsilon(X^{1/2+\varepsilon}) & \text{if} \quad \ell = 0 \\ O_{\varepsilon,\ell}(X^{1/2+\varepsilon}) & \text{if} \quad \ell \neq 0 \end{cases},$$

Quartic Gauss sums at integral argument

What about

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv \beta \bmod 4}} \widetilde{g}_4(c) R\left(\frac{N(c)}{X}\right)?$$

Quartic Gauss sums at integral argument

What about

$$\sum_{\substack{c\in\mathbb{Z}[i]\\ c\equiv\beta \bmod 4}} \widetilde{g}_4(c) R\left(\frac{N(c)}{X}\right) = \frac{1}{2\pi i}\int_{(\sigma)} \psi_\beta^{(4)}(s)\, X^s\, \widehat{R}(s) ds$$

where $\psi_\beta^{(4)}(s) = \sum_{\substack{c\in\mathbb{Z}[i]\\ c\equiv\beta \bmod 4}} \frac{\widetilde{g}_4(c)}{N(c)^s}$ cvgs absolutely for $\Re(s) > 1$.

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$$\widetilde{g}_4(c_1c_2) = \sum_{a \bmod c_1c_2} \left(\frac{a}{c_1c_2}\right)_4 \mathbf{e}\left(\frac{a}{c_1c_2}\right)$$

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$$= \left(\frac{c_1}{c_2}\right)_4 \left(\frac{c_2}{c_1}\right)_4 \widetilde{g}_4(c_1)\widetilde{g}_4(c_2).$$

Metaplectic forms

- Weil (1953) observed that the (complex) θ-function which transforms as

$$\theta\left(\frac{az+b}{cz+d}\right) = \epsilon_d \left(\frac{c}{d}\right) \sqrt{cz+d}\, \theta(z), \quad \epsilon_d = \begin{cases} 1 & d \equiv 1 \bmod 4 \\ i & d \equiv 3 \bmod 4 \end{cases}$$

can be thought as an automorphic form on $\widetilde{\mathrm{GL}}_2$, the two-fold metaplectic cover of GL_2.

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- For quartic Gauss sums, Suzuki (1983) computed the functional equation and the residue of the pole at $s = \frac{3}{4}$ in certain cases.

Shifted quartic Gauss sums

Let

$$g_4(\nu, c) = \sum_{a \bmod c} \left(\frac{a}{c}\right)_4 \mathbf{e}\left(\frac{\nu a}{q}\right)$$

$$\tilde{\psi}_{\beta}^{(4)}(s, \nu) := \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv \beta \bmod 4}} \frac{\tilde{g}_4(\nu, c)}{N(c)^s}$$

which converges absolutely for $\Re(s) > 1$.

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Let

$$\psi_\beta^{(4)}(\nu) := \mathrm{Res}_{s=3/4} \tilde{\psi}_\beta^{(4)}(s, \nu) = \mathrm{Res}_{s=5/4} \psi_\beta^{(4)}(s, \nu).$$

Functional Equation

Theorem

The functions $\psi_{i1}^{(4)}(s,\nu)$, $0 \neq \nu \in \mathbb{Z}[i]$, *and* $i = 1, \ldots, 24$ *can be meromorphically extended to* $\mathbb{C}$, *with at most two simple poles at* $s = 5/4$ *and* $s = 3/4$. *The functions are bounded in vertical strips and satisfy the functional equation*

$$\psi_{i1}^{(4)}(s,\nu) = N(\nu)^{1-s} \sum_{i=1}^{24} A_{ji}(2^{-s}) \psi_{i1}^{(4)}(2-s,\nu).$$

For $\varepsilon > 0$, *we have for* $1 + \varepsilon < \sigma < \frac{3}{2} + \varepsilon$ *and* $\left|s - \frac{5}{4}\right| > \frac{1}{8}$,

$$\psi_{i1}^{(4)}(\nu, s) \ll_{\varepsilon, ord_\lambda(\nu)} N(\nu)^{\frac{1}{2}(\frac{3}{2}-\sigma)+\varepsilon}(|s|+1)^{\frac{3}{2}(\frac{3}{2}-\sigma)+\varepsilon}$$

$$\psi_{i1}^{(4)}(\nu) \ll N(\nu)^{\frac{1}{8}}$$

Can we do better than convexity?

By the work of Suzuki (1983), for m square-free and $(m, \nu) = 1$,

$$\psi_\beta^{(4)}(m^4\nu) = \psi_\beta^{(4)}(\nu)$$

$$\psi_\beta^{(4)}(m^3\nu) = 0$$

$$\psi_\beta^{(4)}(m^2\nu) = \begin{cases} \dfrac{\overline{\widetilde{g}_4(\nu, m)}}{N(m)^{\frac{1}{4}}}\ \psi_\beta^{(4)}(\nu) & m \equiv 1 \bmod 4 \\[2ex] \dfrac{\overline{\widetilde{g}_4(\nu, m)}}{N(m)^{\frac{1}{4}}}\ \psi_\beta^{(4)}(\nu) & m \equiv 1 + \lambda^3 \bmod 4 \end{cases}$$

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It is conjectured that for all $m \in \mathbb{Z}[i]$ square-free,

$$|\psi_\beta^{(4)}(m)| = \frac{1}{N(m)^{\frac{1}{8}}}.$$

Back to quartic Gauss sums at integral argument

Let $m \in \mathbb{Z}[i]$ be square-free, then

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv \beta \bmod 4}} \widetilde{g}_4(m^2, c) R\left(\frac{N(c)}{X}\right) = \frac{1}{2\pi i} \int_{(\sigma)} \psi_\beta^{(4)}(s + \tfrac{1}{2}, m^2) X^s \widehat{R}(s) ds = \frac{c_{\beta,m}}{N(m^2)^{\frac{1}{8}}} X^{\frac{3}{4}} + O\left(X^{\frac{1}{2}+\varepsilon} N(m^2)^{\frac{1}{4}+\varepsilon}\right)$$

From integers to primes : Vaughan's identity

Let

$$H_\beta(X) = \sum_{\substack{c\in\mathbb{Z}[i]\\ c\equiv\beta \bmod 4}} \Lambda(c)\widetilde{g}(c)R\left(\frac{N(c)}{X}\right)$$

$$\Sigma_{j,\beta}(X,u) = \sum_{a,b,c} \Lambda(a)\mu(b)\widetilde{g}(abc)R\left(\frac{N(abc)}{X}\right)$$

where $a, b, c \in \mathbb{Z}[i]$ such that $abc \equiv \beta \bmod 4$, and some j-conditions on the size of a, b, c..
Then,

$$H_\beta(X) + \Sigma_{2'}(X,u) + \Sigma_{2''}(X,u) + \Sigma_3(X,u) = \Sigma_1(X,u).$$

Type 1 and Type 2 sums

$$\Sigma_{j,\beta}(X,u) = \sum_{\substack{a,b,c\in\mathbb{Z}[i] \\ a,b,c\equiv 1 \bmod \lambda^3 \\ abc\equiv\beta \bmod 4}} \Lambda(a)\mu(b)\widetilde{g}(abc)R\left(\frac{N(abc)}{X}\right)$$

where for $1 \le u \le X^{1/2}$,

$$\begin{aligned} &N(b) \le u && \text{for } j = 1, \\ &N(ab) \le u && \text{for } j = 2', \\ &N(a), N(b) \le u < N(ab) && \text{for } j = 2'', \\ &N(b) \le u < N(a), N(bc) && \text{for } j = 3, \end{aligned}$$

Bounding Type 1 sums

For Type 1 sums, using Patterson's and Suzuki's work, and an extra averaging using the quadratic large sieve, we get

$$\begin{aligned}
&\Sigma_{1,\beta}(X,u),\ \ \Sigma_{2',\beta}(X,u) \\
&\ll X^{\varepsilon} \sum_{N(\alpha)\le u} \mu^2(\alpha) \sum_{\substack{c\in\mathbb{Z}[i] \\ c\equiv\beta \bmod 4 \\ c\equiv 0 \bmod \alpha}} \tilde{g}_4(c) R\left(\frac{N(c)}{X}\right) \\
&\ll_{\varepsilon} X^{\frac{3}{4}+\varepsilon}
\end{aligned}$$

which is equivalent to Heath-Brown for cubic (2000): $X^{\frac{5}{6}+\varepsilon}$.

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By properties of quartic Gauss sums, for $(\alpha, c) = 1$,

$$\begin{aligned}
g_4(\nu, \alpha c) &= \left(\frac{c}{\alpha}\right)_4 \left(\frac{\alpha}{c}\right)_4 g_4(\nu,\alpha) g_4(\nu, c) \\
&= (-1)^{C(\alpha,c)} g_4(\nu,\alpha) g_4(\nu\alpha^2, c)
\end{aligned}$$

Bounding Type 2 sums

For Type 2 sums, using the Quadratic Large Sieve over $\mathbb{Q}(i)$, we have

$$\Sigma_{2'',\beta}(X,u), \Sigma_{3,\beta}(X,u) \ll X^{\epsilon}\left(X^{\frac{1}{2}}u + Xu^{-\frac{1}{2}}\right) \ll X^{\frac{5}{6}+\varepsilon},$$

taking $u = X^{\frac{1}{3}}$.