SIGN-CHANGING SOLUTIONS FOR CRITICAL EQUATIONS WITH HARDY POTENTIAL

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ABSTRACT. We consider the following perturbed critical Dirichlet problem involving the Hardy-Schrödinger operator on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, with $0 \in \Omega$:

$$\begin{cases} -\Delta u - \gamma \frac{u}{|x|^2} - \epsilon u = |u|^{\frac{4}{N-2}} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when $\epsilon > 0$ is small and $\gamma < \frac{(N-2)^2}{4}$. Setting $\gamma_j = \frac{(N-2)^2}{4}\left(1 - \frac{j(N-2+j)}{N-1}\right) \in (-\infty, 0]$ for $j \in \mathbb{N}$, we show that if $\gamma \leq \frac{(N-2)^2}{4} - 1$ and $\gamma \neq \gamma_j$ for any j, then for small ϵ, the above equation has a positive –non variational– solution that develops a bubble at the origin. If moreover $\gamma < \frac{(N-2)^2}{4} - 4$, then for any integer $k \geq 2$, the equation has for small enough ϵ, a sign-changing solution that develops into a superposition of k bubbles with alternating sign centered at the origin. The above result is optimal in the radial case, where the condition that $\gamma \neq \gamma_j$ is not necessary. Indeed, it is known that, if $\gamma > \frac{(N-2)^2}{4} - 1$ and Ω is a ball B, then there is no radial positive solution for $\varepsilon > 0$ small. We complete the picture here by showing that, if $\gamma \geq \frac{(N-2)^2}{4} - 4$, then the above problem has no radial sign-changing solutions for $\varepsilon > 0$ small. These results recover and improve what is known in the non-singular case, i.e., when $\gamma = 0$.

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1. Introduction

We consider existence issues for the following Dirichlet problem:

$$\begin{cases} -\Delta u - \gamma \frac{u}{|x|^2} - \lambda u = |u|^{\frac{4}{N-2}} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain with $0 \in \Omega$, $\gamma < \frac{(N-2)^2}{4}$ and $\lambda \in \mathbb{R}$. Problem (1.1) is the Euler-Lagrange equation of the following action functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\gamma}{2}\int_\Omega \frac{u^2}{|x|^2} - \frac{\lambda}{2}\int_\Omega u^2 - \frac{N-2}{2N}\int_\Omega |u|^{\frac{2N}{N-2}}, \quad u \in H_0^1(\Omega).$$

Since $\frac{(N-2)^2}{4}$ is the best constant in the classical Hardy inequality:

$$\frac{(N-2)^2}{4} = \inf\left\{\int_\Omega |\nabla u|^2 : \ u \in H_0^1(\Omega) \text{ s.t. } \int_\Omega \frac{u^2}{|x|^2} = 1\right\}$$

see [22], we have that

$$\int_\Omega |\nabla u|^2 - \gamma \int_\Omega \frac{u^2}{|x|^2} \geq \left(1 - \frac{4\gamma}{(N-2)^2}\right)\int_\Omega |\nabla u|^2 \qquad \forall\ u \in H_0^1(\Omega). \tag{1.2}$$

It is then useful to equip the Hilbert space $H_0^1(\Omega)$ with the inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v - \gamma \int_\Omega \frac{uv}{|x|^2},$$

and the assumption $\gamma < \frac{(N-2)^2}{4}$ guarantees that the induced norm $\|\cdot\|$ is equivalent to the usual one in view of (1.2). Letting $L_\gamma = -\Delta - \frac{\gamma}{|x|^2}$ be the Hardy operator, let us denote by $0 < \lambda_1 \leq \lambda_2 \leq \dots$ the eigenvalues of L_γ.

For $\lambda < \lambda_1$ positive solutions of (1.1) can be found through the minimization problem:

$$S_{\gamma,\lambda}(\Omega) = \inf\left\{\|u\|^2 - \lambda\int_\Omega u^2 : \ u \in H_0^1(\Omega) \text{ s.t. } \int_\Omega |u|^{\frac{2N}{N-2}} = 1\right\}.$$

When $\lambda = 0$, it is classical to see that $S_{\gamma,0}(\Omega) = S_{\gamma,0}(\mathbb{R}^N)$ and is never attained, the difficulty being here that (1.1) is doubly critical for the presence of the Hardy potential $\frac{1}{|x|^2}$ and the nonlinearity $|u|^{\frac{4}{N-2}}u$. Extremals for $S_{\gamma,0}(\mathbb{R}^N)$ exist for $\gamma \geq 0$ and have the form (up to a multiplicative constant)

$$U_\mu(x) = \mu^{-\frac{N-2}{2}} U\left(\frac{x}{\mu}\right) = \frac{\alpha_N \mu^\Gamma}{|x|^{\beta_-}(\mu^{\frac{4\Gamma}{N-2}} + |x|^{\frac{4\Gamma}{N-2}})^{\frac{N-2}{2}}}, \quad \mu > 0, \tag{1.3}$$

where

$$U(x) = \frac{\alpha_N}{|x|^{\beta_-}(1 + |x|^{\frac{4\Gamma}{N-2}})^{\frac{N-2}{2}}} = \frac{\alpha_N}{\left(|x|^{\frac{2}{N-2}\beta_-} + |x|^{\frac{2}{N-2}\beta_+}\right)^{\frac{N-2}{2}}} \tag{1.4}$$

with

$$\Gamma = \sqrt{\frac{(N-2)^2}{4} - \gamma}, \quad \beta_\pm = \frac{N-2}{2} \pm \Gamma, \quad \alpha_N = \left[\frac{4\Gamma^2 N}{N-2}\right]^{\frac{N-2}{4}}, \tag{1.5}$$

see [9, 12, 30]. For $\gamma < 0$ the problem is even more difficult since $S_{\gamma,0}(\mathbb{R}^N) = S_{0,0}(\mathbb{R}^N)$ is not attained, even though (1.3) is still a family of positive solutions to

$$-\Delta U - \gamma \frac{U}{|x|^2} = U^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{1.6}$$

As in the classical Brézis-Nirenberg problem [3], on a bounded domain Ω the presence of a linear perturbation with $0 < \lambda < \lambda_1$ results in a symmetry breaking which is responsible for the existence of minimizers for $S_{\gamma,\lambda}(\Omega)$ [20, 25, 29]. More precisely, a positive ground-state solution for (1.1) is found when

- $\gamma \le 0$ and either
$$N = 3 \text{ and the "Robin" function } R_{\gamma,\lambda} > 0 \text{ somewhere}$$
or
$$N \ge 4, \quad \lambda > |\gamma| \inf \left\{ \frac{1}{|x|^2} : \ x \in \Omega \right\}$$
- $0 < \gamma \le \frac{(N-2)^2}{4} - 1$
- $\max \left\{ 0, \frac{(N-2)^2}{4} - 1 \right\} < \gamma < \frac{(N-2)^2}{4}$ and "mass" $m_{\gamma,\lambda} > 0$.

The question has been completely settled in [20], which we refer to for a precise definition of $R_{\gamma,\lambda}$ and $m_{\gamma,\lambda}$, and the ranges displayed above are essentially optimal for the attainability of $S_{\gamma,\lambda}(\Omega)$, see also the recent survey [18]. Notice that the cases $\gamma < 0$ and $\gamma = 0$, $N = 3$ always require λ to be sufficiently away from zero.

By Pohozaev identity [28] equation (1.1) has no solution when $\lambda \le 0$ on domains which are strictly starshaped w.r.t. 0. Since solutions of (1.1) can't have a given sign when $\lambda \ge \lambda_1$, to attack existence issues for general λ's one needs to search for sign-changing solutions. We can summarize the available results in literature [5, 6, 7, 10, 11, 16] as:

- if $0 \le \gamma < \frac{(N-2)^2}{4} - 4$ there are infinitely many sign-changing solutions for all $\lambda > 0$
- if $\max \left\{ 0, \frac{(N-2)^2}{4} - 4 \right\} \le \gamma < \frac{(N-2)^2}{4} - \frac{(N+2)^2}{N^2}$ there exists a sign-changing solution for all $\lambda \ge \lambda_1$
- if $\max \left\{ 0, \frac{(N-2)^2}{4} - \frac{(N+2)^2}{N^2} \right\} \le \gamma \le \frac{(N-2)^2}{4} - 1$ there exists a sign-changing solution for all $\lambda \in \bigcup_{k=1}^{\infty} (\lambda_k, \lambda_{k+1})$
- if $\gamma \ge 0$ and $\frac{(N-2)^2}{4} - 1 < \gamma < \frac{(N-2)^2}{4}$ there exist n_k sign-changing solutions for all λ in a suitable left open neighborhood of λ_k, $k \ge 2$, where n_k is the multiplicity of λ_k.

Assumption $\gamma \ge 0$ allows here to use U_μ, which are extremals of $S_{\gamma,0}(\mathbb{R}^N)$, as an helpful family of test functions in a variational approach.

Hereafter, we restrict our attention to the regime $\lambda = \varepsilon$, with $\varepsilon > 0$ small:
$$\begin{cases} -\Delta u - \gamma \frac{u}{|x|^2} - \varepsilon u = |u|^{\frac{4}{N-2}} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

When $\gamma = 0$ $S_{0,\varepsilon}(\Omega)$ is not achieved [3, 14, 15] for $N = 3$, and (1.7) in the ball $B = B_1(0)$ admits no positive solutions for $N = 3$ [3] and no radial sign-changing solutions for $N = 3, 4, 5, 6$ [1, 2]. In the singular case, a similar situation arises depending now on γ: $S_{\gamma,\varepsilon}(\Omega)$ is not achieved [20] when either $\gamma < 0$ or $\gamma > \frac{(N-2)^2}{4} - 1$, and (1.7) in B admits no radial positive solutions [8] for $\gamma > \frac{(N-2)^2}{4} - 1$. Our first main result, along with Theorem 1.2 below, completes the picture in a radial setting:

Theorem 1.1. *When $\gamma \ge \frac{(N-2)^2}{4} - 4$ problem (1.7) has no radial sign-changing solutions in B for $\varepsilon > 0$ small.*

Theorem 1.1 is based on a fine asymptotic analysis combined with Pohozaev identities. In this way we also recover, see the precise statement in Corollary 2.3, the results in [1, 2] and [8] concerning the regular case $\gamma = 0$ and the singular case $\gamma > \frac{(N-2)^2}{4} - 1$, respectively. Moreover, when $\gamma < \frac{(N-2)^2}{4} - 4$ the analysis shows that radial sign-changing solutions need to develop in a very precise way a bubble of alternating towers centered at 0 as $\varepsilon \to 0^+$, recovering and improving the discussion in [23] concerning the asymptotics of radial least-energy sign-changing solutions in the regular case $\gamma = 0$ when $N \ge 7$. Once the radial case is well understood, we can attack by a perturbative approach the case of a general domain Ω leading to the following result, which is optimal in the radial case.

Theorem 1.2. *Let*

$$\gamma_j = \frac{(N-2)^2}{4}\left(1 - \frac{j(N-2+j)}{N-1}\right) \in (-\infty, 0]\ , \quad j \in \mathbb{N}. \tag{1.8}$$

Assume that either Ω is a general domain with $\gamma \neq \gamma_j$ for all $j \in \mathbb{N}$ or Ω is $j-$admissible (see Definition 3.3 and Remark 3.4) with $\gamma = \gamma_j$ for some $j \in \mathbb{N}$.
i) Let $\gamma \leq \frac{(N-2)^2}{4} - 1$. Then there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$ problem (1.7) has a positive solution u_ε developing a bubble at the origin.
ii) Let $\gamma < \frac{(N-2)^2}{4} - 4$. For any integer $k \geq 2$ there exists $\varepsilon_k > 0$ such that for all $\varepsilon \in (0, \varepsilon_k)$ problem (1.7) has a sign-changing solution u_ε, which looks like the superposition of k bubbles with alternating sign centered at the origin.

Theorem 1.2-(i) provides positive solutions of (1.7) for $\gamma < 0$ which are not minimizers for $S_{\gamma,\varepsilon}(\Omega)$, exactly as U_μ are solutions of (1.6) which are not extremals for $S_{\gamma,0}(\mathbb{R}^N)$. More generally, our result allows to consider the case $\gamma < 0$ which cannot be dealt in a variational way when $\varepsilon > 0$ is small. When $0 \leq \gamma < \frac{(N-2)^2}{4} - 4$ the solutions we found likely coincide with the infinitely many ones found in [7, 11].

The paper is organized as follows. In Section 2 we discuss the asymptotic behavior for radial solutions of problem (1.7) in B with $\epsilon \to 0^+$, establishing in particular the validity of Theorem 1.1. In Sections 3 and 4 we deduce Theorem 1.2 by developing a very delicate perturbative approach where a crucial splitting of the remainder term is performed, see [24, 26] for related results. In the Appendix 5 we collect several technical estimates.

2. Asymptotic analysis in the radial case: proof of Theorem 1.1

In this section we will consider the case when Ω is the unit ball B. From now on, for any function $u \in L^q(A)$, $1 \leq q \leq +\infty$, we let $|u|_{q,A} = \left(\int_A |u|^q dx\right)^{1/q}$ and $|u|_q = |u|_{q,\Omega}$. We will denote by c, C various positive constants which can vary from lines to lines.

Let $u \in H_0^1(B)$ be a radial solution of (1.1). The function

$$v(r) = (\frac{N-2}{2\Gamma})^{\frac{N-2}{2}} r^{\frac{N-2}{2}(\frac{N-2}{2\Gamma}-1)} u(r^{\frac{N-2}{2\Gamma}}) \tag{2.1}$$

is in $H_0^1(B)$ and is a radial solution of

$$-\Delta v = |v|^{\frac{4}{N-2}} v + \varepsilon |x|^\alpha v \text{ in } B \setminus \{0\},\ v = 0 \text{ on } \partial B, \tag{2.2}$$

where $\alpha = \frac{N-2}{\Gamma} - 2$ and $\varepsilon = (\frac{N-2}{2\Gamma})^2 \lambda$. We have the following simple description of nodal regions:

Lemma 2.1. *Given $\alpha > -2$, any non-trivial radial solution $v \in H_0^1(B)$ of (2.2) is in $C(\overline{B}) \cap C^2(\overline{B} \setminus \{0\})$ and, if $\epsilon > 0$ and $v(0) > 0$, there exist an integer $k = k(v) \geq 1$ and $R_0 = r_1 = 0 < R_1 < r_2 < \cdots < R_{k-1} < r_k < R_k = r_{k+1} = 1$ so that for all $j = 1, \ldots, k$*

$$(-1)^{j-1} v > v(R_j) = 0 \text{ in } (R_{j-1}, R_j), \quad (-1)^j v' > v'(r_j) = 0 \text{ in } (r_j, r_{j+1}),$$

with the convention $v'(0) = 0$. Moreover, there exists $\varepsilon_0 > 0$ small, independent on v, so that for all $0 < \varepsilon \leq \varepsilon_0$ there holds

$$\int_A |v|^{\frac{2N}{N-2}} \geq (\frac{S}{2})^{\frac{N}{2}} \tag{2.3}$$

for any nodal region A of v, where $S = S_{0,0}(\mathbb{R}^N)$ is the Sobolev constant.

Proof. Since $\alpha > -2$, we have that

$$|x|^\alpha \in L^p(B) \text{ for some } p > \frac{N}{2}. \tag{2.4}$$

Since by the Sobolev embedding theorem $v \in L^{\frac{2N}{N-2}}(B)$, for any $\eta > 0$ we can decompose $|v|^{\frac{4}{N-2}} + \varepsilon|x|^\alpha$ as $f_1 + f_2$ with $|f_1|_{\frac{N}{2}} \leq \eta$ and $f_2 \in L^\infty(B)$ in view of (2.4). We can re-write (2.2) as

$$v - (-\Delta)^{-1}(f_1 v) = (-\Delta)^{-1}(f_2 v).$$

By elliptic regularity theory and the Sobolev embedding $W^{2,\frac{Ns}{N+2s}}(B) \hookrightarrow L^s(B)$ we have that

$$|(-\Delta)^{-1}(f_1 v)|_s \leq C\|(-\Delta)^{-1}(f_1 v)\|_{W^{2,\frac{Ns}{N+2s}}} \leq C|f_1 v|_{\frac{Ns}{N+2s}} \leq C\eta|v|_s \tag{2.5}$$

in view of the Hölder's inequality and $|f_1|_{\frac{N}{2}} \leq \eta$. Equivalently $H : v \in L^s(B) \to (-\Delta)^{-1}(f_1 v) \in L^s(B)$ has operatorial norm $\leq C\eta$, and then the operator $\mathrm{Id} - H : L^s(B) \to L^s(B)$ is invertible for all $s > 1$ and η sufficiently small. Arguing as in (2.5), we have that

$$|v|_{\frac{Ns}{N-2s}} \leq \|(\mathrm{Id} - H)^{-1}\||(-\Delta)^{-1}(f_2 v)|_{\frac{Ns}{N-2s}} \leq C|f_2 v|_s \leq C|f_2|_\infty |v|_s$$

when $s < \frac{N}{2}$ and for all $q > 1$

$$|v|_q \leq \|(\mathrm{Id} - H)^{-1}\||(-\Delta)^{-1}(f_2 v)|_q \leq C|f_2 v|_s \leq C|f_2|_\infty |v|_s$$

when $s \geq \frac{N}{2}$. Starting from $v \in L^{\frac{2N}{N-2}}(B)$ we iteratively prove that $v \in L^s(B)$ for all $s > 1$, and then $|v|^{\frac{4}{N-2}} v + \varepsilon|x|^\alpha v \in L^{\frac{N+2p}{4}}(B) \cap L^s_{\text{loc}}(\overline{B} \setminus \{0\})$ for all $s > 1$, where p is given in (2.4). Since $\frac{N+2p}{4} > \frac{N}{2}$, by elliptic regularity theory we deduce that $v \in C(\overline{B}) \cap C^2(\overline{B} \setminus \{0\})$. Moreover, we claim that

$$\lim_{r \to 0} r^{N-1} v'(r) = 0. \tag{2.6}$$

Indeed, let us write equation (2.2) in radial coordinates as

$$-\frac{1}{r^{N-1}}(r^{N-1} v')' = |v|^{\frac{4}{N-2}} v + \varepsilon|x|^\alpha v \qquad r \in (0, 1). \tag{2.7}$$

Since v is non-trivial, then $v(0) \neq 0$ and then, by continuity of v, the R.H.S. in (2.7) has a given sign near 0. By (2.7) we deduce that the function $r^{N-1} v'(r)$ is monotone in r and then has limit as $r \to 0$: $\lim\limits_{r \to 0} r^{N-1} v'(r) = l$. However, $l \neq 0$ would imply a discontinuity of v at 0, and then (2.6) is established.

Take $\epsilon > 0$ and assume w.l.o.g. $v(0) > 0$. Given R so that $\lim\limits_{r \to R} r^{N-1} v'(r) = 0$, observe that the integration of (2.7) in (R, r) gives

$$v'(r) = -\frac{1}{r^{N-1}} \int_R^r s^{N-1}(|v|^{\frac{4}{N-2}} v + \epsilon s^\alpha v) ds \tag{2.8}$$

for all $r > 0$. Since $v(0) > 0$ and $v' < 0$ near 0 in view of (2.8) with $R = 0$, let us define

$$R_1 = \sup\{r \in (0, 1) : \ v > 0 \text{ in } (R_0, r)\}, \quad r_2 = \sup\{r \in (0, 1) : v' < 0 \text{ in } (r_1, r)\}.$$

If $R_1 = 1$, then $r_2 = 1$ and the choice $k = 1$ completes the proof. If $R_1 < 1$, by (2.8) with $R = 0$ and $v(1) = 0$ we deduce that $R_1 < r_2 < 1$, $v'(r_2) = 0$ and

$$v > v(R_1) = 0 \text{ in } (R_0, R_1), \quad v' < v'(r_1) = 0 \text{ in } (r_1, r_2).$$

In an iterative way, for $i \geq 2$ assume to have found $R_0 = r_1 = 0 < R_1 < r_2 < \cdots < R_{i-1} < r_i < 1$ so that $v'(r_i) = 0$ and for all $j = 1, \ldots, i-1$

$$(-1)^{j-1} v > v(R_j) = 0 \text{ in } (R_{j-1}, R_j), \quad (-1)^j v' > v'(r_j) = 0 \text{ in } (r_j, r_{j+1}).$$

Define

$$R_i = \sup\{r \in (0, 1) : \ (-1)^{i-1} v > 0 \text{ in } (R_{i-1}, r)\}, \quad r_{i+1} = \sup\{r \in (0, 1) : (-1)^i v' > 0 \text{ in } (r_i, r)\}.$$

Since $(-1)^{i-1}v' > 0$ in (r_{i-1}, r_i) and $R_{i-1} \in (r_{i-1}, r_i)$, we have that $r_i < R_i \le 1$, and by (2.8) with $R = r_i$ it follows that $(-1)^i v' > 0$ in $(r_i, R_i]$. If $R_i = 1$, then $r_{i+1} = 1$ and the choice $k = i$ completes the proof. If $R_i < 1$, the boundary condition $v(1) = 0$ implies that $R_i < r_{i+1} < 1$, which in turn leads to $v'(r_{i+1}) = 0$ and

$$(-1)^{i-1}v > v(R_i) = 0 \text{ in } (R_{i-1}, R_i), \quad (-1)^i v' > v'(r_i) = 0 \text{ in } (r_i, r_{i+1}).$$

Such a process needs to stop after k steps. Otherwise, we would find an increasing sequence R_i, $i \in \mathbb{N}$, so that $v(R_i) = v(R_{i+1}) = 0$. Letting $R = \lim\limits_{i\to+\infty} R_i \in (0, 1]$, we would have that $\lim\limits_{i\to+\infty} r_i = R$ in view of $R_{i-1} < r_i < R_i$. Since $v \in C^2(\overline{B} \setminus \{0\})$, we would deduce that $v(R) = v'(R) = 0$, and then by the uniqueness for the ODE $v = 0$, a contradiction.

Finally, let us integrate (2.2) against v on a nodal region A to get

$$\begin{aligned} S\left(\int_A |v|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} &\le \int_A |\nabla v|^2 = \int_A |v|^{\frac{2N}{N-2}} + \epsilon \int_A |x|^\alpha v^2 \\ &\le \int_A |v|^{\frac{2N}{N-2}} + \epsilon \||x|^\alpha\|_{\frac{N}{2}} \left(\int_A |v|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \end{aligned}$$

thanks to the Hölder's inequality and to the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ with Sobolev constant S. Setting $\varepsilon_0 = \frac{S}{2\||x|^\alpha\|_{\frac{N}{2}}}$, the validity of (2.3) easily follows for all $0 < \varepsilon \le \varepsilon_0$. $\square$

Let $v_n \in H_0^1(B)$ be a sequence of non-trivial radial solutions to (2.2) with $\alpha > -2$. Up to a subsequence, we can assume that there exist $k \ge 1$ and sequences $R_0^n = r_1^n = 0 < R_1^n < r_2^n < \cdots < R_{k-1}^n < r_k^n < R_k^n = r_{k+1}^{n+1} \le 1$ so that for all $j = 1, \dots, k$

$$(-1)^{j-1}v_n > v_n(R_j^n) = 0 \quad \text{in } (R_{j-1}^n, R_j^n), \quad (-1)^j v_n' > v_n'(r_j^n) = 0 \text{ in } (r_j^n, r_{j+1}^n). \tag{2.9}$$

Notice that such an assumption simply means that all the v_n's have at least k nodal regions. The case of positive solutions v_n corresponds to take $k = 1$ and $R_1^n = 1$, whereas for sign-changing solutions we can always choose a subsequence with at least $k \ge 2$ nodal regions. Set $\delta_j^n = |v_n(r_j^n)|^{-\frac{2}{N-2}}$, where

$$|v_n|(r_j^n) = \max_{[R_{j-1}^n, R_j^n]} |v_n|. \tag{2.10}$$

Blow-up phenomena for (2.2) are described in terms of the limiting problem

$$-\Delta V = V^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N, \tag{2.11}$$

whose bounded solutions are completely classified [4, 21]. In particular, every radial positive and bounded solution of (2.11) is given by

$$V_\delta(x) = \delta^{-\frac{N-2}{2}} V(\frac{x}{\delta}) = \left(\frac{\delta}{\delta^2 + a_N|x|^2}\right)^{\frac{N-2}{2}} \tag{2.12}$$

for some $\delta > 0$, where $a_N = \frac{1}{N(N-2)}$ and

$$V(x) = \left(\frac{1}{1 + a_N|x|^2}\right)^{\frac{N-2}{2}}. \tag{2.13}$$

The asymptotic behavior of v_n is described in the following main result:

Theorem 2.2. *As $n \to +\infty$ there hold*

$$\frac{r_j^n}{\delta_j^n} \to 0, \quad \frac{R_j^n}{\delta_j^n} \to +\infty, \quad V_j^n(x) = (-1)^{j-1}(\delta_j^n)^{\frac{N-2}{2}} v_n(\delta_j^n x) \to V \text{ in } C^1_{loc}(\mathbb{R}^N \setminus \{0\}) \tag{2.14}$$

for all $j = 1, \dots, k$. Moreover, $\alpha \le N - 4$ if $k = 1$ and $\alpha < \frac{N-6}{2}$ if $k \ge 2$. If in addition

$$R_{k-1}^n \to 0 \text{ and } R_k^n \to R_k > 0 \tag{2.15}$$

as $n \to +\infty$, there hold $R_k^n = 1$ and for all $j = 1, \dots k-1$

$$R_j^n \sim \left[\frac{\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}}}{(N-2)\omega_{N-1}} \right]^{\frac{1}{N-2}} \sqrt{\delta_j^n \delta_{j+1}^n} \tag{2.16}$$

$$\delta_j^n \sim \left[\frac{(\alpha+2)\int_{\mathbb{R}^N} |x|^\alpha V^2}{(N-2)\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}}} \varepsilon_n \right]^{\frac{(N-2)(\frac{N-2}{N-6-2\alpha})^{k-j} - (N-4-\alpha)}{(2+\alpha)(N-4-\alpha)}} \left[\frac{(N-2)\omega_{N-1}}{\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}}} \right]^{\frac{1}{N-4-\alpha}(\frac{N-2}{N-6-2\alpha})^{k-j}}$$

$$\delta_k^n \sim \left[\frac{(\alpha+2)\omega_{N-1}\int_{\mathbb{R}^N} |x|^\alpha V^2}{(\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}})^2} \varepsilon_n \right]^{\frac{1}{N-4-\alpha}}$$

as $n \to +\infty$ provided $\alpha < N-4$.

Asymptotics for radial least-energy sign-changing solutions of (2.2) with $\alpha = 0$ and $N \geq 7$ has been already considered in [23] and corresponds to the case $k = 2$. Here we develop the asymptotic analysis in a completely general way by refining the results in [23] for $k = 2$, by covering the situation $\alpha \neq 0$ and including the case $k \geq 3$. Several new difficulties arise:

- in each nodal region v_n might develop multiple bubbles, but the Pohozaev identity will show crucial to prevent the interaction between bubbles of same sign;
- the limiting problem admits positive radial solutions also on annuli or complements of balls, but none of them can be limit of V_j^n, as we will prove by a matching condition on $v_n'(R_j^n)$ as computed from the left and the right;
- the precise law of δ_j^n is prescribed by the Pohozaev identity in terms of ε_n and R_j^n, but the asymptotic behavior of R_j^n has to be determined according to a tricky compatibility condition between $v_n'(R_j^n)$ and $v_n(r_j^n)$.

Given Γ in (1.5), let

$$\sigma_j = \frac{1}{2}\frac{\Gamma}{\Gamma-1}\left(\frac{\Gamma}{\Gamma-2}\right)^{j-1} - \frac{1}{2}. \tag{2.17}$$

For $\mu = [\sqrt{N(N-2)}\delta]^{\frac{N-2}{2\Gamma}}$, notice that the solution U_μ of (1.6) given by (1.3) corresponds through (2.1) to the solution V_δ of (2.11) given by (2.12). Setting $M_{k-j+1}^n = (R_j^n)^{\frac{2\Gamma}{N-2}}$ and $\mu_j^n = [\sqrt{N(N-2)}\delta_{k-j+1}^n]^{\frac{N-2}{2\Gamma}}$, by Theorem 2.2 with $\alpha = \frac{N-2}{\Gamma} - 2$ we deduce the following:

Corollary 2.3. *Let u_n be a sequence of radial solutions for (1.7) in B with $\varepsilon_n \to 0^+$ as $n \to +\infty$.*

(i) *If u_n are positive functions, then $\gamma \leq \frac{(N-2)^2}{4} - 1$ and*

$$\mu_1^n = d_1 \varepsilon_n^{\sigma_1}(1+o(1)), \quad U_1^n(x) = (\mu_1^n)^{\frac{N-2}{2}} u_n(\mu_1^n x) \to U \text{ in } C^1_{loc}(\mathbb{R}^N \setminus \{0\})$$

as $n \to +\infty$ when $\gamma < \frac{(N-2)^2}{4} - 1$.

(ii) *If u_n are sign-changing solutions, then $\gamma < \frac{(N-2)^2}{4} - 4$.*

(iii) *If u_n have precisely $k-1$ shrinking nodal regions with nodes*

$$0 = M_{k+1}^n < M_k^n < \cdots < M_2^n \to 0, \qquad M_1^n \to M_1 \in (0,1]$$

as $n \to +\infty$, then there exist $\mu_j^n > 0$, $j = 1, \dots, k$, so that as $n \to +\infty$:

$$\mu_j^n = d_j \varepsilon_n^{\sigma_j}(1+o(1)), \quad U_j^n(x) = (\mu_j^n)^{\frac{N-2}{2}} u_n(\mu_j^n x) \to U \text{ in } C^1_{loc}(\mathbb{R}^N \setminus \{0\})$$

for all $j = 1, \dots, k$ and

$$M_1^n = 1, \quad M_j^n = A(\mu_{j-1}^n \mu_j^n)^{\frac{2\Gamma}{(N-2)^2}}(1+o(1))$$

for all $j = 2, \dots, k$.

Here U is given in (1.4) and $A, d_j > 0$ are explicit constants.

Let us discuss first the behavior of v_n in $(0,R_1^n)$. Notice that the function $V_1^n=(\delta_1^n)^{\frac{N-2}{2}}v_n(\delta_1^n x)$ solves

$$\begin{cases} -\Delta V_1^n=(V_1^n)^{\frac{N+2}{N-2}}+\varepsilon_n(\delta_1^n)^{2+\alpha}|x|^\alpha V_1^n & \text{in } B_{\frac{R_1^n}{\delta_1^n}}(0)\\ 0<V_1^n\le V_1^n(0)=1 & \text{in } B_{\frac{R_1^n}{\delta_1^n}}(0)\end{cases}$$

in view of

$$0<(-1)^{j-1}v_n\le(-1)^{j-1}v_n(r_j^n)=\frac{1}{(\delta_j^n)^{\frac{N-2}{2}}}\qquad\text{in } (R_{j-1}^n,R_j^n),\tag{2.18}$$

a simple re-writing of (2.10) through (2.9). By elliptic estimates we deduce that V_1^n is uniformly bounded in $C^{0,\gamma}_{\text{loc}}(\mathbb{R}^N)\cap C^{1,\gamma}_{\text{loc}}(\mathbb{R}^N\setminus\{0\})$, $\gamma\in(0,1)$, in view of (2.4). By the Ascoli-Arzelá's Theorem and a diagonal process, we have that, up to a subsequence, $V_1^n\to V$ in $C_{\text{loc}}(\mathbb{R}^N)\cap C^1_{\text{loc}}(\mathbb{R}^N\setminus\{0\})$, where V solves

$$-\Delta V=V^{\frac{N+2}{N-2}}\text{ in }\mathbb{R}^N,\qquad 0<V\le V(0)=1\text{ in }\mathbb{R}^n$$

and has the form (2.13) [4, 21]. We have used that

$$\frac{R_1^n}{\delta_1^n}\to+\infty\tag{2.19}$$

as $n\to+\infty$. Indeed, if $\frac{\delta_1^n}{R_1^n}$ were bounded away from zero, then $\tilde V_1^n(x)=(R_1^n)^{\frac{N-2}{2}}v_n(R_1^n x)$ would be uniformly bounded in B in view of (2.18). Since $\tilde V_1^n>0$ solves

$$-\Delta\tilde V_1^n=(\tilde V_1^n)^{\frac{N+2}{N-2}}+\varepsilon_n(R_1^n)^{2+\alpha}|x|^\alpha\tilde V_1^n\text{ in }B,\qquad \tilde V_1^n=0\text{ on }\partial B,$$

by elliptic estimates, as before, we would deduce that, up to a subsequence, $\tilde V_1^n\to\tilde V_1$ in $C(\overline{B})\cap C^1_{\text{loc}}(\overline{B}\setminus\{0\})$, where $\tilde V_1\ge0$ is a bounded solution of

$$-\Delta\tilde V_1=(\tilde V_1)^{\frac{N+2}{N-2}}\text{ in }B\setminus\{0\},\quad \tilde V_1=0\text{ on }\partial B.$$

Let us recall the Pohozaev identity [28] in a radial form: given a solution v of (2.2) and a radial domain $A\subset B$, multiply (2.2) by $\langle x,\nabla v\rangle=|x|v'$ and integrate in A to get

$$(\alpha+2)\varepsilon\int_A|x|^\alpha v^2=\int_{\partial A}\left[(v')^2+\frac{N-2}{|x|}vv'+\frac{N-2}{N}|v|^{\frac{2N}{N-2}}+\varepsilon|x|^\alpha v^2\right]\langle x,\nu\rangle.\tag{2.20}$$

Since 0 is a removable singularity in view of $\tilde V_1\in L^\infty(\{0\})$, by (2.20) with $\epsilon=0$ on $A=B$ we would get that $\tilde V_1=0$ and then

$$\int_B|\tilde V_1^n|^{\frac{2N}{N-2}}\to0$$

as $n\to+\infty$, in contradiction with (2.3) in view of $\varepsilon_n(R_1^n)^{2+\alpha}\to0$ as $n\to+\infty$.

We aim to show that there is no superposition of bubbles of same sign in $[0,R_1^n]$. Interaction between bubbles of same sign can be ruled out by the Pohozaev identity (2.20). Letting

$$J=\{j=1,\dots,k:\ (2.14)\text{ holds}\},\tag{2.21}$$

notice that $1\in J$ according to (2.19). We have the following general result:

Proposition 2.4. *There exists $C>0$ so that*

$$|v_n|\le CV_{\delta_j^n}\qquad\textit{in }[R_{j-1}^n,R_j^n]\tag{2.22}$$

for all $j\in J$, where V_δ is given by (2.12).

Proof. The presence of other bubbles in $[R_{j-1}^n,R_j^n]$ can be detected by the behavior of $r^{\frac{N-2}{2}}v_n(r)$. Notice that the function $r^{\frac{N-2}{2}}V(r)=(\frac{r}{1+a_Nr^2})^{\frac{N-2}{2}}$ satisfies

$$r^{\frac{N-2}{2}}V\Big|_{r=a_N^{-\frac12}}=(\frac{N(N-2)}{4})^{\frac{N-2}{4}},\quad \lim_{r\to+\infty}r^{\frac{N-2}{2}}V(r)=0\tag{2.23}$$

and

$$[r^{\frac{N-2}{2}}V(r)]' = \frac{N-2}{2}\frac{r^{\frac{N-4}{2}}(1-a_N r^2)}{(1+a_N r^2)^{\frac{N}{2}}} < 0 \text{ in } (a_N^{-\frac{1}{2}}, +\infty). \tag{2.24}$$

Thanks to (2.23) let us fix $M > a_N^{-\frac{1}{2}}$ so that

$$M^{\frac{N-2}{2}}V(M) = \min\{[\frac{N(N-2)}{16}]^{\frac{N-2}{4}}, [\frac{(N-2)^2(N+1)}{2(N+2)^2}]^{\frac{N-2}{4}}\}. \tag{2.25}$$

We claim that for n large

$$(-1)^{j-1}[r^{\frac{N-2}{2}}v_n]' < 0 \text{ in } [M\delta_j^n, R_j^n]. \tag{2.26}$$

Indeed, if (2.26) were not true, we could find $M_n \in [M\delta_j^n, R_j^n]$ so that

$$(-1)^{j-1}[r^{\frac{N-2}{2}}v_n]' < [r^{\frac{N-2}{2}}v_n]'(M_n) = 0 \text{ in } [M\delta_j^n, M_n), \qquad \frac{M_n}{\delta_j^n} \to 0 \text{ as } n \to +\infty, \tag{2.27}$$

as it follows by (2.24) and

$$(-1)^{j-1}\delta_j^n[r^{\frac{N-2}{2}}v_n]'(r\delta_j^n) = [r^{\frac{N-2}{2}}V_j^n]' \to [r^{\frac{N-2}{2}}V]'$$

locally uniformly in $(0,+\infty)$ as $n \to +\infty$ in view of (2.14). By (2.20) applied to v_n on $A = B_{M_n}(0)$ we get that

$$[\frac{M_n v_n'(M_n)}{v_n(M_n)}]^2 + (N-2)\frac{M_n v_n'(M_n)}{v_n(M_n)} + \frac{N-2}{N}M_n^2|v_n(M_n)|^{\frac{4}{N-2}} + \varepsilon_n M_n^{2+\alpha} > 0 \tag{2.28}$$

in view of $\alpha > -2$. Since by (2.27)

$$M_n v_n'(M_n) = -\frac{N-2}{2}v_n(M_n),$$

we deduce that

$$-\frac{(N-2)^2}{4} + \frac{N-2}{N}M_n^2|v_n(M_n)|^{\frac{4}{N-2}} + \varepsilon_n M_n^{2+\alpha} > 0.$$

Since

$$(-1)^{j-1}M_n^{\frac{N-2}{2}}v_n(M_n) \le (-1)^{j-1}(M\delta_j^n)^{\frac{N-2}{2}}v_n(M\delta_j^n) = M^{\frac{N-2}{2}}V_j^n(M) \to M^{\frac{N-2}{2}}V(M)$$

as $n \to +\infty$ in view of (2.14) and (2.27), by (2.25) we deduce that

$$-\frac{(N-2)^2}{4} + \frac{N-2}{N}M_n^2|v_n(M_n)|^{\frac{4}{N-2}} + \varepsilon_n M_n^{2+\alpha} \le -\frac{(N-2)^2}{8} + \varepsilon_n < 0$$

for n large, a contradiction with (2.28). The claim (2.26) is established.

Once (2.26) is established, we can prove the validity of (2.22). First, since $(-1)^{j-1}v_n$ is a positive solution of $L_n v_n = 0$ in $[R_{j-1}^n, R_j^n]$, the operator $L_n = -\Delta - |v_n|^{\frac{4}{N-2}} - \varepsilon_n|x|^\alpha$ satisfies the minimum principle in $[R_{j-1}^n, R_j^n]$, and we can compare $(-1)^{j-1}v_n$ with $\varphi_n = \frac{M^{\frac{(N-2)(N+1)}{N+2}}(\delta_j^n)^{\frac{N(N-2)}{2(N+2)}}}{r^{\frac{(N-2)(N+1)}{N+2}}}$ in $[M\delta_j^n, R_j^n]$. Since

$$\begin{aligned} L_n\varphi_n &= M^{\frac{(N-2)(N+1)}{N+2}}(\delta_j^n)^{\frac{N(N-2)}{2(N+2)}}r^{-\frac{N^2+N+2}{N+2}}\left[\frac{(N-2)^2(N+1)}{(N+2)^2} - r^2|v_n(r)|^{\frac{4}{N-2}} - \varepsilon_n r^{2+\alpha}\right] \\ &\ge M^{\frac{(N-2)(N+1)}{N+2}}(\delta_j^n)^{\frac{N(N-2)}{2(N+2)}}r^{-\frac{N^2+N+2}{N+2}}\left[\frac{(N-2)^2(N+1)}{(N+2)^2} - M^2|V_j^n(M)|^{\frac{4}{N-2}} - \varepsilon_n\right] \end{aligned}$$

in $[M\delta_j^n, R_j^n]$ in view of (2.26), we have that $L_n\varphi_n > 0$ in $[M\delta_j^n, R_j^n]$ for n large in view of (2.14) and (2.25). Since

$$(-1)^{j-1}v_n(M\delta_j^n) \le \frac{1}{(\delta_j^n)^{\frac{N-2}{2}}} = \varphi_n(M\delta_j^n), \quad (-1)^{j-1}v_n(R_j^n) = 0 < \varphi_n(R_j^n)$$

in view of (2.18), we have that

$$|v_n|(r) = (-1)^{j-1} v_n(r) \le \frac{M^{\frac{(N-2)(N+1)}{N+2}} (\delta_j^n)^{\frac{N(N-2)}{2(N+2)}}}{r^{\frac{(N-2)(N+1)}{N+2}}} \quad \text{in } [M\delta_j^n, R_j^n],$$

or equivalently

$$V_j^n(r) \le \frac{M^{\frac{(N-2)(N+1)}{N+2}}}{r^{\frac{(N-2)(N+1)}{N+2}}} \quad \text{in } [M, \frac{R_j^n}{\delta_j^n}]. \tag{2.29}$$

By (2.8) with $R = r_j^n$ we get that in $[R_{j-1}^n, R_j^n]$

$$\begin{aligned} (-1)^j v_n'(r) &= \frac{1}{r^{N-1}} \int_{r_j^n}^r s^{N-1}(|v_n|^{\frac{N+2}{N-2}} + \epsilon_n s^\alpha |v_n|) ds \\ &= \frac{(\delta_j^n)^{\frac{N-2}{2}}}{r^{N-1}} \int_{\frac{r_j^n}{\delta_j^n}}^{\frac{r}{\delta_j^n}} s^{N-1} (V_j^n)^{\frac{N+2}{N-2}} + \frac{\epsilon_n}{r^{N-1}} \int_{r_j^n}^r s^{N-1+\alpha} |v_n| ds. \end{aligned} \tag{2.30}$$

Inserting (2.29) into (2.30) we deduce that

$$\begin{aligned} |v_n'(r)| &\le \frac{(\delta_j^n)^{\frac{N-2}{2}}}{r^{N-1}} \left[\frac{M^N}{N} + M^{N+1} \int_M^\infty \frac{1}{s^2} \right] \\ &\quad + \frac{\epsilon_n}{r^{N-1}} \left[\frac{M^{N+\alpha}}{N+\alpha} (\delta_j^n)^{\alpha + \frac{N+2}{2}} + \frac{1}{\alpha+2} \sup_{[M\delta_j^n, R_j^n]} r^{N-2} |v_n|(r) \right] \\ &\le \frac{C}{r^{N-1}} [(\delta_j^n)^{\frac{N-2}{2}} + \epsilon_n \sup_{[M\delta_j^n, R_j^n]} r^{N-2} |v_n|(r)] \end{aligned}$$

for $M\delta_j^n \le r \le R_j^n$ in view of (2.18) and $\alpha + \frac{N+2}{2} > \frac{N-2}{2}$. Integrating in $[r, R_j^n]$ we get that

$$r^{N-2} |v_n(r)| \le r^{N-2} \int_r^{R_j^n} |v_n'| \le C (\delta_j^n)^{\frac{N-2}{2}}$$

in $[M\delta_j^n, R_j^n]$, and then

$$|v_n|(r) \le C \frac{(\delta_j^n)^{\frac{N-2}{2}}}{r^{N-2}} \le C V_{\delta_j^n} \qquad \text{in } [M\delta_j^n, R_j^n] \tag{2.31}$$

for n large. By (2.18) there holds that

$$|v_n| \le \frac{1}{(\delta_j^n)^{\frac{N-2}{2}}} \le C V_{\delta_j^n} \quad \text{in } [R_{j-1}^n, M\delta_j^n]$$

which, combined with (2.31), completes the proof. □

Thanks to Proposition 2.4 we are now in position to establish Theorem 2.2.

Proof (*of Theorem* 2.2). Let $j \in J$, J given in (2.21), so that Proposition 2.4 applies. By (2.18) and (2.31) we deduce that

$$\varepsilon_n \int_{r_j^n}^{R_j^n} s^{N-1+\alpha} |v_n| ds = O(\varepsilon_n (\delta_j^n)^{\alpha + \frac{N+2}{2}} + \varepsilon_n (\delta_j^n)^{\frac{N-2}{2}}) = o((\delta_j^n)^{\frac{N-2}{2}}) \tag{2.32}$$

as $n \to +\infty$ in view of $\alpha + \frac{N+2}{2} > \frac{N-2}{2}$, and (2.22) can be re-written as

$$|V_j^n| \le CV \quad \text{in } [\frac{R_{j-1}^n}{\delta_j^n}, \frac{R_j^n}{\delta_j^n}]. \tag{2.33}$$

Inserting (2.32) into (2.30), by the Lebesgue's Theorem we have that

$$(-1)^j (\delta_j^n)^{-\frac{N-2}{2}} (R_j^n)^{N-1} v_n'(R_j^n) \to \int_0^\infty s^{N-1} V^{\frac{N+2}{N-2}} \tag{2.34}$$

for all $j \in J$ as $n \to +\infty$, in view of $V_j^n \to V$ in $C_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ and (2.33).

Since $1 \in J$, let us apply (2.20) to v_n on $B_{R_1^n}(0)$ if $j = 1$ or on $B_{R_j^n}(0) \setminus B_{R_{j-1}^n}(0)$ if $j \geq 2$ with $j-1,\ j \in J$. As $n \to +\infty$ we get that

$$(\alpha+2)\varepsilon_n \int_{R_{j-1}^n}^{R_j^n} r^{N-1+\alpha} v_n^2 = (R_j^n)^N (v_n'(R_j^n))^2 - (R_{j-1}^n)^N (v_n'(R_{j-1}^n))^2$$
$$= \left(\int_0^\infty r^{N-1} V^{\frac{N+2}{N-2}}\right)^2 [(\frac{\delta_j^n}{R_j^n})^{N-2}(1+o(1)) - (\frac{\delta_{j-1}^n}{R_{j-1}^n})^{N-2}(1+o(1))] \tag{2.35}$$

in view of (2.34), with the convention $\frac{\delta_0^n}{R_0^n} = 0$. The LHS above has the following asymptotic behavior: if $\alpha > N-4$ there holds

$$\int_{R_{j-1}^n}^{R_j^n} r^{N-1+\alpha} v_n^2 \leq C^2 [N(N-2)]^{N-2} (\delta_j^n)^{N-2} \int_{R_{j-1}^n}^{R_j^n} r^{3+\alpha-N} = O((\delta_j^n)^{N-2}) \tag{2.36}$$

in view of (2.22); if $-2 < \alpha \leq N-4$ there holds

$$\begin{aligned}\int_{R_{j-1}^n}^{R_j^n} r^{N-1+\alpha} v_n^2 &= (\delta_j^n)^{2+\alpha} \int_{\frac{R_{j-1}^n}{\delta_j^n}}^{\frac{R_j^n}{\delta_j^n}} r^{N-1+\alpha} (V_j^n)^2 \\ &= \begin{cases} (\delta_j^n)^{2+\alpha} \int_0^{+\infty} r^{N-1+\alpha} V^2 (1+o(1)) & \text{if } \alpha < N-4 \\ O((\delta_j^n)^{N-2} |\log \frac{R_j^n}{\delta_j^n}|) & \text{if } \alpha = N-4 \end{cases}\end{aligned} \tag{2.37}$$

in view of (2.14), (2.33) and the Lebesgue's Theorem.

We have some useful properties to establish.

<u>1^{st} **Claim**</u>: We have that

$$j-1 \in J,\ R_{j-1}^n < 1 \ \Rightarrow \max_{[R_{j-1}^n, R_j^n]} |v_n| \to +\infty \text{ as } n \to +\infty. \tag{2.38}$$

Up to a subsequence, assume that $R_{j-1}^n \to R_{j-1}$ and $R_j^n \to R_j$ as $n \to +\infty$. If $\max\limits_{[R_{j-1}^n, R_j^n]} |v_n| \leq C$, by $\varepsilon_n \to 0$ as $n \to +\infty$, (2.3) and elliptic estimates we deduce that $R_{j-1} < R_j$ and, up to a subsequence, $(-1)^{j-1} v_n \to v$ in $C^2_{\text{loc}}(A)$, $A = B_{R_j}(0) \setminus \overline{B_{R_{j-1}}(0)}$, as $n \to +\infty$, where $v > 0$ is a bounded solution of

$$-\Delta v = v^{\frac{N+2}{N-2}} \text{ in } A, \quad v = 0 \text{ on } \partial A \setminus \{0\}. \tag{2.39}$$

We have that $R_{j-1} > 0$, since otherwise v would be a solution of (2.39) in the whole $B_{R_j}(0)$, 0 being a removable singularity, and then would vanish by the Pohozaev identity (2.20). Up to a subsequence, by elliptic estimates $\tilde{v}_n(r) = (-1)^{j-1} (R_{j-1}^n)^{\frac{N-2}{2}} v_n(r R_{j-1}^n) \to \tilde{v}$ in $C^2_{\text{loc}}(A)$, $A = B_{\frac{R_j}{R_{j-1}}}(0) \setminus B$, as $n \to +\infty$, where $\tilde{v} > 0$ is a bounded solution of

$$-\Delta \tilde{v} = \tilde{v}^{\frac{N+2}{N-2}} \text{ in } A, \quad \tilde{v} = 0 \text{ on } \partial A.$$

In particular, $\tilde{v}_n'(1) = (-1)^{j-1} (R_{j-1}^n)^{\frac{N}{2}} v_n'(R_{j-1}^n) \to \tilde{v}'(1) > 0$, in contradiction with (2.34) when $j-1 \in J$ and $R_{j-1}^n \to R_{j-1} > 0$ as $n \to +\infty$. Then (2.38) is established and the Claim is proved. □

<u>2^{nd} **Claim**</u>: We have that

$$j-1 \in J,\ R_{j-1}^n < 1 \ \Rightarrow \sup \frac{r_j^n}{\delta_j^n} < +\infty. \tag{2.40}$$

If $\frac{r_j^n}{\delta_j^n} \to +\infty$ as $n \to +\infty$, then $j \geq 2$ and the function $\tilde{V}_j^n(r) = (-1)^{j-1}(\delta_j^n)^{\frac{N-2}{2}} v_n(r_j^n + \delta_j^n r)$ solves

$$\begin{cases} -(\tilde{V}_j^n)'' - (N-1)\frac{\delta_j^n}{r_j^n+\delta_j^n r}(\tilde{V}_j^n)' = (\tilde{V}_j^n)^{\frac{N+2}{N-2}} + \varepsilon_n(\delta_j^n)^2(r_j^n + \delta_j^n r)^\alpha \tilde{V}_j^n & \text{in } I_n = \left(-\frac{r_j^n - R_{j-1}^n}{\delta_j^n}, \frac{R_j^n - r_j^n}{\delta_j^n}\right) \\ 0 < \tilde{V}_j^n \leq \tilde{V}_j^n(0) = 1 & \text{in } I_n \\ \tilde{V}_j^n = 0 & \text{on } \partial I_n \end{cases}$$

in view of (2.18). Up to a subsequence, assume that

$$\frac{r_j^n - R_{j-1}^n}{\delta_j^n} \to L_1 \in [0, +\infty], \quad \frac{R_j^n - r_j^n}{\delta_j^n} \to L_2 \in [0, +\infty]$$

as $n \to +\infty$. As we will justify later, we have that

$$L_1, L_2 > 0. \tag{2.41}$$

Notice that

$$(\delta_j^n)^2(r_j^n + \delta_j^n r)^\alpha = (\frac{\delta_j^n}{r_j^n + \delta_j^n r})^2 (r_j^n + \delta_j^n r)^{2+\alpha} \leq (\frac{\delta_j^n}{r_j^n + \delta_j^n r})^2 \to 0 \tag{2.42}$$

as $n \to +\infty$ in $C_{\text{loc}}(-L_1, L_2)$, in view of $\frac{r_j^n}{\delta_j^n} \to +\infty$ as $n \to +\infty$. Up to a subsequence, by elliptic estimates we have that $\tilde{V}_j^n \to \tilde{V}_j$ in $C^1_{\text{loc}}(-L_1, L_2)$, where $\tilde{V}_j$ is a solution of

$$\begin{cases} -(\tilde{V}_j)'' = (\tilde{V}_j)^{\frac{N+2}{N-2}} & \text{in } (-L_1, L_2) \\ 0 < \tilde{V}_j \leq \tilde{V}_j(0) = 1 & \text{in } (-L_1, L_2). \end{cases}$$

Since by energy conservation there holds

$$\frac{N}{N-2}(\tilde{V}_j')^2 + (\tilde{V}_j)^{\frac{2N}{N-2}} = 1,$$

the property $\tilde{V}_j > 0$ implies that $L_1, L_2 < +\infty$. By (2.8) with $R = r_j^n$ and $r = R_{j-1}^n$ we get

$$\begin{aligned} (-1)^{j-1}(\delta_j^n)^{\frac{N}{2}} v_n'(R_{j-1}^n) &= \frac{(\delta_j^n)^{\frac{N}{2}}}{(R_{j-1}^n)^{N-1}} \int_{R_{j-1}^n}^{r_j^n} s^{N-1}(|v_n|^{\frac{N+2}{N-2}} + \epsilon_n s^\alpha |v_n|) ds \\ &= \int_{-\frac{r_j^n - R_{j-1}^n}{\delta_j^n}}^0 (\frac{r_j^n + \delta_j^n s}{R_{j-1}^n})^{N-1} [(\tilde{V}_j^n)^{\frac{N+2}{N-2}} + \epsilon_n (\delta_j^n)^2 (r_j^n + \delta_j^n s)^\alpha \tilde{V}_j^n] ds \\ &\to \int_{-L_1}^0 (\tilde{V}_j)^{\frac{N+2}{N-2}} ds \end{aligned} \tag{2.43}$$

in view of $\tilde{V}_j^n \leq 1$, (2.42) and

$$\frac{r_j^n}{\delta_j^n} \to +\infty, \ \frac{r_j^n - R_{j-1}^n}{\delta_j^n} \to L_1 \in [0, +\infty) \ \Rightarrow \ \frac{r_j^n}{R_{j-1}^n} = 1 + \frac{\frac{r_j^n - R_{j-1}^n}{\delta_j^n}}{\frac{r_j^n}{\delta_j^n} - \frac{r_j^n - R_{j-1}^n}{\delta_j^n}} \to 1 \tag{2.44}$$

as $n \to +\infty$. When $j-1 \in J$, (2.43) is in contradiction with (2.34) since

$$\frac{(\delta_{j-1}^n)^{\frac{N-2}{2}}}{(R_{j-1}^n)^{N-1}} = (\frac{\delta_{j-1}^n}{R_{j-1}^n})^{\frac{N-2}{2}} (\frac{r_j^n}{R_{j-1}^n})^{\frac{N}{2}} (\frac{\delta_j^n}{r_j^n})^{\frac{N}{2}} \frac{1}{(\delta_j^n)^{\frac{N}{2}}} = o\left(\frac{1}{(\delta_j^n)^{\frac{N}{2}}}\right)$$

as $n \to +\infty$, as it follows by (2.44), $j-1 \in J$ and $\frac{r_j^n}{\delta_j^n} \to +\infty$ as $n \to +\infty$. Then (2.40) is established.

To complete the proof of the Claim, we need to establish (2.41). Apply (2.8) with $R = r_j^n$ to get by (2.18) that

$$|v_n'(r)| \leq (\frac{r_j^n}{r})^{N-1} (\delta_j^n)^{-\frac{N-2}{2}} \left[\frac{r_j^n - R_{j-1}^n}{(\delta_j^n)^2} + \varepsilon_n \frac{(r_j^n)^{\alpha+1}}{N+\alpha}\right] \tag{2.45}$$

for all $R_{j-1}^n \le r \le r_j^n$ and

$$|v_n'(r)| \le (\delta_j^n)^{-\frac{N-2}{2}} \left[\frac{r - r_j^n}{(\delta_j^n)^2} + \varepsilon_n \frac{r^{\alpha+1}}{N+\alpha} \right] \tag{2.46}$$

for all $r_j^n \le r \le R_j^n$. We deduce the following estimates by integrating (2.45) in $[R_{j-1}^n, r_j^n]$:

$$(\delta_j^n)^{-\frac{N-2}{2}} = |\int_{R_{j-1}^n}^{r_j^n} v_n'| \le (\frac{r_j^n}{R_{j-1}^n})^{N-1} (\delta_j^n)^{-\frac{N-2}{2}} \left[(\frac{r_j^n - R_{j-1}^n}{\delta_j^n})^2 + \frac{\varepsilon_n}{N+\alpha} \right], \tag{2.47}$$

and (2.46) in $[r_j^n, R_j^n]$:

$$(\delta_j^n)^{-\frac{N-2}{2}} = |\int_{r_j^n}^{R_j^n} v_n'| \le (\delta_j^n)^{-\frac{N-2}{2}} \left[(\frac{R_j^n - r_j^n}{\delta_j^n})^2 + \frac{\varepsilon_n}{N+\alpha} \right], \tag{2.48}$$

in view of $\alpha + 2 > 0$ and $\int_0^1 r^{\alpha+1} dr < +\infty$. Therefore we have shown that

$$\frac{R_j^n - r_j^n}{\delta_j^n}, \ \frac{r_j^n - R_{j-1}^n}{\delta_j^n} \ge \delta > 0 \tag{2.49}$$

for some $\delta > 0$ in view of (2.44), and the validity of (2.41) follows. □

When $k = 1$, we can apply (2.35) with $j = 1$ to get $\alpha \le N - 4$. Indeed, $\alpha > N - 4$ would imply, by inserting (2.36) into (2.35), that $1 = O(\varepsilon_n (R_1^n)^{N-2})$, yielding a contradiction in view of $\varepsilon_n (R_1^n)^{N-2} \to 0$ as $n \to +\infty$. If in addition $R_1^n \to R_1 > 0$ as $n \to +\infty$, by (2.38) for $j = 2$ condition $R_1^n < 1$ would imply $\delta_2^n \to 0$ and then $\frac{r_2^n}{\delta_2^n} \to +\infty$ as $n \to +\infty$, in contradiction with (2.40) for $j = 2$. Hence $R_1^n = 1$ for n large and, when $\alpha < N - 4$, by inserting (2.37) into (2.35) for $j = 1$ we get that

$$\delta_1^n = \left[\frac{(\alpha+2)\omega_{N-1} \int_{\mathbb{R}^N} |x|^\alpha V^2}{(\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}})^2} \varepsilon_n \right]^{\frac{1}{N-4-\alpha}} (1 + o(1)),$$

completing the proof for $k = 1$.

When $k \ge 2$, by (2.38) and (2.40) for $j = 2$ we can assume, up to a subsequence, that $\delta_2^n \to 0$ and $\frac{r_2^n}{\delta_2^n} \to L \in [0, +\infty)$ as $n \to +\infty$.

<u>$3^{\mathbf{rd}}$ **Claim**</u>: There holds

$$\lim_{n \to +\infty} \frac{r_2^n}{\delta_2^n} = 0. \tag{2.50}$$

Assume by contradiction that $L > 0$. Since

$$\frac{r_2^n}{R_1^n} = 1 + \frac{\frac{r_2^n - R_1^n}{\delta_2^n}}{\frac{r_2^n}{\delta_2^n} - \frac{r_2^n - R_1^n}{\delta_2^n}} \to 1$$

if $\frac{r_2^n - R_1^n}{\delta_2^n} \to 0$ as $n \to +\infty$, by (2.47)-(2.48) we can still deduce the validity of (2.49) for $j = 2$. Up to a subsequence, we can then assume that

$$\frac{R_1^n}{\delta_2^n} \to L_1 \in [0, L), \qquad \frac{R_2^n}{\delta_2^n} \to L_2 \in (L, +\infty].$$

The function V_2^n does solve

$$\begin{cases} -\Delta V_2^n = (V_2^n)^{\frac{N+2}{N-2}} + \varepsilon_n (\delta_2^n)^{2+\alpha} |x|^\alpha V_2^n & \text{in } I_n = (\frac{R_1^n}{\delta_2^n}, \frac{R_2^n}{\delta_2^n}) \\ 0 < V_2^n \le V_2^n(\frac{r_2^n}{\delta_2^n}) = 1 & \text{in } I_n \\ V_2^n = 0 & \text{on } \partial I_n \end{cases}$$

in view of (2.18). Arguing as above, by elliptic estimates we have that, up to a subsequence, $V_2^n \to V_2$ in $C^1_{\rm loc}(A)$, $A = B_{L_2}(0) \setminus \overline{B_{L_1}(0)}$, where V_2 solves

$$-\Delta V_2 = (V_2)^{\frac{N+2}{N-2}} \text{ in } A, \qquad 0 < V_2 \le V_2(L) = 1 \text{ in } A.$$

By (2.30) it follows that

$$\begin{aligned} -(\delta_2^n)^{-\frac{N-2}{2}}(R_1^n)^{N-1}v_n'(R_1^n) &= \int_{\frac{R_1^n}{\delta_2^n}}^{\frac{r_2^n}{\delta_2^n}} s^{N-1}(V_2^n)^{\frac{N+2}{N-2}} + \varepsilon_n(\delta_2^n)^{2+\alpha}\int_{\frac{R_1^n}{\delta_2^n}}^{\frac{r_2^n}{\delta_2^n}} s^{N-1+\alpha}V_2^n \\ &\to \int_{L_1}^{L} s^{N-1}(V_2)^{\frac{N+2}{N-2}} \end{aligned} \tag{2.51}$$

as $n \to +\infty$ in view of $V_2^n \le 1$. Since $1 \in J$, by (2.34) and (2.51) we get that $\delta_1^n \sim \delta_2^n$ as $n \to +\infty$, in contradiction with

$$\frac{\delta_2^n}{\delta_1^n} \ge \frac{1}{2L}\frac{r_2^n}{\delta_1^n} \ge \frac{1}{2L}\frac{R_1^n}{\delta_1^n} \to +\infty$$

as $n \to +\infty$ as it follows by (2.19). Then (2.50) is established and the Claim is proved. $\square$

Once (2.50) is established, we proceed as follows. Since $0 \le \frac{r_2^n - R_1^n}{\delta_2^n} \le \frac{r_2^n}{\delta_2^n} \to 0$ observe that

$$\frac{R_1^n}{r_2^n} \to 0 \text{ as } n \to +\infty \tag{2.52}$$

in view of (2.47). Up to a subsequence, we can assume that $\frac{R_2^n}{\delta_2^n} \to L_2 \in (0, +\infty]$ in view of (2.48), and, arguing as above, deduce by elliptic estimates that $V_2^n \to V_2$ in $C^1_{\rm loc}(B_{L_2}(0) \setminus \{0\})$ as $n \to +\infty$, where V_2 solves

$$-\Delta V_2 = (V_2)^{\frac{N+2}{N-2}} \text{ in } B_{L_2}(0), \qquad 0 \le V_2 \le 1 \text{ in } B_{L_2}(0)$$

with $V_2(L_2) = 0$ if $L_2 < +\infty$. Since by (2.46) there holds

$$|(V_2^n)'|(r) \le r - \frac{r_2^n}{\delta_2^n} + \varepsilon_n(\delta_2^n)^{\alpha+2}\frac{r^{\alpha+1}}{N+\alpha}$$

for all $\frac{r_2^n}{\delta_2^n} \le r \le \frac{R_2^n}{\delta_2^n}$, we have that

$$V_2^n(r) = 1 + \int_{\frac{r_2^n}{\delta_2^n}}^{r} (V_2^n)' \ge 1 - \frac{1}{2}(r - \frac{r_2^n}{\delta_2^n})^2 - \varepsilon_n(\delta_2^n)^{\alpha+2}\int_0^r \frac{s^{\alpha+1}}{N+\alpha}$$

for all $\frac{r_2^n}{\delta_2^n} \le r \le \frac{R_2^n}{\delta_2^n}$, and then as $n \to +\infty$ we deduce that $1 \ge V_2(r) \ge 1 - \frac{1}{2}r^2$ for all $0 < r < L_2$. Hence $V_2(0) = 1$, $L_2 = +\infty$ by Pohozaev identity (2.20) and $V_2 = V$, where V is given by (2.13).

So far we have shown that $1 \in J \Rightarrow 2 \in J$. As already explained, the new estimate (2.16) becomes crucial here. The difficulty is that very few is known about v_n in the range $[R_1^n, r_2^n]$, a problem which can be by-passed through the following trick. The key remark is that

$$\frac{1}{r^{N-1}}\int_{R_1^n}^{r} s^{N-1}(|v_n|^{\frac{N+2}{N-2}} + \varepsilon_n s^\alpha |v_n|)ds = (\delta_2^n)^{-\frac{N-2}{2}} O\left(\frac{r_2^n}{(\delta_2^n)^2} + \varepsilon_n r^{\alpha+1}\right) \tag{2.53}$$

for all $r \in [R_1^n, r_2^n]$ in view of (2.18). By integrating (2.7) for v_n in (R_1^n, r) we get that

$$v_n'(r) = \frac{(R_1^n)^{N-1}v_n'(R_1^n)}{r^{N-1}} - \frac{1}{r^{N-1}}\int_{R_1^n}^{r} s^{N-1}(|v_n|^{\frac{N+2}{N-2}} + \epsilon s^\alpha |v_n|)ds \tag{2.54}$$

for all $r \in [R_1^n, r_2^n]$. Inserting (2.34) with $j = 1$ and (2.53) into (2.54) we deduce that

$$v_n'(r) = -\frac{(\delta_1^n)^{\frac{N-2}{2}}}{r^{N-1}}\int_0^\infty s^{N-1}V^{\frac{N+2}{N-2}}[1 + o(1)] + (\delta_2^n)^{-\frac{N-2}{2}} O\left(\frac{r_2^n}{(\delta_2^n)^2} + \varepsilon_n r^{\alpha+1}\right)$$

for all $r \in [R_1^n, r_2^n]$, and then

$$\begin{aligned}(\delta_2^n)^{-\frac{N-2}{2}} &= -\int_{R_1^n}^{r_2^n} v_n' \\ &= \frac{(\delta_1^n)^{\frac{N-2}{2}}}{N-2}\int_0^\infty s^{N-1}V^{\frac{N+2}{N-2}}[1+o(1)][\frac{1}{(R_1^n)^{N-2}}-\frac{1}{(r_2^n)^{N-2}}] \\ &\quad +(\delta_2^n)^{-\frac{N-2}{2}}O\left((\frac{r_2^n}{\delta_2^n})^2+\varepsilon_n\int_0^1 r^{\alpha+1}\right)\end{aligned} \tag{2.55}$$

as $n \to +\infty$. Since $\frac{R_1^n}{r_2^n}, \frac{r_2^n}{\delta_2^n} \to 0$ as $n \to +\infty$ in view of (2.14) with $j=2$ and (2.52), by (2.55) we deduce the validity of (2.16) for R_1^n.

We already have that $\alpha \le N-4$. The case $\alpha = N-4$ can be excluded since (2.37) into (2.35) for $j=1$ would provide $1 = O(\varepsilon_n (R_1^n)^{N-2}|\log \frac{R_1^n}{\delta_1^n}|)$, a contradiction in view of $\varepsilon_n, R_1^n \to 0$ and

$$\frac{R_1^n}{\delta_1^n} = \frac{\delta_2^n}{R_1^n}\frac{(R_1^n)^2}{\delta_1^n\delta_2^n} = O(\frac{\delta_2^n}{R_1^n}) = O(\frac{1}{R_1^n})$$

as $n \to +\infty$, thanks to (2.16) for R_1^n. Hence $\alpha < N-4$ and (2.37) into (2.35) provides that

$$(\alpha+2)\varepsilon_n(\delta_1^n)^{2+\alpha}\int_0^{+\infty} r^{N-1+\alpha}V^2 = \left(\int_0^\infty r^{N-1}V^{\frac{N+2}{N-2}}\right)^2 (\frac{\delta_1^n}{R_1^n})^{N-2}(1+o(1)). \tag{2.56}$$

In view of (2.16) for R_1^n, (2.56) gives that

$$(\delta_1^n)^{\frac{N-6-2\alpha}{2}} \sim \varepsilon_n(\delta_2^n)^{\frac{N-2}{2}} \to 0$$

as $n \to +\infty$, which necessarily requires $\alpha < \frac{N-6}{2}$.

We can easily iterate the above procedure to show that $J = \{1, \dots, k\}$ and (2.16) does hold for all $j = 1, \dots, k-1$. If (2.15) does hold, condition $R_k^n < 1$ would imply the existence of $R_k^n < r_{k+1}^n < R_{k+1}^n \le 1$ so that $v_n(R_{k+1}^n) = 0$ and

$$|v_n|(r_{k+1}^n) = \max_{[R_k^n, R_{k+1}^n]} |v_n|.$$

Setting $\delta_{k+1}^n = |v_n(r_{k+1}^n)|^{-\frac{2}{N-2}}$, by (2.38) with $j=k$ we would deduce that $\delta_{k+1}^n \to 0$ and then $\frac{r_{k+1}^n}{\delta_{k+1}^n} \to +\infty$ as $n \to +\infty$, in contradiction with (2.40) for $j=k$. Hence $R_k^n = 1$ for n large.

Since by (2.14)

$$\frac{\delta_j^n}{\delta_{j+1}^n} = \frac{\delta_j^n}{R_j^n}\frac{R_j^n}{\delta_{j+1}^n} < \frac{\delta_j^n}{R_j^n}\frac{r_{j+1}^n}{\delta_{j+1}^n} \to 0 \quad \text{as } n \to +\infty$$

for all $j = 1, \dots, k-1$, by (2.35) and (2.37) we get that

$$(\alpha+2)\varepsilon_n(\delta_j^n)^{2+\alpha}\int_0^{+\infty} r^{N-1+\alpha}V^2 = \left(\int_0^\infty r^{N-1}V^{\frac{N+2}{N-2}}\right)^2 (\frac{\delta_j^n}{R_j^n})^{N-2}(1+o(1)). \tag{2.57}$$

For $j=k$ by (2.57) we have that

$$\delta_k^n = \left[\frac{(\alpha+2)\omega_{N-1}\int_{\mathbb{R}^N}|x|^\alpha V^2}{(\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}})^2}\varepsilon_n\right]^{\frac{1}{N-4-\alpha}}(1+o(1)) \tag{2.58}$$

as $n \to +\infty$ in view of $R_k^n = 1$. For $j = 1, \dots, k-1$, by inserting (2.16) into (2.57) we have that

$$(\alpha+2)\int_{\mathbb{R}^N}|x|^\alpha V^2\varepsilon_n(\delta_{j+1}^n)^{\frac{N-2}{2}} = (N-2)\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}}(\delta_j^n)^{\frac{N-6-2\alpha}{2}}(1+o(1))$$

as $n \to +\infty$. We finally deduce that

$$\delta_j^n = \left[\frac{(\alpha+2)\int_{\mathbb{R}^N} |x|^\alpha V^2}{(N-2)\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}}}\right]^{\frac{2}{N-6-2\alpha}} (\varepsilon_n)^{\frac{2}{N-6-2\alpha}} (\delta_{j+1}^n)^{\frac{N-2}{N-6-2\alpha}} (1+o(1)) \tag{2.59}$$

as $n \to +\infty$ for all $j = 1, \dots, k-1$, or equivalently

$$\delta_j^n \sim \left[\frac{(\alpha+2)\int_{\mathbb{R}^N} |x|^\alpha V^2}{(N-2)\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}}}\varepsilon_n\right]^{\frac{(N-2)(\frac{N-2}{N-6-2\alpha})^{k-j}-(N-4-\alpha)}{(2+\alpha)(N-4-\alpha)}} \left[\frac{(N-2)\omega_{N-1}}{\int_{\mathbb{R}^N} V^{\frac{N+2}{N-2}}}\right]^{\frac{1}{N-4-\alpha}(\frac{N-2}{N-6-2\alpha})^{k-j}}$$

as it follows iteratively by (2.58)-(2.59). This completes the proof. □

3. A perturbative approach: setting of the problem

In this section we provide a very delicate perturbative scheme in order to prove Theorem 1.2. The main ingredient in our construction are the Euclidean bubbles defined in (1.3) which are all the solutions to the critical equation (1.6) with Hardy potential in the Euclidean space.

It turns out to be useful to rewrite problem (1.7) as follows. We let $\imath^* : L^{\frac{2N}{N+2}}(\Omega) \to H_0^1(\Omega)$ be the adjoint operator of the embedding $\imath : H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, i.e. for any $w \in L^{\frac{2N}{N+2}}(\Omega)$ the function $u = \imath^*(w) \in H_0^1(\Omega)$ is the unique solution of

$$L_\gamma u = -\Delta u - \gamma\frac{u}{|x|^2} = w \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{3.1}$$

By continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, we get

$$\|\imath^*(w)\| \le C\,|w|_{\frac{2N}{N+2}}$$

for some $C > 0$. We rewrite problem (1.7) as

$$u = \imath^*\left[|u|^{\frac{4}{N-2}}u + \varepsilon u\right],\ u \in H_0^1(\Omega). \tag{3.2}$$

3.1. **The projection of the bubble.** To get a good approximation of our solution, it is necessary to project the bubble U_μ onto the space $H_0^1(\Omega)$. More precisely, letting $PU_\mu = \imath^*\left(U_\mu^{\frac{N+2}{N-2}}\right)$, according to (3.1) PU_μ solves

$$L_\gamma PU_\mu = L_\gamma U_\mu = U_\mu^{\frac{N+2}{N-2}} \text{ in } \Omega,\ PU_\mu = 0 \text{ on } \partial\Omega \tag{3.3}$$

in view of (1.6) for U_μ. Since $U_\mu^{\frac{N+2}{N-2}} \ge 0$ in Ω and $PU_\mu \in H_0^1(\Omega)$, by the weak maximum principle we have that $PU_\mu \ge 0$ in Ω. To get the expansion of PU_μ with respect to μ, we make use of some tools introduced by Ghossoub and Robert [18, 20]. First, let us recall the existence of a positive singular solution $G_\gamma \in C^2(\bar\Omega \setminus \{0\})$ to

$$\begin{cases} L_\gamma G_\gamma = 0 & \text{in } \Omega \setminus \{0\} \\ G_\gamma = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

having near the origin the following expansion:

$$G_\gamma(x) = \frac{c_1}{|x|^{\beta_+}} - \frac{c_2}{|x|^{\beta_-}} + o\left(\frac{1}{|x|^{\beta_-}}\right) \qquad \text{as } x \to 0, \tag{3.5}$$

where $c_1,\ c_2 > 0$ and $\beta_\pm$ are given in (1.5). The function $H_\gamma = \frac{c_1}{|x|^{\beta_+}} - G_\gamma$ in turn satisfies

$$\begin{cases} L_\gamma H_\gamma = 0 & \text{in } \Omega \setminus \{0\} \\ H_\gamma = \dfrac{c_1}{|x|^{\beta_+}} & \text{on } \partial\Omega \end{cases} \tag{3.6}$$

with

$$H_\gamma(x) \sim \frac{c_2}{|x|^{\beta_-}} \qquad \text{as } x \to 0. \tag{3.7}$$

By Theorem 9 in [20] observe that $H_\gamma \in H_0^1(\Omega)$, whereas $G_\gamma \notin H_0^1(\Omega)$. The quantity $m = m_{\gamma,0} = \frac{c_2}{c_1} > 0$ is referred to as the *Hardy interior mass* of Ω associated to L_γ and w.l.o.g. we can assume $c_1 = 1$.

We have the following estimates.

Lemma 3.1. *There hold*

(i) $0 \le PU_\mu \le U_\mu$ *in* Ω

(ii) $PU_\mu = U_\mu - \alpha_N \mu^\Gamma H_\gamma + O\left(\frac{\mu^{\frac{N+2}{N-2}\Gamma}}{|x|^{\beta_-}}\right)$ *uniformly in* Ω *as* $\mu \to 0$

(iii) $PU_\mu = U_\mu + O\left(\frac{\mu^\Gamma}{|x|^{\beta_-}}\right)$ *uniformly in* Ω *as* $\mu \to 0$.

Proof. (i) The function $\varphi_\mu = U_\mu - PU_\mu$ solves

$$\begin{cases} L_\gamma \varphi_\mu = 0 & \text{in } \Omega \setminus \{0\} \\ \varphi_\mu = U_\mu & \text{on } \partial\Omega. \end{cases}$$

Since $U_\mu \ge 0$ and $\varphi_\mu \in H^1(\Omega)$, by the weak maximum principle it follows that $\varphi_\mu \ge 0$ and (i) holds.

(ii) Let $W_\mu = U_\mu - PU_\mu - \alpha_N \mu^\Gamma H_\gamma$. Then W_μ satisfies the following problem

$$\begin{cases} L_\gamma W_\mu = 0 & \text{in } \Omega \setminus \{0\} \\ W_\mu = \frac{\alpha_N \mu^\Gamma}{|x|^{\beta_-}(\mu^{\frac{4\Gamma}{N-2}} + |x|^{\frac{4\Gamma}{N-2}})^{\frac{N-2}{2}}} - \frac{\alpha_N \mu^\Gamma}{|x|^{\beta_+}} = O\left(\mu^{\frac{N+2}{N-2}\Gamma}\right) & \text{on } \partial\Omega. \end{cases}$$

Since $W_\mu \in H^1(\Omega)$, by weak comparison principle it follows that

$$W_\mu = O\left(\mu^{\frac{N+2}{N-2}\Gamma} H_\gamma\right) = O\left(\frac{\mu^{\frac{N+2}{N-2}\Gamma}}{|x|^{\beta_-}}\right) \qquad \text{in } \Omega \setminus \{0\}$$

in view of (3.7), and (ii) follows.

(iii) It follows immediately by (ii) and (3.7). $\square$

3.2. **The linearized operator.** It is important to linearize the problem (1.6) around the solution U defined in (1.4). More precisely, let us consider the linear problem

$$\begin{cases} -\Delta Z - \gamma \frac{Z}{|x|^2} = \frac{N+2}{N-2} U^{\frac{4}{N-2}} Z \quad \text{in } \mathbb{R}^N \\ Z \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{3.8}$$

Dancer, Gladiali and Grossi in [13] classified all the solutions to (3.8):

Lemma 3.2 (Lemma 1.3, [13])**.** *Let* $\gamma < \frac{(N-2)^2}{4}$ *so that* $\gamma \ne \gamma_j$ *for all* $j \in \mathbb{N}$*, where* γ_j *is given by* (1.8)*. Then the space of solutions to* (3.8) *has dimension 1 and is spanned by*

$$Z^\gamma(x) = \frac{1 - |x|^{\frac{4\Gamma}{N-2}}}{|x|^{\beta_-}\left(1 + |x|^{\frac{4\Gamma}{N-2}}\right)^{\frac{N}{2}}}, \ x \in \mathbb{R}^N.$$

If $\gamma = \gamma_j$ *for some* $j \in \mathbb{N}$*, then the space of solutions to* (3.8) *has dimension* $1 + \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!}$ *and is spanned by*

$$Z^\gamma(x) \quad \text{and} \quad Z_i^\gamma(x) = \frac{|x|^{\frac{N\Gamma}{N-2} - \frac{N-2}{2}} P_{j,i}(x)}{\left(1 + |x|^{\frac{4\Gamma}{N-2}}\right)^{\frac{N}{2}}}, \ i = 1, \ldots, \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!},$$

where $\{P_{j,i}\}$ is a basis for the space $\mathbb{P}_j(\mathbb{R}^N)$ of $j-$homogeneous harmonic polynomials in $\mathbb{R}^N$.

Given $\mathcal{G}$ a closed subgroup in the space of linear isometries $\mathcal{O}(N)$ of $\mathbb{R}^N$, we say that a domain $\Omega \subset \mathbb{R}^N$ is $\mathcal{G}-$invariant if $\mathcal{G}x \subset \Omega$ for any $x \in \Omega$ and a function $u : \Omega \to \mathbb{R}$ is $\mathcal{G}-$invariant if $u(gx) = u(x)$ for any $x \in \Omega$ and $g \in \mathcal{G}$.

Definition 3.3. If $\gamma = \gamma_j$ for some $j \in \mathbb{N}$ (see (1.8)), Ω is said to be a $j-$admissible domain if Ω is $\mathcal{G}_j-$invariant for some closed subgroup $\mathcal{G}_j \subset \mathcal{O}(N)$ so that $\int\limits_{\mathbb{R}^N} Z_i^\gamma(x)\phi(x)dx = 0$ for any i and any $\mathcal{G}_j-$invariant function $\phi \in D^{1,2}(\mathbb{R}^N)$.

Remark 3.4. A ball is $j-$admissible for all $j \in \mathbb{N}$ by taking $\mathcal{G}_j = O(N)$. Any even domain Ω (i.e. $x \in \Omega$ iff $-x \in \Omega$) is $j-$admissible for all $j \in \mathbb{N}$ odd by taking $\mathcal{G}_j = \{Id, -Id\}$, since any homogeneous harmonic polynomials of odd degree is odd.

Remark 3.5. In the following we will work in a setting where the space of solutions to (3.8) is simply generated by Z^γ. In a general domain, we will require either $\gamma > 0$ or $\gamma \le 0$ with $\gamma \neq \gamma_j$ for all $j \in \mathbb{N}$. If $\gamma = \gamma_j$ for some $j \in \mathbb{N}$, we will assume that Ω is a $j-$admissible domain and we will work in the space of $\mathcal{G}_j-$invariant functions. Indeed, by Lemma 3.2 we immediately deduce that the space of $\mathcal{G}_j-$invariant solutions to (3.8) is spanned by Z^γ.

From now on we let $Z = Z^\gamma$ and we omit the dependence on γ. It is clear that the function

$$Z_\mu(x) = \mu^{-\frac{N-2}{2}} Z\left(\frac{x}{\mu}\right) = \frac{\mu^\Gamma(\mu^{\frac{4\Gamma}{N-2}} - |x|^{\frac{4\Gamma}{N-2}})}{|x|^{\beta_-}(\mu^{\frac{4\Gamma}{N-2}} + |x|^{\frac{4\Gamma}{N-2}})^{\frac{N}{2}}}, \quad x \in \mathbb{R}^N,$$

solves the linear problem

$$-\Delta Z_\mu - \gamma\frac{Z_\mu}{|x|^2} = \frac{N+2}{N-2}U_\mu^{\frac{4}{N-2}}Z_\mu \text{ in } \mathbb{R}^N.$$

We need to project the function Z_μ to fit Dirichlet boundary condition, i.e. we consider the function $PZ_\mu = \imath^*\left(\frac{N+2}{N-2}U_\mu^{\frac{4}{N-2}}Z_\mu\right)$ according to (3.1). We need an expansion of PZ_μ with respect to μ.

Lemma 3.6. *As $\mu \to 0$ there hold uniformly in Ω*

(i) $PZ_\mu = Z_\mu + \mu^\Gamma H_\gamma + O\left(\frac{\mu^{\frac{N+2}{N-2}\Gamma}}{|x|^{\beta_-}}\right)$

(ii) $PZ_\mu = Z_\mu + O\left(\frac{\mu^\Gamma}{|x|^{\beta_-}}\right)$.

Proof. We argue as in the proof of Lemma 3.1. ☐

3.3. **The tower.** Let $k \ge 1$ be a fixed integer. We look for solutions to (1.7), or equivalently to (3.2), of the form

$$u = \sum_{j=1}^k (-1)^j PU_{\mu_j} + \Phi, \tag{3.9}$$

where

$$\mu_1 = e^{-\frac{d_1}{\varepsilon}} \tag{3.10}$$

when $\Gamma = 1$ and

$$\mu_j = d_j\varepsilon^{\sigma_j},\ j = 1, \dots, k, \tag{3.11}$$

when $\Gamma > 1$, with $d_1, \dots, d_k \in (0, +\infty)$ and σ_j given by (2.17). The choice (3.10)-(3.11) of the concentration rates is motivated by the validity of the following crucial relations: for $\Gamma = 1$

$$\mu_1^2 \sim \varepsilon\mu_1^2 \log\frac{1}{\mu_1} \tag{3.12}$$

and for $\Gamma > 1$

$$\mu_1^{2\Gamma} \sim \varepsilon\mu_1^2 \quad \text{and} \quad \left(\frac{\mu_j}{\mu_{j-1}}\right)^{\Gamma} \sim \varepsilon\mu_j^2,\ j = 2,\ldots,k. \tag{3.13}$$

To build solutions of given sign with a simple blow-up point at the origin, we need to assume $\Gamma \geq 1$ and consider the case $k = 1$. The assumption $\Gamma > 2$ is necessary when constructing sign-changing solutions, i.e. $k \geq 2$, to guarantee $\sigma_1,\ldots,\sigma_k > 0$.

The remainder term Φ shall be splitted into the sum of k terms of different order:

$$\Phi = \sum_{\ell=1}^{k} \phi_\ell, \tag{3.14}$$

where each remainder term ϕ_ℓ only depends on $\mu_1,\ldots,\mu_\ell$ and belongs to the space $\mathcal{K}_\ell^\perp$ defined as follows. For any $\ell = 1,\ldots,k$ we define the subspace $\mathcal{K}_\ell = \text{Span}\,\{PZ_{\mu_1},\ldots,PZ_{\mu_\ell}\}$ and either

$$\mathcal{K}_\ell^\perp = \left\{\phi \in H_0^1(\Omega)\ :\ \langle\phi, PZ_{\mu_i}\rangle = 0,\ i = 1,\ldots,\ell\right\}$$

when Ω is a general domain and $\gamma \neq \gamma_j$ for all $j \in \mathbb{N}$ or

$$\mathcal{K}_\ell^\perp = \left\{\phi \in H_0^1(\Omega)\ :\ \phi \text{ is } \mathcal{G}_j\text{−invariant},\ \langle\phi, PZ_{\mu_i}\rangle = 0,\ i = 1,\ldots,\ell\right\}$$

when Ω is j−admissible and $\gamma = \gamma_j$ for some $j \in \mathbb{N}$ (see Remark 3.5). We also define Π_ℓ and $\Pi_\ell^\perp$ as the projections of the Sobolev space $H_0^1(\Omega)$ onto the respective subspaces $\mathcal{K}_\ell$ and $\mathcal{K}_\ell^\perp$.

In order to solve (3.2), we shall solve the system

$$\begin{aligned} \Pi_k^\perp\left\{u - \imath^*\left[|u|^{\frac{4}{N-2}}u + \varepsilon u\right]\right\} &= 0 \\ \Pi_k\left\{u - \imath^*\left[|u|^{\frac{4}{N-2}}u + \varepsilon u\right]\right\} &= 0 \end{aligned} \tag{3.15}$$

for u given as in (3.9). For sake of simplicity, for any $j = 1,\ldots,k$ we set $U_j = U_{\mu_j}$ and $Z_j = Z_{\mu_j}$.

4. The Ljapunov-Schmidt procedure

In this section we give an outline for the proof of Theorem 1.2. To make the presentation more clear, all the results are stated without proofs, which are postponed into the Appendix.

4.1. **The remainder term: solving equation (3.15).**

In order to find the remainder term Φ, we shall find functions ϕ_ℓ, $\ell = 1,\ldots,k$, which solve the following system:

$$\begin{cases} \mathcal{E}_1 + \mathcal{L}_1(\phi_1) + \mathcal{N}_1(\phi_1) = 0 \\ \mathcal{E}_2 + \mathcal{L}_2(\phi_2) + \mathcal{N}_2(\phi_1,\phi_2) = 0 \\ \ldots \\ \ldots \\ \mathcal{E}_k + \mathcal{L}_k(\phi_k) + \mathcal{N}_k(\phi_1,\ldots,\phi_k) = 0. \end{cases} \tag{4.1}$$

Setting $f(u) = |u|^{\frac{4}{N-2}}u$, the error terms $\mathcal{E}_\ell$ are defined by

$$\mathcal{E}_\ell = \Pi_\ell^\perp\left\{(-1)^\ell PU_\ell - \imath^*\left[f\left(\sum_{j=1}^{\ell}(-1)^j PU_j\right) - f\left(\sum_{j=1}^{\ell-1}(-1)^j PU_j\right) + \varepsilon(-1)^\ell PU_\ell\right]\right\}$$

and the linear operators $\mathcal{L}_\ell$ are given by

$$\mathcal{L}_\ell(\phi) = \Pi_\ell^\perp\left\{\phi - \imath^*\left[f'\left(\sum_{j=1}^{\ell}(-1)^j PU_j\right)\phi + \varepsilon\phi\right]\right\},$$

with the convention that a sum over an empty set of indices is zero. The nonlinear terms $\mathcal{N}_\ell$ have the form

$$
\begin{aligned}
&\mathcal{N}_\ell(\phi_1,\dots,\phi_\ell) = \\
&\Pi_\ell^\perp\left\{-\imath^*\left[f\left(\sum_{j=1}^{\ell}\left((-1)^j PU_j+\phi_j\right)\right)-f\left(\sum_{j=1}^{\ell}(-1)^j PU_j\right)-f'\left(\sum_{j=1}^{\ell}(-1)^j PU_j\right)\phi_\ell\right.\right.\\
&\left.\left.-f\left(\sum_{j=1}^{\ell-1}\left((-1)^j PU_j+\phi_j\right)\right)+f\left(\sum_{j=1}^{\ell-1}(-1)^j PU_j\right)\right]\right\}.
\end{aligned}
\tag{4.2}
$$

In order to solve system (4.1), first we need to evaluate the $H_0^1(\Omega)-$ norm of the error terms $\mathcal{E}_\ell$.

Lemma 4.1. *For any $\ell = 1,\dots,k$ and any compact subset $A_\ell \subset (0,+\infty)^\ell$ there exist $C,\ \varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and for any $(d_1,\dots,d_\ell) \in A_\ell$ there holds*

$$
\|\mathcal{E}_1\| = \begin{cases} O\left(\varepsilon\mu_1^\Gamma\right) & \text{if } 1\le\Gamma<2\\ O\left(\varepsilon\mu_1^2\log^{\frac{N+2}{2N}}\frac{1}{\mu_1}\right) & \text{if } \Gamma=2\\ O\left(\varepsilon\mu_1^2\right) & \text{if } \Gamma>2\end{cases}
+\begin{cases} O\left(\mu_1^{2\Gamma}\right) & \text{if } 3\le N\le5\\ O\left(\mu_1^{2\Gamma}\log^{\frac{2}{3}}\frac{1}{\mu_1}\right) & \text{if } N=6\\ O\left(\mu_1^{\frac{N+2}{N-2}\Gamma}\right) & \text{if } N\ge7\end{cases}
\tag{4.3}
$$

and

$$
\|\mathcal{E}_\ell\| = O(\varepsilon\mu_\ell^2)+\begin{cases} O\left(\left(\frac{\mu_\ell}{\mu_{\ell-1}}\right)^\Gamma\right) & \text{if } 3\le N\le5\\ O\left(\left(\frac{\mu_\ell}{\mu_{\ell-1}}\right)^{\frac{N+2}{N-2}\frac{\Gamma}{2}}\log^{\frac{2}{3}}\frac{1}{\mu_\ell}\right) & \text{if } N\ge6\end{cases}
\tag{4.4}
$$

for any $l = 2,\dots,k$, when $k \ge 2$ and $\Gamma > 2$.

Next, we need to understand the invertibility of the linear operators $\mathcal{L}_\ell$. This is done in the following lemma whose proof can be carried out as in [27].

Lemma 4.2. *For any $\ell = 1,\dots,k$ and any compact subset $A_\ell \subset (0,+\infty)^\ell$ there exist $C,\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and for any $(d_1,\dots,d_\ell) \in A_\ell$ there holds*

$$
\|\mathcal{L}_\ell(\phi_\ell)\| \ge C\|\phi_\ell\| \text{ for any } \phi_\ell \in \mathcal{K}_\ell^\perp. \tag{4.5}
$$

In particular $\mathcal{L}_\ell^{-1} : \mathcal{K}_\ell^\perp \to K_\ell^\perp$ is well defined for $\varepsilon \in (0,\varepsilon_0)$ and $(d_1,\dots,d_\ell) \in A_\ell$ and has uniformly bounded operatorial norm.

Finally, we are able to solve system (4.1). This is done in the following proposition, whose proof in the Appendix relies on a sophisticated contraction mapping argument.

Proposition 4.3. *Given $A \subset (0,+\infty)^k$ compact, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ there exist C^1-maps $(d_1,\dots,d_k) \in A \to \phi_{\ell,\varepsilon} = \phi_{\ell,\varepsilon}(d_1,\dots,d_\ell) \in \mathcal{K}_\ell^\perp$, $\ell = 1,\dots,k$, which solve (4.1) and satisfy uniform estimates:*

$$
\|\phi_{1,\varepsilon}\| = O(\|\mathcal{E}_1\|),\quad \|\phi_{\ell,\varepsilon}\| = O\left(\left(\frac{\mu_\ell}{\mu_{\ell-1}}\right)^\Gamma+\left(\frac{\mu_\ell}{\mu_{\ell-1}}\right)^{\frac{\Gamma}{2}+1}+\left(\frac{\mu_\ell}{\mu_{\ell-1}}\right)^{\frac{N+2}{2(N-2)}\Gamma}\log^{\frac{2}{3}}\frac{1}{\mu_\ell}\right) \tag{4.6}
$$

for $l \ge 2$ and

$$
\|\nabla_{(d_1,\dots,d_\ell)}\phi_{\ell,\varepsilon}\| = o(1) \quad \ell = 1,\dots,k. \tag{4.7}
$$

Moreover, there exists $\rho > 0$ so that

$$
|\phi_{\ell,\varepsilon}(x)| = O\left(\frac{1}{\mu_\ell^\Gamma|x|^{\beta_-}}\right) \text{ if } x \in B_{\rho\mu_\ell}(0). \tag{4.8}
$$

4.2. **The reduced problem: proof of Theorem 1.2.** Let us recall the expression for the energy functional $J_\varepsilon : H_0^1(\Omega) \to \mathbb{R}$:

$$J_\epsilon(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} - \epsilon u^2 \right) dx - \frac{N-2}{2N}\int_\Omega |u|^{\frac{2N}{N-2}} dx,$$

whose critical points are solutions to the problem (1.7). Let us introduce the reduced energy as

$$J_\varepsilon(\mu_1, \dots, \mu_k) = J_\varepsilon\left(\sum_{j=1}^k (-1)^j PU_j \right).$$

Given Φ_ε according to (3.14) and Proposition 4.3, the following result is the main core of the finite dimensional reduction of our problem.

Proposition 4.4. *Given (3.10)-(3.11), we have that*

$$J_\varepsilon(\mu_1) = A_1 + \begin{cases} A_2 m\mu_1^2 - A_3\varepsilon\mu_1^2 \log\frac{1}{\mu_1} & \text{if } \Gamma = 1 \\ A_2 m \mu_1^{2\Gamma} - A_3\varepsilon\mu_1^2 & \text{if } \Gamma > 1 \end{cases} + \Upsilon_1(\mu_1) \tag{4.9}$$

and when $\Gamma > 2$

$$\begin{aligned} J_\varepsilon(\mu_1, \dots, \mu_k) =& kA_1 + A_2 m\mu_1^{2\Gamma} - A_3\varepsilon\mu_1^2 + \sum_{\ell=2}^k \left[A_4 (\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma - A_3\varepsilon\mu_\ell^2 \right] \\ &+ \sum_{\ell=1}^k \Upsilon_\ell(\mu_1, \dots, \mu_\ell), \end{aligned} \tag{4.10}$$

where $|\Upsilon_1| = o(\mu_1^{2\Gamma})$ *and* $|\Upsilon_\ell| = o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \right)$, $\ell = 2, \dots, k$, *do hold as* $\varepsilon \to 0$ *locally uniformly for* $(d_1, \dots, d_k)$ *in* $(0, +\infty)^k$. *Here* $A_1, \dots, A_4 > 0$ *and* $m > 0$ *is the Hardy interior mass of* Ω *associated to* L_γ. *Moreover, critical points of*

$$\widetilde{J}_\varepsilon(\mu_1, \dots, \mu_k) = J_\varepsilon\left(\sum_{j=1}^k (-1)^j PU_j + \Phi_\varepsilon \right) = J_\varepsilon(\mu_1, \dots, \mu_k) + \sum_{\ell=1}^k \widetilde{\Upsilon}_\ell(\mu_1, \dots, \mu_\ell)$$

give rise to solutions $\sum_{j=1}^k (-1)^j PU_j + \Phi_\varepsilon$ *of (1.7), where* $\widetilde{\Upsilon}_\ell$ *satisfies the same estimate as* Υ_ℓ.

Proof of Theorem 1.2. By (3.10)-(3.11) and Proposition 4.4 it is sufficient to find a critical point of

$$F_\varepsilon(d_1) = e^{-\frac{2d_1}{\varepsilon}} \left(A_2 m - A_3 d_1 + o_\ell(1) \right)$$

when $\Gamma = 1$ and

$$F_\varepsilon(d_1, \dots, d_k) = \sum_{\ell=1}^k \varepsilon^{2\sigma_\ell + 1} \left(G_\ell(d_1, \dots, d_\ell) + o_\ell(1) \right)$$

when $\Gamma > 1$, where

$$G_1(d_1) = A_2 m d_1^{2\Gamma} - A_3 d_1^2, \qquad G_\ell(d_1, \dots, d_\ell) = A_4 (\frac{d_\ell}{d_{\ell-1}})^\Gamma - A_3 d_\ell^2, \ \ell = 2, \dots, k.$$

Here $o_\ell(1)$ only depends on $d_1, \dots, d_\ell$ and $o_\ell(1) \to 0$ as $\varepsilon \to 0$ locally uniformly for $(d_1, \dots, d_\ell)$ in $(0, +\infty)^\ell$. For $k = 1$ it is easily found an interval

$$I = \begin{cases} (\frac{A_2}{A_3} m + \frac{\varepsilon}{4}, \frac{A_2}{A_3} m + \varepsilon) & \text{if } \Gamma = 1 \\ \left(\frac{1}{2} (\frac{A_3}{A_2 m \Gamma})^{\frac{1}{2(\Gamma-1)}}, 2 (\frac{A_3}{A_2 m\Gamma})^{\frac{1}{2(\Gamma-1)}} \right) & \text{if } \Gamma > 1 \end{cases} \subset (0, +\infty)$$

so that

$$\inf_I F_\varepsilon < \inf_{\partial I} F_\varepsilon$$

for ε small, which guarantees the existence of a minimum point $d_\varepsilon \in I$ of F_ε. For $k \geq 2$ it is still possible to show that F_ε has a minimum point but the proof is more involved. Since it can be carried out as in [26], we omit the details. □

5. Appendix

All the technical proofs can be carried out as in [26]. Since they are quite involved, we rewrite some of them here by re-adapting the arguments to the present situation.

5.1. The rate of the error: proof of Lemma 4.1. By the property of $\imath^*$, we get

$$\|\mathcal{E}_1\| = O\left(|(U_1)^{\frac{N+2}{N-2}} - (PU_1)^{\frac{N+2}{N-2}}|_{\frac{2N}{N+2}}\right) + O\left(\varepsilon |PU_1|_{\frac{2N}{N+2}}\right). \tag{5.1}$$

By Lemma 3.1 and scaling $x = \mu_1 y$ we have that

$$|PU_1|_{\frac{2N}{N+2}} \leq |U_1|_{\frac{2N}{N+2}} = \mu_1^2 |U|_{\frac{2N}{N+2}, \frac{\Omega}{\mu_1}} = \begin{cases} O\left(\mu_1^\Gamma\right) & \text{if } 1 \leq \Gamma < 2 \\ O\left(\mu_1^2 \log^{\frac{N+2}{2N}} \frac{1}{\mu_1}\right) & \text{if } \Gamma = 2 \\ O\left(\mu_1^2\right) & \text{if } \Gamma > 2 \end{cases} \tag{5.2}$$

in view of $\frac{2\beta_-}{N+2} < 1$ and $\frac{2\beta_+}{N+2} = \frac{N-2+2\Gamma}{N+2}$. Since $|a+b|^{\frac{N+2}{N-2}} - |a|^{\frac{N+2}{N-2}} = O(|a|^{\frac{4}{N-2}}|b| + |b|^{\frac{N+2}{N-2}})$ for all $a, b \in \mathbb{R}$, we deduce that

$$\left|(U_1)^{\frac{N+2}{N-2}} - (PU_1)^{\frac{N+2}{N-2}}\right|_{\frac{2N}{N+2}} = O\left(\left|U_1^{\frac{4}{N-2}}(PU_1 - U_1)\right|_{\frac{2N}{N+2}} + \left|PU_1 - U_1\right|_{\frac{2N}{N-2}}^{\frac{N+2}{N-2}}\right). \tag{5.3}$$

By Lemma 3.1 and scaling $x = \mu_1 y$ we have that

$$\begin{aligned} \left|U_1^{\frac{4}{N-2}}(PU_1 - U_1)\right|_{\frac{2N}{N+2}} &\leq c\mu_1^\Gamma \left|\frac{U_1^{\frac{4}{N-2}}}{|x|^{\beta_-}}\right|_{\frac{2N}{N+2}} = c(\mu_1)^{2\Gamma} \left|\frac{U^{\frac{4}{N-2}}}{|y|^{\beta_-}}\right|_{\frac{2N}{N+2}, \frac{\Omega}{\mu_1}} \\ &= \begin{cases} O\left(\mu_1^{2\Gamma}\right) & \text{if } 3 \leq N \leq 5 \\ O\left(\mu_1^{2\Gamma} \log^{\frac{2}{3}} \frac{1}{\mu_1}\right) & \text{if } N = 6 \\ O\left(\mu_1^{\frac{N+2}{N-2}\Gamma}\right) & \text{if } N \geq 7 \end{cases} \end{aligned} \tag{5.4}$$

and

$$\left|PU_1 - U_1\right|_{\frac{2N}{N-2}}^{\frac{N+2}{N-2}} = O\left(|\frac{\mu_1^\Gamma}{|x|^{\beta_-}}|_{\frac{2N}{N-2}}^{\frac{N+2}{N-2}}\right) = O\left(\mu_1^{\frac{N+2}{N-2}\Gamma}\right), \tag{5.5}$$

in view of $\frac{2\beta_-}{N-2} < 1$ and

$$\frac{2N}{N+2}(\beta_- + \frac{4\beta_+}{N-2}) = N - \frac{2N(N-6)}{N^2-4}\Gamma. \tag{5.6}$$

Inserting (5.4)-(5.5) into (5.3), by (5.1)-(5.2) we deduce the validity of (4.3).

Let us now consider the case $k \geq 2$ and assume $\Gamma > 2$. For $\ell \geq 2$ we have that

$$\begin{aligned} \|\mathcal{E}_\ell\| = &\underbrace{O(|| \sum_{j=1}^{\ell} (-1)^j PU_j|^{\frac{4}{N-2}} \sum_{j=1}^{\ell} (-1)^j PU_j - |\sum_{j=1}^{\ell-1} (-1)^j PU_j|^{\frac{4}{N-2}} \sum_{j=1}^{\ell-1} (-1)^j PU_j - (-1)^l (PU_\ell)^{\frac{N+2}{N-2}}|_{\frac{2N}{N+2}})}_{(I)} \\ &+ \underbrace{O\left(\left|(U_\ell)^{\frac{N+2}{N-2}} - (PU_\ell)^{\frac{N+2}{N-2}}\right|_{\frac{2N}{N+2}} + \varepsilon\, |PU_\ell|_{\frac{2N}{N+2}}\right)}_{(II)}. \end{aligned}$$

(II) is estimated as in (4.3) with μ_1 replaced by μ_l. As for (I), let us introduce disjoint annuli $\mathcal{A}_h$ as

$$\mathcal{A}_0 = \Omega \setminus B_r(0), \quad \mathcal{A}_h = B_{\sqrt{\mu_{h-1}\mu_h}}(0) \setminus B_{\sqrt{\mu_h \mu_{h+1}}}(0), \ h = 1, \ldots, \ell, \tag{5.7}$$

where μ_0 satisfies $\mu_0\mu_1 = r^2$ with $r = \frac{1}{2}\text{dist}(0, \partial\Omega)$ and $\mu_{\ell+1} = 0$. Moreover define μ_{-1} so that $\mu_{-1}\mu_0 = (\text{diam }\Omega)^2$, in order to get $\mathcal{A}_0 \subset B_{\sqrt{\mu_{-1}\mu_0}}(0) \setminus B_{\sqrt{\mu_0\mu_1}}(0)$. Since

$$|a+b|^{\frac{4}{N-2}}(a+b) - |a|^{\frac{4}{N-2}}a - \frac{N+2}{N-2}|a|^{\frac{4}{N-2}}b = O(|b|^{\frac{N+2}{N-2}}) + \underbrace{O(|a|^{\frac{6-N}{N-2}}b^2)}_{\text{if } 3\leq N\leq 5} \tag{5.8}$$

for all $a, b \in \mathbb{R}$, we have that

$$\begin{aligned}
&\Big|\Big|\sum_{j=1}^{\ell}(-1)^j PU_j\Big|^{\frac{4}{N-2}}\sum_{j=1}^{\ell}(-1)^j PU_j - \Big|\sum_{j=1}^{\ell-1}(-1)^j PU_j\Big|^{\frac{4}{N-2}}\sum_{j=1}^{\ell-1}(-1)^j PU_j - (-1)^l (PU_\ell)^{\frac{N+2}{N-2}}\Big|_{\frac{2N}{N+2},\mathcal{A}_h}\\
&= \begin{cases} O\left(\sum_{j=1}^{\ell-1}|(PU_j)^{\frac{4}{N-2}}PU_\ell|_{\frac{2N}{N+2},\mathcal{A}_h} + |PU_\ell|^{\frac{N+2}{N-2}}_{\frac{2N}{N-2},\mathcal{A}_h}\right) & \text{if } h = 0,\ldots,l-1\\ O\left(\sum_{j=1}^{\ell-1}|(PU_\ell)^{\frac{4}{N-2}}PU_j|_{\frac{2N}{N+2},\mathcal{A}_\ell} + \sum_{j=1}^{\ell-1}|PU_j|^{\frac{N+2}{N-2}}_{\frac{2N}{N-2},\mathcal{A}_\ell}\right) & \text{if } h = l.\end{cases}
\end{aligned} \tag{5.9}$$

Hereafter we will repeatedly use that $\mu_1 >> \ldots >> \mu_k$. Since $\frac{2\beta_-}{N-2} < 1 < \frac{2\beta_+}{N-2}$, by Lemma 3.1 and scaling $x = \mu_i y$ we have that

$$|PU_j|_{\frac{2N}{N-2},\mathcal{A}_h} \leq |U_j|_{\frac{2N}{N-2},\mathcal{A}_h} = |U|_{\frac{2N}{N-2},\frac{\mathcal{A}_h}{\mu_j}} = \begin{cases} O((\frac{\mu_\ell}{\sqrt{\mu_h\mu_{h+1}}})^\Gamma) & \text{if } j=\ell\\ O((\frac{\sqrt{\mu_{l-1}\mu_l}}{\mu_j})^\Gamma) & \text{if } h=\ell\end{cases} = O((\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{\Gamma}{2}}) \tag{5.10}$$

for any $j = 1,\ldots,\ell$ and $h = 0,\ldots,\ell$ with $\max\{j,h\} = \ell$, $j \neq h$. Since $|x| >> \mu_l$ in $\mathcal{A}_h$, for any $j = 1,\ldots,\ell-1$ and $h = 0,\ldots,\ell-1$ by Lemma 3.1 we have

$$\begin{aligned}
\left|(PU_j)^{\frac{4}{N-2}}PU_\ell\right|_{\frac{2N}{N+2},\mathcal{A}_h} &\leq \left|U_j^{\frac{4}{N-2}}U_\ell\right|_{\frac{2N}{N+2},\mathcal{A}_h} \leq c\mu_l^\Gamma\left|\frac{U_j^{\frac{4}{N-2}}}{|x|^{\beta_+}}\right|_{\frac{2N}{N+2},\mathcal{A}_h}\\
&= c(\frac{\mu_l}{\mu_j})^\Gamma\left|\frac{U^{\frac{4}{N-2}}}{|y|^{\beta_+}}\right|_{\frac{2N}{N+2},\frac{\mathcal{A}_h}{\mu_j}} = O\left((\frac{\mu_l}{\mu_{l-1}})^\Gamma\right)
\end{aligned} \tag{5.11}$$

when $3 \leq N \leq 5$ and

$$\begin{aligned}
&\left|(PU_j)^{\frac{4}{N-2}}PU_\ell\right|_{\frac{2N}{N+2},\mathcal{A}_h} \leq \left|U_j^{\frac{4}{N-2}}U_\ell\right|_{\frac{2N}{N+2},\mathcal{A}_h} \leq c\mu_j^{-\frac{4\Gamma}{N-2}}\left|\frac{U_l}{|x|^{\frac{4\beta_-}{N-2}}}\right|_{\frac{2N}{N+2},\mathcal{A}_h}\\
&= c(\frac{\mu_l}{\mu_j})^{\frac{4\Gamma}{N-2}}\left|\frac{U}{|y|^{\frac{4\beta_-}{N-2}}}\right|_{\frac{2N}{N+2},\frac{\mathcal{A}_h}{\mu_l}} \leq c(\frac{\mu_l}{\mu_j})^{\frac{4\Gamma}{N-2}}\begin{cases}\log^{\frac{2}{3}}\frac{\sqrt{\mu_{h-1}\mu_h}}{\mu_l} & if N=6\\ (\frac{\mu_l}{\sqrt{\mu_h\mu_{h+1}}})^{\frac{N-6}{N-2}\Gamma} & \text{if } N\geq 7\end{cases}\\
&= O\left((\frac{\mu_l}{\mu_{l-1}})^{\frac{N+2}{N-2}\frac{\Gamma}{2}}\log^{\frac{2}{3}}\frac{1}{\mu_l}\right)
\end{aligned} \tag{5.12}$$

when $N \geq 6$, in view of $\frac{2\beta_-}{N-2} < 1 < \frac{2\beta_+}{N-2}$ and

$$\frac{2N}{N+2}(\frac{4\beta_-}{N-2} + \beta_+) = N + \frac{2N(N-6)}{N^2-4}\Gamma.$$

Similarly, for $j = 1, \ldots, l-1$ we have that

$$\left|(PU_l)^{\frac{4}{N-2}} PU_j\right|_{\frac{2N}{N+2},\mathcal{A}_l} \le \left|U_l^{\frac{4}{N-2}} U_j\right|_{\frac{2N}{N+2},\mathcal{A}_l} \le c\mu_j^{-\Gamma} \left|\frac{U_l^{\frac{4}{N-2}}}{|x|^{\beta_-}}\right|_{\frac{2N}{N+2},\mathcal{A}_l}$$
$$= c(\frac{\mu_l}{\mu_j})^{\Gamma} \left|\frac{U^{\frac{4}{N-2}}}{|y|^{\beta_-}}\right|_{\frac{2N}{N+2},\frac{\mathcal{A}_l}{\mu_l}} = O\left((\frac{\mu_l}{\mu_{l-1}})^{\Gamma}\right) \tag{5.13}$$

when $3 \le N \le 5$ and

$$\left|(PU_l)^{\frac{4}{N-2}} PU_j\right|_{\frac{2N}{N+2},\mathcal{A}_l} \le \left|U_l^{\frac{4}{N-2}} U_j\right|_{\frac{2N}{N+2},\mathcal{A}_l} \le c\mu_l^{\frac{4\Gamma}{N-2}} \left|\frac{U_j}{|x|^{\frac{4\beta_+}{N-2}}}\right|_{\frac{2N}{N+2},\mathcal{A}_l}$$
$$= c(\frac{\mu_l}{\mu_j})^{\frac{4\Gamma}{N-2}} \left|\frac{U}{|y|^{\frac{4\beta_+}{N-2}}}\right|_{\frac{2N}{N+2},\frac{\mathcal{A}_l}{\mu_j}} \le c(\frac{\mu_l}{\mu_i})^{\frac{4\Gamma}{N-2}} \begin{cases} \log^{\frac{2}{3}} \frac{\mu_j}{\sqrt{\mu_{l-1}\mu_l}} & if N = 6 \\ (\frac{\sqrt{\mu_{l-1}\mu_l}}{\mu_j})^{\frac{N-6}{N-2}\Gamma} & \text{if } N \ge 7 \end{cases} \tag{5.14}$$
$$= O\left((\frac{\mu_l}{\mu_{l-1}})^{\frac{N+2}{N-2}\frac{\Gamma}{2}} \log^{\frac{2}{3}} \frac{1}{\mu_l}\right)$$

when $N \ge 6$ in view of $\frac{2\beta_-}{N-2} < 1$ and (5.6). By inserting (5.10)-(5.14) into (5.9) we deduce an estimate of (I) which, along with the estimate on (II) in terms of μ_ℓ, leads to the validity of (4.4).

5.2. **The reduced energy: proof of (4.9)-(4.10).** To get an expansion of $J_\varepsilon(\mu_1, \ldots, \mu_k)$, let us first write that

$$J_\varepsilon(\sum_{\ell=1}^k (-1)^\ell PU_\ell) = \sum_{\ell=1}^k J_\varepsilon(PU_\ell) + \sum_{i<\ell} (-1)^{i+\ell} \int_\Omega [U_\ell^{\frac{N+2}{N-2}} - \varepsilon PU_\ell - (PU_\ell)^{\frac{N+2}{N-2}}] PU_i \, dx$$
$$- \frac{N-2}{2N} \int_\Omega [|\sum_{\ell=1}^k (-1)^\ell PU_\ell|^{\frac{2N}{N-2}} - \sum_{\ell=1}^k (PU_\ell)^{\frac{2N}{N-2}} - \frac{2N}{N-2} \sum_{i<\ell} (-1)^{i+\ell} (PU_\ell)^{\frac{N+2}{N-2}} PU_i] \, dx$$

in view of $PU_\ell = \imath^* \left(U_\ell^{\frac{N+2}{N-2}}\right)$. Introducing the quantities

$$a_\ell = J_\varepsilon(PU_\ell) + \sum_{i=1}^{\ell-1} (-1)^{i+\ell} \int_\Omega [U_\ell^{\frac{N+2}{N-2}} - \varepsilon PU_\ell - (PU_\ell)^{\frac{N+2}{N-2}}] PU_i \, dx$$
$$- \frac{N-2}{2N} \int_\Omega [|\sum_{i=1}^{\ell} (-1)^i PU_i|^{\frac{2N}{N-2}} - |\sum_{i=1}^{\ell-1} (-1)^i PU_i|^{\frac{2N}{N-2}} - (PU_\ell)^{\frac{2N}{N-2}} - \frac{2N}{N-2} \sum_{i=1}^{\ell-1} (-1)^{i+\ell} (PU_\ell)^{\frac{N+2}{N-2}} PU_i] \, dx$$

for any $\ell = 1, \ldots, k$, let us notice that each a_ℓ only depends on $d_1, \ldots, d_\ell$ and the following decomposition does hold:

$$J_\varepsilon(\sum_{\ell=1}^k (-1)^\ell PU_\ell) = \sum_{\ell=1}^k a_\ell. \tag{5.15}$$

We claim that

$$a_1 = A_1 + A_2 m \mu_1^{2\Gamma}(1 + o(1)) - A_3 \varepsilon (1 + o(1)) \begin{cases} \mu_1^2 \log \frac{1}{\mu_1} & \text{if } \Gamma = 1 \\ \mu_1^2 & \text{if } \Gamma > 1 \end{cases} \tag{5.16}$$

and

$$a_\ell = A_1 + A_4 \left(\frac{\mu_\ell}{\mu_{\ell-1}}\right)^\Gamma (1 + o(1)) - A_3 \varepsilon \mu_\ell^2 (1 + o(1)), \ \ell = 2, \ldots, k, \tag{5.17}$$

where $m > 0$ is the *Hardy interior mass* of Ω associated to L_γ and $A_1, \ldots, A_4 > 0$. Inserting (5.16)-(5.17) into (5.15), we deduce the validity of (4.9)-(4.10).

To compute $J_\varepsilon(PU_\ell)$, let us first write

$$
\begin{aligned}
J_\varepsilon(PU_\ell) = \frac{1}{N}\int_\Omega U_\ell^{\frac{2N}{N-2}}dx - \frac{1}{2}\int_\Omega U_\ell^{\frac{N+2}{N-2}}(PU_\ell - U_\ell)dx - \frac{\varepsilon}{2}\int_\Omega PU_\ell^2 dx \\
- \frac{N-2}{2N}\int_\Omega [(PU_\ell)^{\frac{2N}{N-2}} - U_\ell^{\frac{2N}{N-2}} - \frac{2N}{N-2}U_\ell^{\frac{N+2}{N-2}}(PU_\ell - U_\ell)]dx
\end{aligned}
\tag{5.18}
$$

in view of $PU_\ell = \imath^*\left(U_\ell^{\frac{N+2}{N-2}}\right)$. We have that

$$
\int_\Omega U_\ell^{\frac{2N}{N-2}}dx = \int_{\mathbb{R}^N} U^{\frac{2N}{N-2}}dy + O(\mu_\ell^{\frac{2N}{N-2}\Gamma}),
\tag{5.19}
$$

and by Lemma 3.1 and (3.7) we deduce that

$$
\begin{aligned}
\int_\Omega U_\ell^{\frac{N+2}{N-2}}(PU_\ell - U_\ell)dx &= -\alpha_N\mu_\ell^\Gamma \int_\Omega U_\ell^{\frac{N+2}{N-2}}[H_\gamma(x) + O(\frac{\mu_\ell^{\frac{4\Gamma}{N-2}}}{|x|^{\beta_-}})]\,dx \\
&= -\alpha_N m\mu_\ell^{2\Gamma}\int_{\mathbb{R}^N}\frac{U^{\frac{N+2}{N-2}}}{|y|^{\beta_-}}dy\ (1+o(1))
\end{aligned}
\tag{5.20}
$$

and

$$
\begin{aligned}
\int_\Omega PU_\ell^2 dx &= \int_\Omega U_\ell^2 dx + O(\int_\Omega U_\ell \frac{\mu_\ell^\Gamma}{|x|^{\beta_-}}\ dx) = \mu_\ell^2\int_{\frac{\Omega}{\mu_\ell}} U^2\ dy + O(\mu_\ell^{2\Gamma}\int_\Omega \frac{dx}{|x|^{N-2}}) \\
&= \begin{cases} \mu_\ell^2 \log\frac{1}{\mu_\ell}[\alpha_N^2\omega_{N-1} + o(1)] & \text{if } \Gamma = 1 \\ \mu_\ell^2[\int_{\mathbb{R}^N} U^2\ dy + o(1)] & \text{if } \Gamma > 1 \end{cases}
\end{aligned}
\tag{5.21}
$$

in view of $\frac{2N\beta_-}{N-2} < N < \beta_- + \frac{N+2}{N-2}\beta_+$ and $2\beta_\pm = N - 2 \pm 2\Gamma$. Since

$$
|a+b|^{\frac{2N}{N-2}} - |a|^{\frac{2N}{N-2}} - \frac{2N}{N-2}|a|^{\frac{4}{N-2}}ab = O(|a|^{\frac{4}{N-2}}b^2 + |b|^{\frac{2N}{N-2}})
\tag{5.22}
$$

for all $a, b \in \mathbb{R}$, by Lemma 3.1 we finally deduce

$$
\begin{aligned}
&\int_\Omega [(PU_\ell)^{\frac{2N}{N-2}} - U_\ell^{\frac{2N}{N-2}} - \frac{2N}{N-2}U_\ell^{\frac{N+2}{N-2}}(PU_\ell - U_\ell)]dx \\
&= O\left(\int_\Omega |PU_\ell - U_\ell|^{\frac{2N}{N-2}}dx + \int_\Omega U_\ell^{\frac{4}{N-2}}(PU_\ell - U_\ell)^2 dx\right) = O(\mu_\ell^{\frac{2N}{N-2}\Gamma} + \mu_\ell^{2\Gamma}\int_\Omega \frac{U_\ell^{\frac{4}{N-2}}}{|x|^{2\beta_-}}) \\
&= \begin{cases} O(\mu_\ell^{\frac{2N}{N-2}\Gamma} + \mu_\ell^{4\Gamma}\int_{B_{\frac{R}{\mu_\ell}}(0)} \frac{U^{\frac{4}{N-2}}}{|y|^{2\beta_-}}\ dy) & \text{if } 3 \le N \le 4 \\ O(\mu_\ell^{\frac{2N}{N-2}\Gamma} + \mu_\ell^{\frac{2N}{N-2}\Gamma}\int_\Omega \frac{dx}{|x|^{\frac{4\beta_+}{N-2}+2\beta_-}}) & \text{if } N \ge 5 \end{cases} = o(\mu_\ell^{2\Gamma})
\end{aligned}
\tag{5.23}
$$

in view of $\frac{2\beta_-}{N-2} < 1$ and $\frac{4\beta_+}{N-2} + 2\beta_- = N - 2\frac{N-4}{N-2}\Gamma$. Inserting (5.19)-(5.21) and (5.23) into (5.18) we get the validity of (5.16) for $a_1 = J_\varepsilon(PU_1)$.

Hereafter let us consider the case $k \ge 2$ with $\Gamma > 2$. As for $\ell = 1$ in (5.16), the following expansion does hold

$$
J_\varepsilon(PU_\ell) = A_1 + A_2 m\mu_\ell^{2\Gamma}(1+o(1)) - A_3\varepsilon\mu_\ell^2(1+o(1)), \quad \ell = 1,\dots,k.
\tag{5.24}
$$

Let $\ell \ge 2$. Since

$$
U_\ell^{\frac{4}{N-2}} = O\left((\frac{\mu_\ell^\Gamma}{|x|^{\beta_+}})^{\frac{5}{2(N-2)}}(\frac{1}{\mu_\ell^\Gamma|x|^{\beta_-}})^{\frac{3}{2(N-2)}}\right) = O(\frac{\mu_\ell^{\frac{\Gamma}{N-2}}}{|x|^{\frac{5}{2(N-2)}\beta_+ + \frac{3}{2(N-2)}\beta_-}}),
$$

by Lemma 3.1 and (5.8) we have that for any $i = 1, \dots, \ell - 1$

$$\begin{aligned}
&\int_\Omega [U_\ell^{\frac{N+2}{N-2}} - \varepsilon PU_\ell - (PU_\ell)^{\frac{N+2}{N-2}}] PU_i \, dx \\
&= O\left(\mu_\ell^{\frac{N+2}{N-2}\Gamma} \int_\Omega \frac{U_i}{|x|^{\frac{N+2}{N-2}\beta_-}} \, dx + \mu_\ell^\Gamma \int_\Omega \frac{U_\ell^{\frac{4}{N-2}} U_i}{|x|^{\beta_-}} \, dx + \varepsilon \int_\Omega U_i U_\ell \, dx \right) \\
&= O\left(\int_\Omega \frac{\mu_\ell^{\frac{N+2}{N-2}\Gamma} \mu_i^\Gamma}{|x|^{\frac{N+2}{N-2}\beta_- + \beta_+}} \, dx + (\frac{\mu_\ell}{\mu_i})^\Gamma \mu_\ell^{\frac{\Gamma}{N-2}} \int_\Omega \frac{dx}{|x|^{\frac{4N-5}{2(N-2)}\beta_- + \frac{5}{2(N-2)}\beta_+}} + \varepsilon (\frac{\mu_\ell}{\mu_i})^\Gamma \int_\Omega \frac{dx}{|x|^{\beta_- + \beta_+}} \right)
\end{aligned}$$

and then

$$\int_\Omega [U_\ell^{\frac{N+2}{N-2}} - \varepsilon PU_\ell - (PU_\ell)^{\frac{N+2}{N-2}}] PU_i \, dx = o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \right) \tag{5.25}$$

in view of (3.10)-(3.11) and

$$\frac{N+2}{N-2}\beta_- + \beta_+ < N, \quad \frac{4N-5}{2(N-2)}\beta_- + \frac{5}{2(N-2)}\beta_+ < N. \tag{5.26}$$

In order to expand the last term in a_ℓ, $\ell = 2, \dots, k$, let us split Ω as $\Omega = \bigcup_{h=0}^{\ell} \mathcal{A}_h$ (see (5.7)), and for $h = 0, \dots, \ell$ set

$$I_h = \int_{\mathcal{A}_h} [|\sum_{i=1}^{\ell} (-1)^i PU_i|^{\frac{2N}{N-2}} - |\sum_{i=1}^{\ell-1} (-1)^i PU_i|^{\frac{2N}{N-2}} - (PU_\ell)^{\frac{2N}{N-2}} - \frac{2N}{N-2} \sum_{i=1}^{\ell-1} (-1)^{i+\ell} (PU_\ell)^{\frac{N+2}{N-2}} PU_i] \, dx.$$

By (5.10), (5.13)-(5.14) and (5.22) we deduce that

$$\begin{aligned}
I_\ell &= O\left(\sum_{i=1}^{\ell-1} \int_{\mathcal{A}_\ell} [(PU_i)^{\frac{2N}{N-2}} + (PU_i)^2 PU_\ell^{\frac{4}{N-2}}] \, dx \right) \\
&= O\left(\sum_{i=1}^{\ell-1} |PU_i|_{\frac{2N}{N-2}, \mathcal{A}_\ell}^{\frac{2N}{N-2}} + \sum_{i=1}^{\ell-1} |PU_i|_{\frac{2N}{N-2}, \mathcal{A}_\ell} |PU_\ell^{\frac{4}{N-2}} PU_i|_{\frac{2N}{N+2}, \mathcal{A}_\ell} \right) = o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \right).
\end{aligned} \tag{5.27}$$

For $h = 0, \dots, \ell - 1$ by (5.8) and (5.22) we get that

$$\begin{aligned}
I_h &= \frac{2N}{N-2} \int_{\mathcal{A}_h} [|\sum_{i=1}^{\ell-1} (-1)^i PU_i|^{\frac{4}{N-2}} \sum_{i=1}^{\ell-1} (-1)^i PU_i] (-1)^\ell PU_\ell \, dx \\
&+ O(\int_{\mathcal{A}_h} \sum_{i=1}^{\ell-1} [(PU_i)^{\frac{4}{N-2}} (PU_l)^2 + (PU_l)^{\frac{N+2}{N-2}} PU_i] \, dx + \int_{\mathcal{A}_h} (PU_l)^{\frac{2N}{N-2}} \, dx) \\
&= -\frac{2N}{N-2} \int_{\mathcal{A}_h} (PU_{\ell-1})^{\frac{N+2}{N-2}} PU_\ell \, dx + O\left(\int_{\mathcal{A}_h} \sum_{i=1}^{\ell-2} [(PU_{\ell-1})^{\frac{4}{N-2}} PU_i PU_\ell + (PU_i)^{\frac{N+2}{N-2}} PU_\ell] \, dx \right) \\
&+ O(\int_{\mathcal{A}_h} \sum_{i=1}^{\ell-1} [(PU_i)^{\frac{4}{N-2}} (PU_l)^2 + (PU_l)^{\frac{N+2}{N-2}} PU_i] \, dx + \int_{\mathcal{A}_h} (PU_l)^{\frac{2N}{N-2}} \, dx).
\end{aligned}$$

Since $\beta_- + \frac{N+2}{N-2}\beta_+ = N + \frac{4\Gamma}{N-2} > N$, by Lemma 3.1 and (5.26) we deduce that

$$\int_{\mathcal{A}_h} (PU_{\ell-1})^{\frac{4}{N-2}} PU_i PU_\ell = O((\frac{\mu_\ell}{\mu_i})^\Gamma \int_{\mathcal{A}_h} \frac{U_{\ell-1}^{\frac{4}{N-2}}}{|x|^{N-2}}\, dx) = O((\frac{\mu_\ell}{\mu_i})^\Gamma \int_{\frac{\mathcal{A}_h}{\mu_{\ell-1}}} \frac{U^{\frac{4}{N-2}}}{|y|^{N-2}}\, dy) = O((\frac{\mu_\ell}{\mu_i})^\Gamma)$$

$$\int_{\mathcal{A}_h} (PU_i)^{\frac{N+2}{N-2}} PU_\ell\, dx = O(\mu_\ell^\Gamma \int_{\mathcal{A}_h} \frac{U_i^{\frac{N+2}{N-2}}}{|x|^{\beta_+}}\, dx) = O((\frac{\mu_\ell}{\mu_i})^\Gamma \int_{\frac{\mathcal{A}_h}{\mu_i}} \frac{U^{\frac{N+2}{N-2}}}{|y|^{\beta_+}}\, dy) = O((\frac{\mu_\ell}{\mu_i})^\Gamma)$$

$$\int_{\mathcal{A}_h} (PU_\ell)^{\frac{N+2}{N-2}} PU_i\, dx = O(\frac{1}{\mu_i^\Gamma} \int_{\mathcal{A}_h} \frac{U_\ell^{\frac{N+2}{N-2}}}{|x|^{\beta_-}}\, dx) = O((\frac{\mu_\ell}{\mu_i})^\Gamma \int_{\frac{\mathcal{A}_h}{\mu_\ell}} \frac{U^{\frac{N+2}{N-2}}}{|y|^{\beta_-}}\, dy) = O((\frac{\mu_\ell}{\mu_i})^\Gamma (\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{2\Gamma}{N-2}})$$

for any $i = 1, \dots, \ell-1$ and $h = 0, \dots, \ell-1$, which inserted into the previous expression for I_h give that

$$\begin{aligned} I_h &= -\frac{2N}{N-2} \int_{\mathcal{A}_h} (PU_{\ell-1})^{\frac{N+2}{N-2}} PU_\ell\, dx + O(\int_{\mathcal{A}_h} \sum_{i=1}^{\ell-1} (PU_i)^{\frac{4}{N-2}} (PU_l)^2\, dx + \int_{\mathcal{A}_h} (PU_l)^{\frac{2N}{N-2}}\, dx) \\ &+ o((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma) = -\frac{2N}{N-2} \int_{\mathcal{A}_h} (PU_{\ell-1})^{\frac{N+2}{N-2}} PU_\ell\, dx \\ &+ O(\int_{\mathcal{A}_h} \sum_{i=1}^{\ell-1} |(PU_i)^{\frac{4}{N-2}} PU_l|_{\frac{2N}{N+2}, \mathcal{A}_h} |PU_l|_{\frac{2N}{N-2}, \mathcal{A}_h} + |PU_l|^{\frac{2N}{N-2}}_{\frac{2N}{N-2}, \mathcal{A}_h}) + o((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma) \\ &= -\frac{2N}{N-2} \int_{\mathcal{A}_h} (PU_{\ell-1})^{\frac{N+2}{N-2}} PU_\ell\, dx + o((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma) \end{aligned} \tag{5.28}$$

for $h = 0, \dots, \ell-1$ in view of (5.10)-(5.12). By (5.8) and Lemma 3.1 we have that

$$\int_{\mathcal{A}_h} (PU_{\ell-1})^{\frac{N+2}{N-2}} PU_\ell\, dx = O((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \int_{\frac{\mathcal{A}_h}{\mu_{\ell-1}}} \frac{U^{\frac{N+2}{N-2}}}{|y|^{\beta_+}}\, dy) = o((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma) \tag{5.29}$$

for $h = 0, \dots, \ell-2$ and

$$\begin{aligned} &\int_{\mathcal{A}_{\ell-1}} (PU_{\ell-1})^{\frac{N+2}{N-2}} PU_\ell\, dx \\ &= \int_{\mathcal{A}_{\ell-1}} U_{\ell-1}^{\frac{N+2}{N-2}} U_\ell\, dx + O(\int_{\mathcal{A}_{\ell-1}} [\mu_{\ell-1}^\Gamma \frac{(U_{\ell-1})^{\frac{4}{N-2}} U_\ell}{|x|^{\beta_-}} + \mu_{\ell-1}^{\frac{N+2}{N-2}\Gamma} \frac{U_\ell}{|x|^{\frac{N+2}{N-2}\beta_-}} + \mu_\ell^\Gamma \frac{U_{\ell-1}^{\frac{N+2}{N-2}}}{|x|^{\beta_-}}]\, dx) \\ &= (\frac{\mu_{\ell-1}}{\mu_\ell})^{\frac{N-2}{2}} \int_{\frac{\mathcal{A}_{\ell-1}}{\mu_{\ell-1}}} U^{\frac{N+2}{N-2}} U(\frac{\mu_{\ell-1}}{\mu_\ell} y)\, dy + O\left(\mu_{\ell-1}^\Gamma \int_{\mathcal{A}_{\ell-1}} \frac{(U_{\ell-1})^{\frac{4}{N-2}} U_\ell}{|x|^{\beta_-}}\, dx\right) \\ &+ O\left(\mu_{\ell-1}^{\frac{N+2}{N-2}\Gamma} \mu_\ell^\Gamma \int_{\mathcal{A}_{\ell-1}} \frac{dx}{|x|^{\frac{N+2}{N-2}\beta_- + \beta_+}}\right) + O(\mu_\ell^\Gamma)(\int_{\mathcal{A}_{\ell-1}} \frac{dx}{|x|^{\frac{2N\beta_-}{N-2}}})^{\frac{N-2}{2N}} \\ &= \alpha_N (\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \int_{\mathbb{R}^N} \frac{U^{\frac{N+2}{N-2}}}{|y|^{\beta_+}}\, dy + o((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma) \end{aligned} \tag{5.30}$$

in view of $\frac{\mu_{\ell-1}}{\mu_\ell} y \geq \sqrt{\frac{\mu_{\ell-1}}{\mu_\ell}} \to +\infty$ for all $y \in \frac{\mathcal{A}_{\ell-1}}{\mu_{\ell-1}}$, (5.26) and

$$\int_{\mathcal{A}_{\ell-1}} \frac{U_{\ell-1}^{\frac{4}{N-2}} U_\ell}{|x|^{\beta_-}} = \begin{cases} O\left(|U_{\ell-1}^{\frac{4}{N-2}} U_\ell|_{\frac{2N}{N+2}, \mathcal{A}_{\ell-1}}\right) = O((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma) & \text{if } 3 \leq N \leq 5 \\ O\left(\frac{\mu_\ell^\Gamma}{\mu_{\ell-1}^{\frac{4\Gamma}{N-2}}} \int_{\mathcal{A}_{\ell-1}} \frac{dx}{|x|^{\frac{N+2}{N-2}\beta_- + \beta_+}}\right) = O((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma) & \text{if } N \geq 6 \end{cases} \tag{5.31}$$

thanks to (5.11). Therefore, inserting (5.29)-(5.30) into (5.28) we have the following expansion:

$$I_h = o((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma), \qquad I_{\ell-1} = -\frac{2N}{N-2}A_4(\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma + o((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma). \tag{5.32}$$

Summing up (5.27) and (5.32) we get that the third term in a_ℓ, $\ell = 2, \dots, k$, takes the form

$$-\frac{N-2}{2N}\sum_{h=0}^{\ell} I_h = A_4(\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma + o((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma),$$

which, along with (5.24)-(5.25), finally establishes the validity of (5.17) for a_ℓ, $\ell = 2, \dots, k$.

5.3. **The remainder term: proof of Proposition 4.3.** We assume that either $\ell = 1$ or $\ell \geq 2$ and C^1-maps $(d_1, \dots, d_k) \in A \to \phi_{j,\varepsilon}(d_1, \dots, d_j) \in \mathcal{K}_j^\perp$ have already been constructed for $j = 1, \dots, \ell-1$ satisfying the properties of Proposition 4.3. By Lemma 4.2 we can rewrite the equation $\mathcal{E}_\ell + \mathcal{L}_\ell(\phi_\ell) + \mathcal{N}_\ell(\phi_{1,\varepsilon}, \dots, \phi_{\ell-1,\varepsilon}, \phi_\ell) = 0$ as

$$\phi_\ell = -\mathcal{L}_\ell^{-1}\left(\mathcal{E}_\ell + \mathcal{N}_\ell(\phi_{1,\varepsilon}, \dots, \phi_{\ell-1,\varepsilon}, \phi_\ell)\right) = \mathcal{T}_\ell(\phi_\ell).$$

Given $R > 0$ large, we show below that $\mathcal{T}_\ell : \mathcal{B}_\ell \to \mathcal{B}_\ell = \{\phi \in \mathcal{K}_\ell^\perp : \ \|\phi\| \leq RR_\ell\}$ is a contraction mapping for ε small, where

$$R_\ell = \begin{cases} \|\mathcal{E}_1\| & \text{if } \ell = 1 \\ (\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma + (\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{\Gamma}{2}+1} + (\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{N+2}{2(N-2)}\Gamma}\log^{\frac{2}{3}}\frac{1}{\mu_\ell} & \text{if } \ell = 2, \dots, k. \end{cases} \tag{5.33}$$

Hence, for $\epsilon > 0$ small it follows the existence of a unique fixed point $\phi_{\ell,\varepsilon}(d_1, \dots, d_\ell) \in \mathcal{B}_\ell$ for any $(d_1, \dots, d_k) \in A$. By the Implicit Function Theorem it is possible to show that $(d_1, \dots, d_k) \in A \to \phi_{\ell,\varepsilon}(d_1, \dots, d_\ell)$ is a C^1-map satisfying also (4.7). Since the proof can be made similarly as in [26] we omit it. The validity of (4.8) will be addressed at the end of this section.

Set $\mathcal{N}_\ell(\phi) = \mathcal{N}_\ell(\phi_{1,\varepsilon}, \dots, \phi_{\ell-1,\varepsilon}, \phi)$. Since by Lemma 4.2

$$\|\mathcal{T}_\ell(\phi)\| \leq c\left(\|\mathcal{E}_\ell\| + \|\mathcal{N}_\ell(\phi)\|\right), \quad \|\mathcal{T}_\ell(\phi_1) - \mathcal{T}_\ell(\phi_2)\| \leq c\|\mathcal{N}_\ell(\phi_1) - \mathcal{N}_\ell(\phi_2)\|,$$

by Lemma 4.1 and (3.10)-(3.13) it is enough to show that

$$\|\mathcal{N}_\ell(\phi)\| = O(R_\ell) + o(\|\phi\|), \quad \|\mathcal{N}_\ell(\phi_1) - \mathcal{N}_\ell(\phi_2)\| = o(1)\|\phi_1 - \phi_2\| \tag{5.34}$$

uniformly for any $\phi, \phi_1, \phi_2 \in \mathcal{B}_\ell$. Let $f(u) = |u|^{\frac{4}{N-2}}u$ and set

$$\begin{aligned} N_\ell \quad = \quad & f\left(\sum_{j=1}^{\ell}(-1)^j PU_j + \sum_{j=1}^{\ell-1}\phi_{j,\varepsilon} + \phi\right) - f\left(\sum_{j=1}^{\ell}(-1)^j PU_j\right) - f'\left(\sum_{j=1}^{\ell}(-1)^j PU_j\right)\phi \\ & -f\left(\sum_{j=1}^{\ell-1}[(-1)^j PU_j + \phi_{j,\varepsilon}]\right) + f\left(\sum_{j=1}^{\ell-1}(-1)^j PU_j\right). \end{aligned}$$

First, by (5.8) for $\ell = 1$ we have that

$$\begin{aligned} \|\mathcal{N}_1(\phi)\| \quad & \leq \quad c|N_1|_{\frac{2N}{N+2}} = c|f(-PU_1 + \phi) - f(-PU_1) - f'(-PU_1)\phi|_{\frac{2N}{N+2}} \\ & \leq \quad c(|\phi|_{\frac{2N}{N-2}}^{\frac{N+2}{N-2}} + \underbrace{|U_1^{\frac{6-N}{N-2}}\phi^2|_{\frac{2N}{N+2}}}_{\text{if } 3 \leq N \leq 5}) = o(\|\phi\|) \end{aligned}$$

and then the first in (5.34) is established. For $\ell \geq 2$, by (5.8) we have the expansion

$$
\begin{aligned}
N_\ell = & f\left(\sum_{j=1}^{\ell}(-1)^j PU_j + \sum_{j=1}^{\ell-1}\phi_{j,\varepsilon}\right) - f\left(\sum_{j=1}^{\ell}(-1)^j PU_j\right) - f\left(\sum_{j=1}^{\ell-1}[(-1)^j PU_j + \phi_{j,\varepsilon}]\right) \\
& + f\left(\sum_{j=1}^{\ell-1}(-1)^j PU_j\right) + \left[f'\left(\sum_{j=1}^{\ell}(-1)^j PU_j + \sum_{j=1}^{\ell-1}\phi_{j,\varepsilon}\right) - f'\left(\sum_{j=1}^{\ell}(-1)^j PU_j\right)\right]\phi \\
& + O(|\phi|^{\frac{N+2}{N-2}}) + \underbrace{O(\sum_{j=1}^{\ell}(PU_j)^{\frac{6-N}{N-2}}\phi^2 + \sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{6-N}{N-2}}\phi^2)}_{\text{if } 3\leq N\leq 5}
\end{aligned}
\tag{5.35}
$$

Letting $\mathcal{A}_h$ be as in (5.7), we have that

$$
\|\mathcal{N}_\ell(\phi)\| \leq c\sum_{h=0}^{\ell}|N_\ell|_{\frac{2N}{N+2},\mathcal{A}_h}. \tag{5.36}
$$

By (5.8) and

$$
|a+b|^{\frac{4}{N-2}} - |a|^{\frac{4}{N-2}} = O(|b|^{\frac{4}{N-2}} + \underbrace{|a|^{\frac{6-N}{N-2}}|b|}_{\text{if } 3\leq N\leq 5}),
$$

for $h = 0, \dots, \ell-1$ we have

$$
\begin{aligned}
|N_\ell|_{\frac{2N}{N+2},\mathcal{A}_h} \leq & c\Big|U_l^{\frac{N+2}{N-2}} + U_l\sum_{j=1}^{\ell-1}[U_j^{\frac{4}{N-2}} + |\phi_{j,\varepsilon}|^{\frac{4}{N-2}}] + |\phi|\sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{4}{N-2}} + |\phi|^{\frac{N+2}{N-2}}\Big|_{\frac{2N}{N+2},\mathcal{A}_h} \\
& \underbrace{+ c\Big|\sum_{j=1}^{\ell}U_j^{\frac{6-N}{N-2}}\phi^2 + \sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{6-N}{N-2}}\phi^2 + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell-1}U_i^{\frac{6-N}{N-2}}|\phi||\phi_{j,\varepsilon}|\Big|_{\frac{2N}{N+2},\mathcal{A}_h}}_{\text{if } 3\leq N\leq 5} \\
= & O\left(R_\ell + \sum_{j=1}^{\ell-1}||\phi_{j,\varepsilon}|^{\frac{4}{N-2}}U_l|_{\frac{2N}{N+2},\mathcal{A}_h}\right) + o(\|\phi\|)
\end{aligned}
\tag{5.37}
$$

and

$$
\begin{aligned}
|N_\ell|_{\frac{2N}{N+2},\mathcal{A}_\ell} \leq & c\Big|U_l^{\frac{4}{N-2}}\sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}| + \sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{N+2}{N-2}} + |\phi|^{\frac{N+2}{N-2}}\Big|_{\frac{2N}{N+2},\mathcal{A}_\ell} \\
& \underbrace{+ c\Big|U_l\sum_{i,j=1}^{\ell-1}U_i^{\frac{6-N}{N-2}}|\phi_{j,\varepsilon}| + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell-1}U_i^{\frac{6-N}{N-2}}\phi_{j,\varepsilon}^2 + \sum_{j=1}^{\ell}U_j^{\frac{6-N}{N-2}}\phi^2 + \sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{6-N}{N-2}}\phi^2\Big|_{\frac{2N}{N+2},\mathcal{A}_\ell}}_{\text{if } 3\leq N\leq 5} \\
= & O\left(\sum_{j=1}^{\ell-1}|U_\ell^{\frac{4}{N-2}}\phi_{j,\varepsilon}|_{\frac{2N}{N+2},\mathcal{A}_\ell} + \sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{N+2}{N-2}}_{\frac{2N}{N-2},\mathcal{A}_\ell} + \sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^2_{\frac{2N}{N-2},\mathcal{A}_\ell}\right) \\
& \underbrace{+ O\left(\sum_{i,j=1}^{\ell-1}|U_\ell^{\frac{4}{N-2}}\phi_{j,\varepsilon}|^{\frac{N-2}{4}}_{\frac{2N}{N+2},\mathcal{A}_\ell}|U_i|^{\frac{6-N}{N-2}}_{\frac{2N}{N-2},\mathcal{A}_\ell}|\phi_{j,\varepsilon}|^{\frac{6-N}{4}}_{\frac{2N}{N-2},\mathcal{A}_\ell}\right)}_{\text{if } 3\leq N\leq 5} + o(\|\phi\|)
\end{aligned}
\tag{5.38}
$$

for any $\phi \in \mathcal{B}_\ell$ in view of (5.10)-(5.12) and Hölder inequality, where R_ℓ is given in (5.33). Notice in the estimate (5.37) we couple the first/second term in the expression (5.35) of N_ℓ with the third/fourth one, while in the estimate (5.38) the first two and the second two terms in (5.35) are coupled.

For $j=1,\dots,\ell-1$ there holds $\mathcal{A}_\ell \subset B_{\rho\mu_j}(0)$ and by (4.8) we deduce that

$$|\phi_{j,\varepsilon}|_{\frac{2N}{N-2},\mathcal{A}_\ell} \le \frac{c}{\mu_j^\Gamma}|\frac{1}{|x|^{\beta_-}}|_{\frac{2N}{N-2},\mathcal{A}_\ell} \le c(\frac{\sqrt{\mu_{\ell-1}\mu_\ell}}{\mu_j})^\Gamma = O\left((\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{\Gamma}{2}}\right) \tag{5.39}$$

and

$$\begin{aligned} |U_\ell^{\frac{4}{N-2}}\phi_{j,\varepsilon}|_{\frac{2N}{N+2},\mathcal{A}_\ell} &\le \frac{c}{\mu_j^\Gamma}|\frac{U_\ell^{\frac{4}{N-2}}}{|x|^{\beta_-}}|_{\frac{2N}{N+2},\mathcal{A}_\ell} \\ &\le \begin{cases} c(\frac{\mu_\ell}{\mu_j})^\Gamma|\frac{U^{\frac{4}{N-2}}}{|y|^{\beta_-}}|_{\frac{2N}{N+2},\frac{\mathcal{A}_\ell}{\mu_\ell}} = O\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma\right) & \text{if } 3\le N\le 5 \\ c(\frac{\mu_\ell}{\mu_j})^\Gamma|\frac{U^{\frac{4}{N-2}}}{|y|^{\beta_-}}|_{\frac{2N}{N+2},\frac{\mathcal{A}_\ell}{\mu_\ell}} = O\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \log^{\frac{2}{3}}\frac{1}{\mu_\ell}\right) & \text{if } N=6 \\ c\frac{\mu_\ell^{\frac{4\Gamma}{N-2}}}{\mu_j^\Gamma}|\frac{1}{|x|^{\beta_-+\frac{4}{N-2}\beta_+}}|_{\frac{2N}{N+2},\mathcal{A}_\ell} = O\left((\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{N+2}{N-2}\frac{\Gamma}{2}}\right) & \text{if } N\ge 7. \end{cases} \end{aligned} \tag{5.40}$$

For $h=0,\dots,\ell-2$ by (5.10) we deduce

$$||\phi_{j,\varepsilon}|^{\frac{4}{N-2}}PU_\ell|_{\frac{2N}{N+2},\mathcal{A}_h} \le c\|\phi_{j,\varepsilon}\|^{\frac{4}{N-2}}|PU_\ell|_{\frac{2N}{N-2},\mathcal{A}_h} \le c\left(\frac{\mu_\ell}{\sqrt{\mu_h\mu_{h+1}}}\right)^\Gamma = O\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma\right) \tag{5.41}$$

for any $j=1,\dots,\ell-1$. Splitting $\mathcal{A}_{\ell-1}$ as $\mathcal{A}'_{\ell-1}\cup\mathcal{A}''_{\ell-1}$, where $\mathcal{A}'_{\ell-1}=\mathcal{A}_{\ell-1}\cap B_{\rho\mu_{\ell-1}}(0)$ and $\mathcal{A}''_{\ell-1}=\mathcal{A}_{\ell-1}\setminus B_{\rho\mu_{\ell-1}}(0)$, by (4.8) and (5.10) we get that

$$\begin{aligned} ||\phi_{j,\varepsilon}|^{\frac{4}{N-2}}PU_\ell|_{\frac{2N}{N+2},\mathcal{A}_{\ell-1}} &\le \frac{c}{\mu_j^{\frac{4\Gamma}{N-2}}}|\frac{PU_\ell}{|x|^{\frac{4\beta_-}{N-2}}}|_{\frac{2N}{N+2},\mathcal{A}'_{\ell-1}} + c\|\phi_{j,\varepsilon}\|^{\frac{4}{N-2}}|PU_\ell|_{\frac{2N}{N-2},\mathcal{A}''_{\ell-1}} \\ &\le \frac{c}{\mu_j^{\frac{4\Gamma}{N-2}}}|PU_\ell|_{\frac{2N}{N+2},\mathcal{A}'_{\ell-1}}\begin{cases} \mu_{\ell-1}^{\frac{4\Gamma}{N-2}-2} & \text{if } \Gamma\ge\frac{N-2}{2} \\ (\mu_{\ell-1}\mu_\ell)^{\frac{2\Gamma}{N-2}-1} & \text{if } \Gamma<\frac{N-2}{2}\end{cases} + c(\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \\ &= O\left((\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{N+2}{N-2}\frac{\Gamma}{2}} + (\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{\Gamma}{2}+1}\right) \end{aligned} \tag{5.42}$$

in view of

$$\begin{aligned} |PU_\ell|_{\frac{2N}{N+2},\mathcal{A}'_{\ell-1}} &\le \mu_\ell^2|U|_{\frac{2N}{N+2},\mathbb{R}^N\setminus B_{\sqrt{\frac{\mu_{\ell-1}}{\mu_\ell}}}(0)} = O\left(\mu_\ell^2(\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{\Gamma}{2}-1}\right) \\ |PU_\ell|_{\frac{2N}{N-2},\mathcal{A}''_{\ell-1}} &\le |U|_{\frac{2N}{N-2},\mathbb{R}^N\setminus B_{\frac{\mu_{\ell-1}}{\mu_\ell}}(0)} = O\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma\right). \end{aligned} \tag{5.43}$$

Estimates (5.10), (5.39)-(5.40) into (5.38) and (5.41)-(5.42) into (5.37) lead to $|N_\ell|_{\frac{2N}{N+2},\mathcal{A}_h} = O(R_\ell)+o(\|\phi\|)$ for any $h=0,\dots,\ell$, which, inserted into (5.36), finally give the validity of the first in (5.34) for $\ell\ge 2$.

Concerning the second one in (5.34), we have that

$$\begin{aligned} \|\mathcal{N}_\ell(\phi_1)-\mathcal{N}_\ell(\phi_2)\| &\le c\left|[\sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{4}{N-2}}+|\phi_1|^{\frac{4}{N-2}}+|\phi_2|^{\frac{4}{N-2}}](\phi_1-\phi_2)\right|_{\frac{2N}{N+2}} \\ &\quad \underbrace{+\,c\left|\sum_{i=1}^{\ell}U_i^{\frac{6-N}{N-2}}[\sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|+|\phi_1|+|\phi_2|](\phi_1-\phi_2)\right|_{\frac{2N}{N+2}}}_{\text{if } 3\le N\le 5} \end{aligned}$$

in view of

$$|a+b+c_1|^{\frac{4}{N-2}}(a+b+c_1)-|a+b+c_2|^{\frac{4}{N-2}}(a+b+c_2)-\frac{N+2}{N-2}|a|^{\frac{4}{N-2}}(c_1-c_2)$$
$$=|c_1-c_2|O\left(|b|^{\frac{4}{N-2}}+|c_1|^{\frac{4}{N-2}}+|c_2|^{\frac{4}{N-2}}+\underbrace{|a|^{\frac{6-N}{N-2}}(|b|+|c_1|+|c_2|)}_{\text{if } 3\le N\le 5}\right)$$

for all $a,b,c_1,c_2\in\mathbb{R}$. Therefore there holds

$$\begin{aligned}\|\mathcal{N}_\ell(\phi_1)-\mathcal{N}_\ell(\phi_2)\|&\le c\left[\sum_{j=1}^{\ell-1}\|\phi_{j,\varepsilon}\|^{\frac{4}{N-2}}+\|\phi_1\|^{\frac{4}{N-2}}+\|\phi_2\|^{\frac{4}{N-2}}\right]\|\phi_1-\phi_2\|\\&\quad\underbrace{+c\sum_{i=1}^{\ell}|U_i|^{\frac{6-N}{N-2}}_{\frac{2N}{N-2}}\left[\sum_{j=1}^{\ell-1}\|\phi_{j,\varepsilon}\|+\|\phi_1\|+\|\phi_2\|\right]\|\phi_1-\phi_2\|}_{\text{if } 3\le N\le 5}\\&=o(1)\|\phi_1-\phi_2\|.\end{aligned}\tag{5.44}$$

in view of $\phi_1,\phi_2\in\mathcal{B}_\ell$. The validity of (5.34) has been fully established.

To prove the validity of (4.8), assume that either $\ell=1$ or $\ell\ge 2$ and C^1-maps $(d_1,\dots,d_k)\in A\to\phi_{j,\varepsilon}(d_1,\dots,d_j)\in\mathcal{K}_j^\perp$ have already been constructed for $j=1,\dots,\ell-1$ satisfying the properties of Proposition 4.3. Setting $u_j=(-1)^jPU_j+\phi_{j,\varepsilon}$, $j=1,\dots,\ell$, we have that u_j satisfies

$$u_j=\imath^*\left[f(\sum_{i=1}^{j}u_i)-f(\sum_{i=1}^{j-1}u_i)+\varepsilon u_j\right]+\Psi_j,\quad \Psi_j\in\mathcal{K}_j,\tag{5.45}$$

for any $j=1,\dots,\ell$, and then

$$\sum_{j=1}^{\ell}u_j=\imath^*\left[f(\sum_{i=1}^{\ell}u_i)+\varepsilon\sum_{j=1}^{\ell}u_j\right]+\sum_{j=1}^{\ell}\lambda_{j,\varepsilon}PZ_j\tag{5.46}$$

in view of $\sum_{j=1}^{\ell}\Psi_j\in\mathcal{K}_\ell$. We claim that $\lambda_{j,\varepsilon}=o(1)$ as $\varepsilon\to 0$ for any $j=1,\dots,\ell$. Indeed, let us take the inner product of (5.46) against PZ_i, $i=1,\dots,\ell$, to get

$$\begin{aligned}\sum_{j=1}^{\ell}\lambda_{j,\varepsilon}\langle PZ_j,PZ_i\rangle&=\int_\Omega[\sum_{j=1}^{\ell}(-1)^jU_j^{\frac{N+2}{N-2}}-f(\sum_{j=1}^{\ell}u_j)]PZ_i\,dx+\sum_{j=1}^{i-1}\langle\phi_{j,\varepsilon},PZ_i\rangle\\&\quad-\varepsilon\sum_{j=1}^{\ell}\int_\Omega u_jPZ_i\,dx\end{aligned}\tag{5.47}$$

in view of $\phi_{j,\varepsilon}\in\mathcal{K}_i^\perp$ for any $j\ge i$ and $PU_j=\imath^*(U_j^{\frac{N+2}{N-2}})$. By Proposition 3.6 and (5.22) we have that

$$\begin{aligned}\int_\Omega(PZ_i)^{\frac{2N}{N-2}}\,dx&=\int_\Omega Z_i^{\frac{2N}{N-2}}\,dx+O\left(\mu_i^\Gamma\int_\Omega\frac{Z_i^{\frac{N+2}{N-2}}}{|x|^{\beta_-}}\,dx+\int_\Omega\frac{\mu_i^{\frac{2N}{N-2}\Gamma}}{|x|^{\frac{2N}{N-2}\beta_-}}\,dx\right)\\&=\int_{\frac{\Omega}{\mu_i}}Z^{\frac{2N}{N-2}}\,dy+O\left(\mu_i^{2\Gamma}\int_{\frac{\Omega}{\mu_i}}\frac{Z^{\frac{N+2}{N-2}}}{|y|^{\beta_-}}\,dy\right)+o(1)=\int_{\mathbb{R}^N}Z^{\frac{2N}{N-2}}\,dy+o(1)\end{aligned}\tag{5.48}$$

and

$$\begin{aligned}\langle PZ_j, PZ_i\rangle &= \frac{N+2}{N-2}\int_\Omega U_j^{\frac{4}{N-2}} Z_j PZ_i\, dx\\ &= \frac{N+2}{N-2}\int_\Omega U_j^{\frac{4}{N-2}} Z_j Z_i\, dx + O\left(\mu_i^\Gamma \int_\Omega \frac{U_j^{\frac{4}{N-2}} Z_j}{|x|^{\beta_-}}\, dx\right)\\ &= \frac{N+2}{N-2}\delta_{ij}\int_{\frac{\Omega}{\mu_j}} U^{\frac{4}{N-2}} Z^2\, dy + O\left(\mu_i^\Gamma \mu_j^\Gamma \int_{\frac{\Omega}{\mu_j}} \frac{U^{\frac{4}{N-2}} Z}{|y|^{\beta_-}}\, dy\right) + o(1)\\ &= \frac{N+2}{N-2}\delta_{ij}\int_{\mathbb{R}^N} U^{\frac{4}{N-2}} Z^2\, dy + o(1)\end{aligned} \tag{5.49}$$

in view of $PZ_j = \imath^*(\frac{N+2}{N-2} U_j^{\frac{4}{N-2}} Z_j)$, $|Z_i| \le U_i$ and

$$|\int_\Omega U_j^{\frac{4}{N-2}} Z_j Z_i\ dx| \le c\int_\Omega U_j^{\frac{N+2}{N-2}} U_i\ dx \le \begin{cases} c(\frac{\mu_j}{\mu_i})^\Gamma \int_{\mathbb{R}^N} \frac{U^{\frac{N+2}{N-2}}}{|y|^{\beta_-}}\, dy & \text{if } j > i\\ c(\frac{\mu_i}{\mu_j})^\Gamma \int_{\mathbb{R}^N} \frac{U^{\frac{N+2}{N-2}}}{|y|^{\beta_+}}\, dy & \text{if } j < i.\end{cases}$$

By inserting (5.48)-(5.49) into (5.47) we get that

$$\begin{aligned}\frac{N+2}{N-2}\left(\int_{\mathbb{R}^N} U^{\frac{4}{N-2}} Z^2\, dy\right)\lambda_{i,\varepsilon} &= \int_\Omega [\sum_{j=1}^\ell (-1)^j (PU_j)^{\frac{N+2}{N-2}} - f(\sum_{j=1}^\ell (-1)^j PU_j)] PZ_i\, dx\\ &\quad + o(\sum_{j=1}^\ell |\lambda_{j,\varepsilon}|) + o(1)\end{aligned} \tag{5.50}$$

in view of (4.6), (5.3)-(5.5), (5.8) and $\|PU_j\| = O(1)$. We have that

$$\begin{aligned}&\int_\Omega [\sum_{j=1}^\ell (-1)^j (PU_j)^{\frac{N+2}{N-2}} - f(\sum_{j=1}^\ell (-1)^j PU_j)] PZ_i\, dx\\ &= -\sum_{j=1}^\ell \int_\Omega \left[f\left(\sum_{i=1}^j (-1)^i PU_i\right) - f\left(\sum_{i=1}^{j-1} (-1)^i PU_i\right) - (-1)^j (PU_j)^{\frac{N+2}{N-2}}\right] PZ_i\, dx\\ &= O\left(\sum_{h=0}^\ell \left| f\left(\sum_{i=1}^j (-1)^i PU_i\right) - f\left(\sum_{i=1}^{j-1} (-1)^i PU_i\right) - (-1)^j (PU_j)^{\frac{N+2}{N-2}}\right|_{\frac{2N}{N+2}, \mathcal{A}_h}\right)\end{aligned}$$

in view of (5.48), with $\mathcal{A}_h$ given as in (5.7). By (5.9)-(5.14) we deduce that

$$\int_\Omega [\sum_{j=1}^\ell (-1)^j (PU_j)^{\frac{N+2}{N-2}} - f(\sum_{j=1}^\ell (-1)^j PU_j)] PZ_i\, dx = o(1),$$

and then (5.50) reduces to

$$\frac{N+2}{N-2}\left(\int_{\mathbb{R}^N} U^{\frac{4}{N-2}} Z^2\, dy\right)\lambda_{i,\varepsilon} = o(\sum_{j=1}^\ell |\lambda_{j,\varepsilon}|) + o(1).$$

This in turn implies that $\sum_{j=1}^\ell |\lambda_{j,\varepsilon}| = o(1)$, and the claim is established.

The function $\mathcal{U}_\ell(y) = \mu_\ell^{\frac{N-2}{2}}(\sum_{j=1}^{\ell} u_j)(\mu_\ell y)$ solves

$$-\Delta \mathcal{U}_\ell - \frac{\gamma}{|y|^2}\mathcal{U}_\ell - \varepsilon \mu_\ell^2 \mathcal{U}_\ell - \mathcal{U}_\ell^{\frac{N+2}{N-2}} = h \quad \text{in } \frac{\Omega}{\mu_\ell} \tag{5.51}$$

in view of (5.46), where

$$h = O\left(\sum_{j=1}^{\ell} |\lambda_{j,\varepsilon}| (\frac{\mu_\ell}{\mu_j})^{\frac{N+2}{2}} \ U^{\frac{N+2}{N-2}}(\frac{\mu_\ell}{\mu_j} y)\right).$$

We have that

$$|y|^\tau |h(y)| = O(\sum_{j=1}^{\ell} |\lambda_{j,\varepsilon}| (\frac{\mu_\ell}{\mu_j})^{\frac{N+2}{N-2}\Gamma}) = O(1)$$

with $\tau = \frac{N+2}{N-2}\beta_- < \beta_- + 2$ and

$$\left(\int_{B_r(0)} |\mathcal{U}_\ell|^{\frac{2N}{N-2}} \ dy\right)^{\frac{N-2}{2N}} \leq \ell \left(\int_{B_r(0)} U^{\frac{2N}{N-2}} \ dy\right)^{\frac{N-2}{2N}} + \sum_{j=1}^{\ell} \|\phi_{j,\epsilon}\|^{\frac{2N}{N-2}} \leq \epsilon$$

in view of $B_{r\mu_\ell} \subset B_{r\mu_j}$ for any $j = 1, \dots, \ell - 1$ and (4.6), for some $r = r(\epsilon)$. We are in position to apply Proposition 5.1 below to get the existence of $\rho, K > 0$ such that

$$|y|^{\beta_-} |\mathcal{U}_\ell(x)| \leq K$$

for all $x \in B_\rho(0)$, or equivalently

$$|x|^{\beta_-} |\sum_{j=1}^{\ell} u_j(x)| \leq \frac{K}{\mu_\ell^\Gamma} \quad \text{ in } B_{\rho\mu_\ell}(0).$$

Since by assumption for any $j = 1, \dots, \ell - 1$

$$|u_j| \leq PU_j + |\phi_{j,\varepsilon}| \leq \frac{C}{\mu_j^\Gamma |x|^{\beta_-}} \leq \frac{C}{\mu_\ell^\Gamma |x|^{\beta_-}}$$

in $B_{\rho\mu_j}(0)$ with $B_{\rho\mu_\ell}(0) \subset B_{\rho\mu_j}(0)$, we deduce that $|u_\ell| \leq \frac{C}{\mu_\ell^\Gamma |x|^{\beta_-}}$ and then $|\phi_{\ell,\varepsilon}| \leq \frac{C}{\mu_\ell^\Gamma |x|^{\beta_-}}$ in $B_{\rho\mu_\ell}(0)$, and (4.8) is established.

The following result is established using the same scheme as in [20] and for convenience we reproduce it here.

Proposition 5.1. *Let $M > 0$ and $\tau < \beta_- + 2$. There exist $\varepsilon, \rho, K > 0$ so that*

$$\sup_{x \in B_\rho(0)} |x|^{\beta_-} |u(x)| \leq K \tag{5.52}$$

does hold for any solution u of

$$-\Delta u - \frac{\gamma}{|x|^2} u = au + |u|^{\frac{4}{N-2}} u + h \text{ in } B_1(0), \quad u \in H^1(B_1(0)), \tag{5.53}$$

with

$$|u|_{\frac{2N}{N-2}, B_1(0)} \leq \varepsilon \tag{5.54}$$

$$|a|_{\infty, B_1(0)} + \sup_{x \in B_1(0)} |x|^\tau |h(x)| \leq M. \tag{5.55}$$

Proof. We need some preliminary facts.

1^{st} Claim: Let $M>0$ and $q>2$ with $\frac{4(q-1)}{q^2}>\frac{4\gamma}{(N-2)^2}$. There exist $\epsilon, K>0$ so that for any $0<\rho_2<\rho_1\le 1$ there holds

$$|u|_{B_{\rho_2}(0),\frac{Nq}{N-2}}\le K\left[|u|_{B_{\rho_1}(0),q}+|h(x)|_{B_{\rho_1}(0),\frac{Nq}{N-2+2q}}\right] \tag{5.56}$$

for any solution $u\in L^q(B_{\rho_1}(0))$ of (5.53) so that (5.54), $h^{\frac{N}{N-2+2q}}\in L^q(B_{\rho_1}(0))$ and $|a|_{\infty,B_{\rho_1}(0)}\le M$ do hold.

Indeed, given $L>0$ define

$$G_L(t)=\begin{cases}|t|^{q-2}t & \text{if } |t|\le L\\ (q-1)L^{q-2}t-(q-2)L^{q-1}\ \text{sign } t & \text{if } |t|>L\end{cases}$$

and

$$H_L(t)=\begin{cases}|t|^{\frac{q}{2}} & \text{if } |t|\le L\\ \frac{q}{2}L^{\frac{q-2}{2}}|t|-\frac{q-2}{2}L^{\frac{q}{2}} & \text{if } |t|>L\end{cases}$$

in such a way that $H_L, G_L\in C^1(\mathbb{R})$ satisfy

$$G_L'(t)=\frac{4(q-1)}{q^2}[H_L'(t)]^2,\quad t\in\mathbb{R}. \tag{5.57}$$

Observe that for all $t\in\mathbb{R}$ there hold

$$0\le tG_L(t)\le H_L^2(t),\qquad |G_L(t)|\le H_L^{\frac{2(q-1)}{q}}(t). \tag{5.58}$$

Given $0<\rho_2<\rho_1\le 1$, let $\eta\in C_c^\infty(\mathbb{R}^N)$ be so that $\eta=1$ in $B_{\rho_2}(0)$ and $\eta=0$ in $\mathbb{R}^N\setminus B_{\rho_1}(0)$. Test (5.53) against $\eta^2G_L(u)$ to get

$$\begin{aligned}&\int_{B_1(0)}\langle\nabla u,\nabla(\eta^2G_L(u))\rangle dx-\int_{B_1(0)}\frac{\gamma}{|x|^2}\eta^2uG_L(u)\,dx\\ &=\lambda\int_{B_1(0)}\eta^2uG_L(u)\,dx+\int_{B_1(0)}\eta^2|u|^{\frac{4}{N-2}}uG_L(u)\,dx+\int_{B_1(0)}\eta^2h(x)G_L(u)\,dx.\end{aligned} \tag{5.59}$$

By (5.57) an integration by parts leads to

$$\begin{aligned}\int_{B_1(0)}\langle\nabla u,\nabla(\eta^2G_L(u))\rangle dx&=\frac{4(q-1)}{q^2}\int_{B_1(0)}|\nabla(\eta H_L(u))|^2\\ &+\frac{4(q-1)}{q^2}\int_{B_1(0)}\eta\Delta\eta H_L^2(u)\,dx-\int_{B_1(0)}\Delta(\eta^2)J_L(u)\,dx\end{aligned} \tag{5.60}$$

where $J_L(t)=\int_0^t G_L(\tau)\,d\tau$. Inserting (5.60) into (5.59) we get

$$\begin{aligned}&\frac{4\alpha}{(\alpha+1)^2}\int_{B_1(0)}|\nabla(\eta H_L(u))|^2\,dx-\int_{B_1(0)}\frac{\gamma}{|x|^2}\eta^2uG_L(u)\,dx\\ &\le K\int_{B_{\rho_1}(0)}[H_L^2(u)+J_L(u)]\,dx+K\int_{B_1(0)}\left\{|u|^{\frac{4}{N-2}}[\eta H_L(u)]^2+\eta^2|h(x)||G_L(u)|\right\}\,dx\end{aligned} \tag{5.61}$$

in view of (5.58), where K denotes a generic constant just depending on q, M, γ, N and ρ_1,ρ_2. By Hölder and Sobolev inequalities we have that

$$\begin{aligned}\int_{B_1(0)}|u|^{\frac{4}{N-2}}[\eta H_L(u)]^2\,dx&\le\left(\int_{B_1(0)}|u|^{\frac{2N}{N-2}}dx\right)^{\frac{2}{N}}\left(\int_{B_1(0)}|\eta H_L(u)|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}}\\ &\le K\varepsilon^{\frac{4}{N-2}}\int_{B_1(0)}|\nabla(\eta H_L(u))|^2\,dx\end{aligned} \tag{5.62}$$

in view of (5.54) and

$$\begin{aligned}&\int_{B_1(0)} \eta^2 |h(x)||G_L(u)|\,dx \le \int_{B_1(0)} |h(x)|[\eta H_L(u)]^{\frac{2(q-1)}{q}}\,dx\\&\le K\left(\int_{B_1(0)} |h(x)|^{\frac{Nq}{N-2+2q}}\,dx\right)^{\frac{N-2+2q}{Nq}}\left(\int_{B_1(0)} |\nabla(\eta H_L(u))|^2\,dx\right)^{\frac{q-1}{q}}\end{aligned} \tag{5.63}$$

in view of (5.58). Plugging (5.62)-(5.63) into (5.61) by (5.58) we get

$$\begin{aligned}&\left[\frac{4(q-1)}{q^2} - K\epsilon^{\frac{4}{N-2}}\right]\int_{B_1(0)} |\nabla(\eta H_L(u))|^2\,dx - \gamma^+\int_{B_1(0)} \frac{1}{|x|^2}[\eta H_L(u)]^2\,dx\\&\le K\int_{B_{\rho_1}(0)} [H_L^2(u)+J_L(u)]\,dx + K\left(\int_{B_1(0)} |h(x)|^{\frac{Nq}{N-2+2q}}\,dx\right)^{\frac{N-2+2q}{Nq}}\left(\int_{B_1(0)} |\nabla(\eta H_L(u))|^2\,dx\right)^{\frac{q-1}{q}}\end{aligned}$$

where $\gamma^+ = \max\{\gamma, 0\}$. By the Hardy inequality we finally deduce that

$$\begin{aligned}&\left[\frac{4(q-1)}{q^2} - K\varepsilon^{\frac{4}{N-2}} - \frac{4\gamma^+}{(N-2)^2}\right]\int_{B_1(0)} |\nabla(\eta H_L(u))|^2\,dx \le K\int_{B_{\rho_1}(0)} [H_L^2(u)+J_L(u)]\,dx\\&+K\left(\int_{B_1(0)} |h(x)|^{\frac{Nq}{N-2+2q}}\,dx\right)^{\frac{N-2+2q}{Nq}}\left(\int_{B_1(0)} |\nabla(\eta H_L(u))|^2\,dx\right)^{\frac{q-1}{q}}\end{aligned}. \tag{5.64}$$

Since $\frac{4(q-1)}{q^2} > \frac{4\gamma}{(N-2)^2}$, for ϵ small we can assume that $\frac{4(q-1)}{q^2} - K\varepsilon^{\frac{4}{N-2}} - \frac{4\gamma+}{(N-2)^2} > 0$. By (5.64) we deduce that

$$\begin{aligned}&\left(\int_{B_1(0)} |\eta H_L(u)|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}} \le K\int_{B_1(0)} |\nabla(\eta H_L(u))|^2\,dx\\&\le K\int_{B_{\rho_1}(0)} [H_L^2(u)+J_L(u)]\,dx + K\left(\int_{B_1(0)} |h(x)|^{\frac{Nq}{N-2+2q}}\,dx\right)^{\frac{N-2+2q}{N}}\end{aligned} \tag{5.65}$$

in view of $\frac{q-1}{q} < 1$ and the Sobolev inequality. Since $0 \le J_L(t) \le tG_L(t) \le H_L^2(t) \le |t|^q$ does hold for all $t \in \mathbb{R}$ in view of (5.58), by (5.65) we get that

$$\left(\int_{B_{\rho_2}(0)} |H_L(u)|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}} \le K\int_{B_{\rho_1}(0)} |u|^q\,dx + K\left(\int_{B_{\rho_1}(0)} |h(x)|^{\frac{Nq}{N-2+2q}}\,dx\right)^{\frac{N-2+2q}{N}}.$$

Taking the power $\frac{1}{q}$ and letting $L \to +\infty$ by the Fatou's Lemma we obtain the validity of (5.56).

<u>2^{nd} **Claim**</u>: Let $1 \le q < Q$, $M > 0$ and $\tau < \beta_- + 2$, where

$$Q = \begin{cases} +\infty & \text{if } \gamma \le 0\\ \frac{N}{\beta_-} & \text{if } \gamma > 0.\end{cases}$$

There exist $\epsilon, K > 0$ so that

$$|u|_{q,B_{\frac{1}{2}}(0)} \le K\left[|u|_{\frac{2N}{N-2},B_1(0)} + 1\right] \tag{5.66}$$

does hold for any solution u of (5.53) so that (5.54)-(5.55) are valid.

Indeed, notice that for $\gamma > 0$ the property $\frac{4(q-1)}{q^2} > \frac{4\gamma}{(N-2)^2}$, $q > 2$, is equivalent to $2 < q < \frac{N-2}{\beta_-} = \frac{N-2}{N}Q$. Since

$$\sup_{q\in[1,\frac{N-2}{N}Q)} \frac{Nq}{N-2+2q} = \begin{cases} \frac{N}{2} & \text{if } \gamma \le 0\\ \frac{N}{\beta_-+2} & \text{if } \gamma > 0\end{cases} < \frac{N}{\tau}$$

if $\tau < \beta_- + 2$, we have that

$$|h|_{\frac{Nq}{N-2+2q}, B_1(0)} \le K(M,\tau) \tag{5.67}$$

for any $q \in [1, \frac{N-2}{N}Q)$, $\tau < \tau_0$ and h satisfying (5.55). Let $q_j = (\frac{N-2}{N})^j q$, $j \in \mathbb{N}$, and r_j be any decreasing sequence so that $r_0 = 1$ and $r_k = \frac{1}{2}$. Since $q_j \to 0$ as $j \to +\infty$, we can find a smallest index $k \in \mathbb{N}$ so that $q_k \le \frac{2N}{N-2}$. Notice that $q_j \le q_1 < \frac{N-2}{N}Q$ for all $j \ge 1$ and $q_k > 2$ in view of $q_{k-1} > \frac{2N}{N-2}$. We can apply the 1^{st} Claim with q_j between r_{j+1} and r_j for $j = 1, \dots, k-1$ and obtain by (5.67) that for $\epsilon > 0$ small

$$|u|_{q, B_{\frac{1}{2}}(0)} \le K\left[|u|_{q_k, B_1(0)} + 1\right] \tag{5.68}$$

does hold for some $K > 0$. We can conclude in view of $q_k \le \frac{2N}{N-2}$ and

$$|u|_{q_k, B_1(0)} \le \omega_N^{\frac{2N-(N-2)q_k}{2Nq_k}} |u|_{\frac{2N}{N-2}, B_1(0)}.$$

<u>3^{rd} **Claim**</u>: Let $\frac{2N}{N-2} < q < Q$, $M > 0$ and $\tau < \beta_- + 2$. There exist $\epsilon, K > 0$ so that

$$\sup_{x \in B_{\frac{1}{4}}(0)} |x|^{\frac{N}{q}} |u(x)| \le K \tag{5.69}$$

does hold for any solution u of (5.53) so that (5.54)-(5.55) are valid.

Given $\frac{2N}{N-2} < q < Q$, $M > 0$ and $\tau < \beta_- + 2$, choose $\epsilon > 0$ small so that the 2^{nd} Claim applies. The function $U(y) = |x|^{\frac{N}{q}} u(|x|y)$ satisfies

$$-\Delta U - \frac{\gamma}{|y|^2} U = |x|^2 a(|x|y) U + |x|^{2 - \frac{4N}{q(N-2)}} |U|^{\frac{4}{N-2}} U + |x|^{\frac{N}{q}+2} h(|x|y) \quad \text{in } B_2(0) \setminus B_{\frac{1}{2}}(0),$$

where

$$\left||x|^2 a(|x|y) U + |x|^{2 - \frac{4N}{q(N-2)}} |U|^{\frac{4}{N-2}} U\right| \le \frac{M}{16}|U| + 4^{\frac{4N}{q(N-2)} - 2} |U|^{\frac{N+2}{N-2}} \tag{5.70}$$

and

$$|x|^{\frac{N}{q}+2} |h(|x|y)| \le |x|^{\frac{N}{q}+2-\tau} \frac{M}{|y|^\tau} \le 4^{2\tau - \frac{N}{q} - 2} M \tag{5.71}$$

for any $|x| \le \frac{1}{4}$ and $\frac{1}{2} \le |y| \le 2$, in view of $\frac{N}{q} + 2 - \tau > \frac{N}{Q} + 2 - \tau \ge \beta_- + 2 - \tau > 0$. Since

$$|U|_{q, B_2(0) \setminus B_1(0)} \le |u|_{q, B_{\frac{1}{2}}(0)},$$

by (5.70)-(5.71) standard elliptic estimates apply for any $\tilde{q} \ge q > \frac{2N}{N-2}$ and through a bootstrap argument yield the validity of (5.69) for some universal constant $K > 0$.

To conclude the proof, let us rewrite (5.53) as

$$-\Delta u - \frac{\gamma + \tilde{a}(x)}{|x|^2} u = h(x), \quad \tilde{a}(x) = |x|^2 a(x) + |x|^2 |u(x)|^{\frac{4}{N-2}}. \tag{5.72}$$

Since $\frac{4N}{Q(N-2)} < 2$, by 3^{rd} Claim and (5.55) it follows that there exists $C_0, \theta > 0$ such that

$$|\tilde{a}(x)| \le C_0 |x|^\theta \tag{5.73}$$

for any $|x| \le \frac{1}{4}$. Since $\tau < \beta_- + 2$, we can fix α so that $\beta_- - \theta < \alpha < \beta_-$ and $\alpha > \tau - 2$. Then we can find $\rho > 0$ small so that $\Phi(x) = |x|^{-\beta_-} - |x|^{-\alpha} \ge \frac{1}{2}|x|^{-\beta_-}$ in $B_\rho(0)$ and satisfies

$$-\Delta \Phi - \frac{\gamma + \tilde{a}}{|x|^2} \Phi = \frac{\alpha^2 - \alpha(N-2) + \gamma}{|x|^{\alpha+2}} - \frac{\tilde{a}}{|x|^2} \Phi \ge \frac{\alpha^2 - \alpha(N-2) + \gamma}{|x|^{2+\alpha}} - \frac{C_0}{|x|^{\beta_- + 2 - \theta}} \ge \frac{M}{|x|^\tau}$$

in $B_\rho(0)$ in view of $\alpha^2 - \alpha(N-2) + \gamma > 0$. Since $|u(x)| \le K\Phi(x)$ for some $K \ge 1$ and any $x \in \partial B_\rho(0)$ in view of (5.69), by (5.55) we can use $K\Phi$ as a supersolution of (5.72) with h and $-h$ to get by the maximum principle $|u(x)| \le K\Phi(x) \le K|x|^{-\beta_-}$ for any $x \in B_\rho(0)$, as desired. $\square$

5.4. **The reduced energy: end of the proof for Proposition 4.4.** Let us first show that $\widetilde{J}_\varepsilon$ has the same expansion as J_ε. Setting $u_\ell = (-1)^\ell PU_\ell + \phi_{\ell,\varepsilon}$, $\ell = 1, \ldots, k$, we have that

$$\begin{aligned}\widetilde{J}_\varepsilon(\mu_1,\ldots,\mu_k) = J_\varepsilon(\mu_1,\ldots,\mu_k) + \sum_{\ell,i=1}^{k}\left[\langle u_\ell, \phi_{i,\varepsilon}\rangle - \varepsilon\int_\Omega u_\ell \phi_{i,\varepsilon}\,dx\right] - \frac{1}{2}\|\sum_{\ell=1}^{k}\phi_{\ell,\varepsilon}\|^2 \\ + \frac{\varepsilon}{2}|\sum_{\ell=1}^{k}\phi_{\ell,\varepsilon}|_2 - \frac{N-2}{2N}\int_\Omega\left[|\sum_{\ell=1}^{k}u_\ell|^{\frac{2N}{N-2}} - |\sum_{\ell=1}^{k}(-1)^\ell PU_\ell|^{\frac{2N}{N-2}}\right]dx\end{aligned} \tag{5.74}$$

in view $\langle u+v, u+v\rangle = \langle u,u\rangle - \langle v,v\rangle + 2\langle u+v, v\rangle$ for any bi-linear form $\langle\cdot,\cdot\rangle$. By multiplying (5.45) against $\phi_{i,\varepsilon} \in \mathcal{K}_\ell^\perp$, $i \geq \ell$, we get that

$$\langle u_\ell, \phi_{i,\varepsilon}\rangle - \varepsilon\int_\Omega u_\ell\phi_{i,\varepsilon}\,dx = \int_\Omega\left[f(\sum_{j=1}^{\ell}u_j) - f(\sum_{j=1}^{\ell-1}u_j)\right]\phi_{i,\varepsilon}\,dx$$

for any $i \geq \ell$. Therefore, (5.74) reads as

$$\begin{aligned}\widetilde{J}_\varepsilon(\mu_1,\ldots,\mu_k) =& J_\varepsilon(\mu_1,\ldots,\mu_k) + \sum_{i<\ell}(-1)^\ell\left[\langle PU_\ell, \phi_{i,\varepsilon}\rangle - \varepsilon\int_\Omega PU_\ell\phi_{i,\varepsilon}\;dx\right] \\ & - \frac{1}{2}\sum_{\ell=1}^{k}\|\phi_{\ell,\varepsilon}\|^2 + \frac{\varepsilon}{2}\sum_{\ell=1}^{k}|\phi_{\ell,\varepsilon}|_2^2 + \sum_{i\geq\ell}\int_\Omega\left[f(\sum_{j=1}^{\ell}u_j) - f(\sum_{j=1}^{\ell-1}u_j)\right]\phi_{i,\varepsilon}\,dx \\ & - \frac{N-2}{2N}\int_\Omega\left[|\sum_{\ell=1}^{k}u_\ell|^{\frac{2N}{N-2}} - |\sum_{\ell=1}^{k}(-1)^\ell PU_\ell|^{\frac{2N}{N-2}}\right]dx.\end{aligned} \tag{5.75}$$

Setting

$$\begin{aligned}\widetilde{\Upsilon}_\ell = (-1)^\ell\left(\langle PU_\ell, \sum_{i=1}^{\ell-1}\phi_{i,\varepsilon}\rangle - \varepsilon\int_\Omega PU_\ell(\sum_{i=1}^{\ell-1}\phi_{i,\varepsilon})\right) - \frac{1}{2}\|\phi_{\ell,\varepsilon}\|^2 + \frac{\varepsilon}{2}|\phi_{\ell,\varepsilon}|_2^2 + \int_\Omega f(\sum_{j=1}^{\ell}u_j)\phi_{\ell,\varepsilon}\,dx \\ - \frac{N-2}{2N}\int_\Omega\left[|\sum_{j=1}^{\ell}u_j|^{\frac{2N}{N-2}} - |\sum_{j=1}^{\ell-1}u_j|^{\frac{2N}{N-2}} - |\sum_{j=1}^{\ell}(-1)^jPU_j|^{\frac{2N}{N-2}} + |\sum_{j=1}^{\ell-1}(-1)^jPU_j|^{\frac{2N}{N-2}}\right],\end{aligned}$$

by (5.75) we have that

$$\widetilde{J}_\varepsilon(\mu_1,\ldots,\mu_k) = J_\varepsilon(\mu_1,\ldots,\mu_k) + \sum_{\ell=1}^{k}\widetilde{\Upsilon}_\ell$$

in view of

$$\sum_{i\geq\ell}\int_\Omega\left[f(\sum_{j=1}^{\ell}u_j) - f(\sum_{j=1}^{\ell-1}u_j)\right]\phi_{i,\varepsilon}\,dx = \sum_{i=1}^{k}\int_\Omega f(\sum_{j=1}^{i}u_j)\phi_{i,\varepsilon}\,dx.$$

Since for $\ell \geq 2$

$$\begin{aligned}&(-1)^\ell\left(\langle PU_\ell, \sum_{i=1}^{\ell-1}\phi_{i,\varepsilon}\rangle - \varepsilon\int_\Omega PU_\ell(\sum_{i=1}^{\ell-1}\phi_{i,\varepsilon})\right) \\ &= (-1)^\ell\int_\Omega PU_\ell^{\frac{N+2}{N-2}}(\sum_{i=1}^{\ell-1}\phi_{i,\varepsilon})\,dx + O\left(|U_\ell^{\frac{N+2}{N-2}} - (PU_\ell)^{\frac{N+2}{N-2}}|_{\frac{2N}{N+2}} + \varepsilon|PU_\ell|_{\frac{2N}{N+2}}\right)\sum_{i=1}^{\ell-1}\|\phi_{i,\varepsilon}\| \\ &= (-1)^\ell\int_\Omega PU_\ell^{\frac{N+2}{N-2}}(\sum_{i=1}^{\ell-1}\phi_{i,\varepsilon})\,dx + o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma + \varepsilon\mu_\ell^2\right)\end{aligned}$$

in view of (4.6) and (5.2)-(5.5) with μ_1 replaced by μ_ℓ, we have that

$$\widetilde{\Upsilon}_1=-\frac{N-2}{2N}\int_\Omega \widetilde{v}_1\,dx+O(\|\mathcal{E}_1\|^2),\quad \widetilde{\Upsilon}_\ell=-\frac{N-2}{2N}\int_\Omega \widetilde{v}_\ell\,dx+o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma+\varepsilon\mu_\ell^2\right)\tag{5.76}$$

for any $\ell=2,\dots,k$, where

$$\begin{aligned}\widetilde{v}_\ell &= |\sum_{j=1}^{\ell}u_j|^{\frac{2N}{N-2}}-|\sum_{j=1}^{\ell-1}u_j|^{\frac{2N}{N-2}}-|\sum_{j=1}^{\ell}(-1)^jPU_j|^{\frac{2N}{N-2}}+|\sum_{j=1}^{\ell-1}(-1)^jPU_j|^{\frac{2N}{N-2}}\\ &\quad -\frac{2N}{N-2}\left[f(\sum_{j=1}^{\ell}u_j)\phi_{\ell,\varepsilon}+(-1)^\ell(PU_\ell)^{\frac{N+2}{N-2}}(\sum_{i=1}^{\ell-1}\phi_{i,\varepsilon})\right].\end{aligned}$$

By (5.8) and (5.22) we have the expansion

$$\begin{aligned}\widetilde{v}_\ell &= |\sum_{j=1}^{\ell}(-1)^jPU_j+\sum_{j=1}^{\ell-1}\phi_{j,\varepsilon}|^{\frac{2N}{N-2}}-|\sum_{j=1}^{\ell-1}u_j|^{\frac{2N}{N-2}}-|\sum_{j=1}^{\ell}(-1)^jPU_j|^{\frac{2N}{N-2}}\\ &\quad+|\sum_{j=1}^{\ell-1}(-1)^jPU_j|^{\frac{2N}{N-2}}-\frac{2N}{N-2}(-1)^\ell(PU_\ell)^{\frac{N+2}{N-2}}(\sum_{i=1}^{\ell-1}\phi_{i,\varepsilon})\\ &\quad+O\left(|\phi_{\ell,\varepsilon}|^{\frac{2N}{N-2}}+\sum_{j=1}^{\ell}(PU_j)^{\frac{4}{N-2}}\phi_{\ell,\varepsilon}^2+\sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{4}{N-2}}\phi_{\ell,\varepsilon}^2\right).\end{aligned}\tag{5.77}$$

We have that

$$\widetilde{\Upsilon}_1=O(\|\mathcal{E}_1\|^2)=o(\mu_1^{2\Gamma})\tag{5.78}$$

in view of (3.10)-(3.13) and (5.76).

Let us now discuss the case $\ell\ge 2$. Given $\mathcal{A}_h$ as in (5.7), by (5.8) and (5.22) for $h=0,\dots,\ell-1$ we have

$$\begin{aligned}|\widetilde{v}_\ell|_{1,\mathcal{A}_h} &\le c\Big|U_l^{\frac{2N}{N-2}}+U_l^2\sum_{j=1}^{\ell-1}[U_j^{\frac{4}{N-2}}+|\phi_{j,\varepsilon}|^{\frac{4}{N-2}}]+U_\ell^{\frac{N+2}{N-2}}\sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|\Big|_{1,\mathcal{A}_h}\\ &\quad+c\Big|[f(\sum_{j=1}^{\ell-1}u_j)-f(\sum_{j=1}^{\ell-1}(-1)^jPU_j)]U_\ell\Big|_{1,\mathcal{A}_h}+O(\|\phi_{\ell,\varepsilon}\|^2)\\ &\le c\sum_{j=1}^{\ell-1}\Big|U_\ell|\phi_{j,\varepsilon}|^{\frac{N+2}{N-2}}\Big|_{1,\mathcal{A}_h}+\sum_{i,j=1}^{\ell-1}\Big|U_\ell U_i^{\frac{4}{N-2}}|\phi_{j,\varepsilon}|\Big|_{1,\mathcal{A}_h}+o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma+\varepsilon\mu_\ell^2\right)\end{aligned}\tag{5.79}$$

and

$$\begin{aligned}|\widetilde{v}_\ell|_{1,\mathcal{A}_\ell} &\le c\Big|\sum_{i=1}^{\ell}\sum_{j=1}^{\ell-1}U_i^{\frac{4}{N-2}}\phi_{j,\varepsilon}^2+\sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{2N}{N-2}}+\sum_{i,j=1}^{\ell-1}U_i^{\frac{N+2}{N-2}}|\phi_{j,\varepsilon}|\Big|_{1,\mathcal{A}_\ell}\\ &\quad+c\Big|[f(\sum_{j=1}^{\ell}(-1)^jPU_j)-f((-1)^\ell PU_\ell)]\sum_{j=1}^{\ell-1}\phi_{j,\varepsilon}\Big|_{1,\mathcal{A}_\ell}+O(\|\phi_{\ell,\varepsilon}\|^2)\\ &\le c\Big|U_\ell^{\frac{4}{N-2}}\sum_{j=1}^{\ell-1}\phi_{j,\varepsilon}^2+\sum_{j=1}^{\ell-1}|\phi_{j,\varepsilon}|^{\frac{2N}{N-2}}\Big|_{1,\mathcal{A}_\ell}+o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma+\varepsilon\mu_\ell^2\right)\end{aligned}\tag{5.80}$$

in view of (4.6), (5.10)-(5.14) and for any $i,j=1,\dots,\ell-1$

$|U_l^2U_j^{\frac{4}{N-2}}|_{1,\mathcal{A}_h}=O(|U_j^{\frac{4}{N-2}}U_\ell|_{\frac{2N}{N+2},\mathcal{A}_h}|U_\ell|_{\frac{2N}{N-2},\mathcal{A}_h})$, $U_\ell^2|\phi_{j,\varepsilon}|^{\frac{4}{N-2}}+U_\ell^{\frac{N+2}{N-2}}|\phi_{j,\varepsilon}|=O(U_\ell|\phi_{j,\varepsilon}|^{\frac{N+2}{N-2}}+U_\ell^{\frac{2N}{N-2}})$, $U_i^{\frac{4}{N-2}}\phi_{j,\varepsilon}^2+U_i^{\frac{N+2}{N-2}}|\phi_{j,\varepsilon}|=O(|\phi_{j,\varepsilon}|^{\frac{2N}{N-2}}+U_i^{\frac{2N}{N-2}})$, $U_\ell^{\frac{4}{N-2}}U_i\phi_{j,\varepsilon}=O([U_\ell^{\frac{4}{N-2}}U_i]^{\frac{2N}{N+2}}+|\phi_{j,\varepsilon}|^{\frac{2N}{N-2}})$.

Notice in the estimate (5.79) we couple the first two and the second two terms in the expression (5.77) of $\widetilde{v}_\ell$, while in the estimate (5.80) the first/second term is coupled with the third/fourth one in (5.77).

For $j = 1, \ldots, \ell - 1$ there holds $\mathcal{A}_\ell \subset B_{\rho\mu_j}(0)$ and by (4.8) we deduce that

$$\begin{aligned}\left| U_\ell^{\frac{4}{N-2}} \phi_{j,\varepsilon}^2 + |\phi_{j,\varepsilon}|^{\frac{2N}{N-2}} \right|_{1,\mathcal{A}_\ell} &\le \frac{c}{\mu_j^{2\Gamma}} \int_{\mathcal{A}_\ell} \frac{U_\ell^{\frac{4}{N-2}}}{|x|^{2\beta_-}}\, dx + \frac{c}{\mu_j^{\frac{2N}{N-2}\Gamma}} \int_{\mathcal{A}_\ell} \frac{dx}{|x|^{\frac{2N}{N-2}\beta_-}} \\ &\le c(\frac{\mu_\ell}{\mu_j})^{2\Gamma} \int_{\frac{\mathcal{A}_\ell}{\mu_\ell}} \frac{U^{\frac{4}{N-2}}}{|y|^{2\beta_-}}\, dy + c(\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{N}{N-2}\Gamma} = O\left((\frac{\mu_\ell}{\mu_{\ell-1}})^{\frac{N}{N-2}\Gamma} \log \frac{1}{\mu_\ell} \right).\end{aligned} \tag{5.81}$$

For $h = 0, \ldots, \ell - 2$ by (4.6) and (5.10) we deduce

$$\begin{aligned}||\phi_{j,\varepsilon}|^{\frac{N+2}{N-2}} U_\ell|_{1,\mathcal{A}_h} + |U_\ell U_i^{\frac{4}{N-2}} \phi_{j,\varepsilon}|_{1,\mathcal{A}_h} &\le c\|\phi_{j,\varepsilon}\|^{\frac{N+2}{N-2}} |U_\ell|_{\frac{2N}{N-2},\mathcal{A}_h} + c\|\phi_{j,\varepsilon}\| |U_\ell|_{\frac{2N}{N-2},\mathcal{A}_h} \\ &= o\left((\frac{\mu_\ell}{\sqrt{\mu_h \mu_{h+1}}})^\Gamma \right) = o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \right)\end{aligned} \tag{5.82}$$

for any $i, j = 1, \ldots, \ell - 1$. Splitting $\mathcal{A}_{\ell-1}$ as $\mathcal{A}'_{\ell-1} \cup \mathcal{A}''_{\ell-1}$, where $\mathcal{A}'_{\ell-1} = \mathcal{A}_{\ell-1} \cap B_{\rho\mu_{\ell-1}}(0)$ and $\mathcal{A}''_{\ell-1} = \mathcal{A}_{\ell-1} \setminus B_{\rho\mu_{\ell-1}}(0)$, by (4.6) and (4.8) we get that

$$\begin{aligned}&||\phi_{j,\varepsilon}|^{\frac{N+2}{N-2}} U_\ell|_{1,\mathcal{A}_{\ell-1}} + |U_\ell U_i^{\frac{4}{N-2}} \phi_{j,\varepsilon}|_{1,\mathcal{A}_{\ell-1}} \\ &\le c\|\phi_{j,\varepsilon}\|^{\frac{2N}{(N-2)(N-1)}} |\phi_{j,\varepsilon}^{\frac{N+1}{N-1}} U_\ell|_{\frac{N-1}{N-2},\mathcal{A}'_{\ell-1}} + c\|\phi_{j,\varepsilon}\|^{\frac{2N}{(N-2)(5N-9)}} |U_\ell U_i^{\frac{4}{N-2}} \phi_{j,\varepsilon}^{\frac{5N^2-21N+18}{(N-2)(5N-9)}}|_{\frac{5N-9}{5N-10},\mathcal{A}'_{\ell-1}} \\ &+ c\left[\|\phi_{j,\varepsilon}\|^{\frac{N+2}{N-2}} + \|\phi_{j,\varepsilon}\| \right] |U_\ell|_{\frac{2N}{N-2},\mathcal{A}''_{\ell-1}} \\ &= o\left[\frac{\mu_\ell^\Gamma}{\mu_j^{\frac{N+1}{N-1}\Gamma}} (\int_{\mathcal{A}'_{\ell-1}} \frac{dx}{|x|^{N-\frac{2\Gamma}{N-2}}})^{\frac{N-2}{N-1}} + \frac{\mu_\ell^\Gamma}{\mu_i^{\frac{4\Gamma}{N-2}} \mu_j^{\frac{5N^2-21N+18}{(N-2)(5N-9)}\Gamma}} (\int_{\mathcal{A}'_{\ell-1}} \frac{dx}{|x|^{N-\frac{18\Gamma}{5(N-2)}}})^{\frac{5N-10}{5N-9}} \right] \\ &+ o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \right) = o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \right)\end{aligned} \tag{5.83}$$

for any $i, j = 1, \ldots, \ell - 1$ in view of (5.43). Inserting (5.82)-(5.83) into (5.79) and (5.81) into (5.80) we deduce that $|\widetilde{v}_\ell|_{1,\mathcal{A}_h} = o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma + \varepsilon \mu_\ell^2 \right)$ for any $h = 0, \ldots, \ell$ and then

$$\widetilde{\Upsilon}_\ell = o\left((\frac{\mu_\ell}{\mu_{\ell-1}})^\Gamma \right) \tag{5.84}$$

for any $\ell \ge 2$ in view of (3.10)-(3.13) and (5.76). Thanks to (5.78) and (5.84) we have established that $\widetilde{\Upsilon}_\ell$ satisfies the same estimate as Υ_ℓ, $\ell = 1, \ldots, k$.

To conclude the proof of Proposition 4.4, let us show that, if $(d_1, \ldots, d_k)$ is a critical point of $\widetilde{J}_\varepsilon$, then $\sum_{\ell=1}^k (-1)^\ell PU_\ell + \Phi_\varepsilon$ is a critical point of the functional J_ε. Assume that

$$0 = \partial_{d_h} \widetilde{J}_\varepsilon(d_1, \ldots, d_k) = \nabla J_\varepsilon \left(\sum_{\ell=1}^k (-1)^\ell PU_\ell + \Phi_\varepsilon \right) \left[(-1)^h \partial_{d_h} PU_h + \partial_{d_h} \Phi_\varepsilon \right]$$

for any $h = 1, \ldots, k$. Since

$$\nabla J_\varepsilon \left(\sum_{\ell=1}^k (-1)^\ell PU_\ell + \Phi_\varepsilon \right) = \sum_{j=1}^k \lambda_{j,\varepsilon} PZ_j,$$

we get that

$$0 = \sum_{j=1}^{k} \lambda_{j,\varepsilon} \left\langle PZ_j, (-1)^h \partial_{d_h} PU_h + \partial_{d_h} \Phi_\varepsilon \right\rangle \tag{5.85}$$

for any $h = 1, \dots, k$. Since by (5.49) there hold

$$\|PZ_h\|^2 = c_0 + o(1), \qquad \langle PZ_j, PZ_h \rangle = o(1)\ \forall\ j \neq h,$$

we have that

$$\langle PZ_j, \partial_{d_h} \Phi_\varepsilon \rangle = \sum_{\ell=h}^{k} \langle PZ_j, \partial_{d_h} \phi_{\ell,\varepsilon} \rangle = O\left(\sum_{\ell=h}^{k} \|PZ_j\| \, \|\partial_{d_h} \phi_{\ell,\varepsilon}\| \right) = o\,(1)\,.$$

in view of (4.7). Since

$$\partial_{d_h} PU_h = -\Gamma \alpha_N PZ_h \times \begin{cases} -\frac{1}{\varepsilon} & \text{if } \Gamma = 1 \\ \frac{1}{d_h} & \text{if } \Gamma > 1, \end{cases}$$

by (5.85) we deduce that $\lambda_{j,\varepsilon} = 0$ for any $j = 1, \dots, k$, or equivalently

$$\nabla J_\varepsilon \left(\sum_{\ell=1}^{k} (-1)^\ell PU_\ell + \Phi_\varepsilon \right) = 0.$$

Then $\sum_{\ell=1}^{k} (-1)^\ell PU_\ell + \Phi_\varepsilon$ is a critical point of the functional J_ε and the proof of Proposition 4.4 is complete.

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