# A Self-dual Variational Approach to Stochastic Partial Differential Equations

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**Abstract**

Unlike many deterministic PDEs, stochastic equations are not amenable to the classical variational theory of Euler-Lagrange. In this paper, we show how self-dual variational calculus leads to solutions of various stochastic partial differential equations driven by monotone vector fields. We construct weak solutions as minima of suitable non-negative and self-dual energy functionals on Itô spaces of stochastic processes. We deal with both additive and non-additive noise. The equations considered in this paper have already been resolved by other methods, starting with the celebrated thesis of Pardoux, and many other subsequent works. This paper is about presenting a new variational approach to this type of problems, hoping it will lead to progress on other still unresolved situations.

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# 1 Introduction

Self-dual variational calculus was developed in the last fifteen years in an effort to construct variational solutions to various partial differential equations and evolutions, which do not fall in the Euler-Lagrange framework of the standard calculus of variations. We refer to the monograph [19] for a comprehensive account of that theory. In this paper, we show how such a calculus can be applied to solve stochastic partial differential equations, which also do not fit in Euler-Lagrange theory, since their solutions are not known to be critical points of energy functionals. We show here that at least for some of these equations, solutions can be obtained as minima of suitable self-dual functionals on Itô spaces of random paths.

The self-dual approach applies to solve stochastic partial differential equations driven by monotone vector fields. These are operators $A : D(A) \subset V \to V^*$ –possibly set-valued– from a possibly infinite dimensional Banach space $V$ into its dual, satisfying

$$\langle p - q, u - v\rangle \geq 0 \quad \text{for all } (u,p) \text{ and } (v,q) \text{ on the graph of } A. \tag{1.1}$$

As a warm-up, we shall tackle basic SPDEs involving additive noise, such as

$$\begin{cases} du(t) = -A(t,u(t))dt + B(t)dW(t) \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ for the Hilbert space $H$, $W(t)$ is a real-valued Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_t$, and where $B : [0,T] \times \Omega \to H$ is a given Hilbert-space valued progressively measurable process. Here $A : \Omega \times [0,T] \times V \to 2^{V^*}$ can be a time-dependent adapted random –possibly set-valued– maximal monotone map, where $V$ is a Banach space such that $V \subset H \subset V^*$ constitute a Gelfand triple.
We will also deal with SPDEs driven by monotone vector fields and involving a non-additive noise. These can take the form

$$\begin{cases} du(t) = -A(t,u(t))dt + B(t,u(t))dW(t) \\ u(0) = u_0, \end{cases} \tag{1.3}$$

where $u \to B(t,u)$ is now a progressively measurable linear or non-linear operator.
They can also come in divergence form such as

$$\begin{cases} du = \mathrm{div}(\beta(\nabla u(t,x)))dt + B(t,u(t))dW(t) & \text{in } [0,T] \times D \\ u(0,x) = u_0 & \text{on } \partial D, \end{cases} \tag{1.4}$$

where here, $\beta$ is a progressively measurable monotone vector field on $\mathbb{R}^n$, $D$ is a bounded domain in $\mathbb{R}^n$, and $B : [0,T] \times \Omega \times H_0^1(D) \to L^2(D)$ is progressively measurable.

By solutions, we shall mean progressively measurable processes $u$, valued in suitable Sobolev spaces, that verify the integral equation

$$u(t) = u_0 - \int_0^t A(s,u(s))ds + \int_0^t B(s,u(s))dW(s),$$

where the last stochastic integral is in the sense of Itô.

The genesis of self-dual variational calculus can be traced to a 1970 paper of Brezis-Ekeland [7, 8] (see also Nayroles [24, 25]), where they proposed a variational principle for the heat equation and other gradient flows for convex energies. The conjecture was eventually verified by Ghoussoub-Tzou [22], who identified and exploited the self-dual nature of the Lagrangians involved. Since then, the theory was developed in many directions [15, 16, 20], so as to provide existence results for several stationary and parabolic -but so far deterministic-PDEs, which may or may not be Euler-Lagrange equations.

While in most examples where the approach was used, the self-dual Lagrangians were explicit, an important development in the theory was the realization [18] that in a prior work, Fitzpatrick [13] had associated a (somewhat) self-dual Lagrangian to any given monotone vector field. That meant that the variational theory could apply to any equation involving such operators. We refer to the monograph [19] for a survey and for applications to existence results for solutions of several PDEs and evolution equations. We also note that since the appearance of this monograph, the theory has been successfully applied to the homogenization of periodic non-self adjoint problems (Ghoussoub-Moameni-Zarate [21]). More recently, the self-dual approach was used in [2, 3] to tackle the more general problem of stochastic homogenization of such equations and to provide valuable quantitative estimates.

The application of the method to solving SPDEs is long overdue, though V. Barbu [5] did use a Brezis-Ekeland approach to address SPDEs driven by gradients of a convex function and additive noise. We shall deal here with more general situations that cannot be reduced to the deterministic case. We note that the equations below have already been solved by other methods, starting with the celebrated thesis of Pardoux [26], and many other subsequent works [10, 27, 28, 29]. This paper is about presenting a new variational approach, hoping it will lead to progress on other unresolved equations.

To introduce the method, we consider the simplest example, where the monotone operator $A$ is given by the gradient $\partial\varphi$ of a (possibly random and progressively measurable) function $\varphi : [0,T] \times H \to \mathbb{R} \cup \{+\infty\}$ such that for every $t \in [0,T]$, the function $\varphi(t,\cdot)$ is convex and lower semi-continuous on a Hilbert space $H$, and the stochastics is driven by a given progressively measurable additive noise coefficient $B : \Omega \times [0,T] \to H$. The equation becomes

$$\begin{cases} du(t) = -\partial\varphi(t,u(t))dt + B(t)dW(t) \\ u(0) = u_0. \end{cases} \tag{1.5}$$

We consider the following *Itô space over* $H$,

$$\mathcal{A}_H^2 = \left\{ u : \Omega_T \to H;\ u(t) = u(0) + \int_0^t \tilde{u}(s)ds + \int_0^t F_u(s)dW(s) \right\},$$

where $u(0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, $\tilde{u} \in L^2(\Omega_T; H)$ and $F_u \in L^2(\Omega_T; H)$, where $\Omega_T = \Omega \times [0,T]$. Here, both the drift $\tilde{u}$ and the diffusive term $F_u$ are progressively measurable. The key idea is that a solution for (1.5) can be obtained by minimizing the following functional on $\mathcal{A}_H^2$,

$$I(u) = \mathbb{E}\Big\{ \int_0^T L_\varphi(t,u(t),-\tilde{u}(t))\,dt + \frac{1}{2}\int_0^T M_B(F_u(t),-F_u(t))\,dt + \ell_{u_0}(u(0),u(T)) \Big\},$$

where

1. $L_\varphi$ is the (possibly random) time-dependent Lagrangian on $H \times H$ given by

$$L_\varphi(t,u,p) = \varphi(w,t,u) + \varphi^*(w,t,p),$$

   where $\varphi^*$ is the Legendre transform of $\varphi$;

2. $\ell_{u_0}$ is the time-boundary random Lagrangian on $H \times H$ given by

$$\ell_{u_0}(a,b) := \ell_{u_0(w)}(a,b) = \frac{1}{2}\|a\|_H^2 + \frac{1}{2}\|b\|_H^2 - 2\langle u_0(w), a\rangle_H + \|u_0(w)\|_H^2;$$

3. $M_B$ is the random time-dependent diffusive Lagrangian on $H \times H$, given by

$$M_B(G_1, G_2) := \Psi_{B(w,t)}(G_1) + \Psi^*_{B(w,t)}(G_2),$$

where $\Psi_{B(w,t)} : H \to \mathbb{R} \cup \{+\infty\}$ is the convex function $\Psi_{B(w,t)}(G) = \frac{1}{2}\|G - 2B(w,t)\|^2_H$.

However, it is not sufficient that $I$ attains its infimum on $\mathcal{A}^2_H$ at some $v$, but one needs to also show that the infimum is actually equal to zero, so as to obtain

$$\begin{aligned}
0 = I(v) = \; & \mathbb{E}\int_0^T \Big(\varphi(t,v) + \varphi^*(t, -\tilde{v}(t))\Big)\, dt \\
& + \mathbb{E}\,\Big(\frac{1}{2}\|v(0)\|^2_H + \frac{1}{2}\|v(T)\|^2_H - 2\langle u_0, v(0)\rangle + \|u_0\|^2_H\Big) \\
& + \mathbb{E}\int_0^T \Big(\frac{1}{2}\,\|F_v(t) - 2B(t)\|^2_H \,+\, \frac{1}{2}\,\|F_v(t)\|^2_H - 2\langle F_v(t), B(t)\rangle\Big)\, dt,
\end{aligned}$$

where we have used the fact that $\Psi^*_B(G) = \frac{1}{2}\,\|G\|^2_H + 2\langle G, B\rangle_H$.

By using Itô's formula, and by adding and subtracting the term $\mathbb{E}\int_0^T \langle v(t), \tilde{v}(t)\rangle dt$, we can rewrite $I(v)$ as the sum of 3 non-negative terms

$$\begin{aligned}
0 = I(v) = \mathbb{E}\int_0^T \Big(\varphi(t,v) + \varphi^*(t, -\tilde{v}(t)) + \langle v(t), \tilde{v}(t)\rangle\Big)\, dt \\
+ 2\,\mathbb{E}\int_0^T \|F_v - B\|^2_H\, dt + \mathbb{E}\,\|v(0) - u_0\|^2_H,
\end{aligned}$$

which yields that for almost all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\varphi(t,v) + \varphi^*(t, -\tilde{v}(t)) + \langle v(t), \tilde{v}(t)\rangle = 0, \text{ hence } -\tilde{v}(t) \in \partial\varphi(v(t)).$$

The two other identities readily give that $B = F_v$ and $v(0) = u_0$. In other words, $v \in \mathcal{A}^2_H$, and satisfies (1.5).

The self-dual variational calculus allows to apply the above approach in much more generality. The special Lagrangians $L_\varphi$, $\ell_{u_0}$ and $M$ can be replaced by much more general self-dual Lagrangians. Moreover, the "tensorization" procedure between the three components can be extended to any number of self-dual Lagrangians.

In section 2, we shall collect –for the convenience of the reader– the elements of self-dual theory that will be needed in the proofs. In section 3, we show how one can lift self-dual Lagrangians from state space to $L^p$-spaces and then to Itô spaces of stochastic processes. In Section 4, we give a variational resolution for Equation 1.2 by using the basic minimization principle for self-dual Lagrangians. Section 5 contains applications to classical SPDEs such as the following stochastic evolution driven by a diffusion and a transport operator,

$$\begin{cases} du = (\Delta u + \mathbf{a}(x)\cdot\nabla u)dt + B(t)dW & \text{on } [0,T] \times D \\ u(0) = u_0 & \text{on } D, \end{cases} \tag{1.6}$$

where $\mathbf{a} : D \to \mathbb{R}^n$ is a smooth vector field with compact support in $D$, such that $\mathrm{div}(\mathbf{a}) \geq 0$. Other examples include the stochastic porous media equation, but also quasi-linear equations involving the p-Laplacian ($2 \leq p < +\infty$), that is

$$\begin{cases} du = (\Delta_p u - u|u|^{p-2})dt + B(t)dW & \text{on } D \times [0,T] \\ u(0) = u_0 & \text{on } \partial D. \end{cases} \tag{1.7}$$

In section 6, we deal with quite general SPDEs driven by a self-dual Lagrangian on $L^\alpha(\Omega_T; V) \times L^\beta(\Omega_T; V^*)$ and a non-additive noise. We then apply this result in Section 7 to resolve equations of the form (1.3) and (1.4), such as

$$\begin{cases} du(t) = \Delta u\, dt + |u|^{q-1}u\, dW(t) \\ u(0,x) = u_0(x), \end{cases} \tag{1.8}$$

where $\frac{1}{2} \leq q < \frac{n}{n-2}$, and more generally,

$$\begin{cases} du = \operatorname{div}(\beta(\nabla u(t,x)))dt + B(u(t))dW(t) & \text{in } [0,T] \times D \\ u(0,x) = u_0(x) & \text{on } \partial D, \end{cases} \tag{1.9}$$

where $D$ is a bounded domain in $\mathbb{R}^n$ and the initial position $u_0$ belongs to $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(D))$. In Equation (1.9), the vector field $\beta$ is a progressively measurable maximal monotone operator on $\mathbb{R}^n$. In a forthcoming paper, we shall deal with more elaborate non-linear SPDEs such as the stochastic Navier-Stokes equations in 2 and 3 dimensions.

# 2 Elements of self-dual variational calculus

If $V$ is a reflexive Banach space and $V^*$ is its dual, then a (jointly) convex lower semi-continuous Lagrangian $L : V \times V^* \to \mathbb{R} \cup \{+\infty\}$ is said to be *self-dual* on $V \times V^*$ if

$$L^*(p,u) = L(u,p), \qquad (u,p) \in V \times V^*, \tag{2.1}$$

where $L^*$ is the Fenchel-Legendre dual of $L$ in both variables, i.e.,

$$L^*(q,v) = \sup\{\langle q,u\rangle + \langle v,p\rangle - L(u,p);\ u \in V, p \in V^*\}.$$

Such Lagrangians satisfy the following basic property

$$L(u,p) - \langle u,p\rangle \geq 0, \qquad \forall\ (u,p) \in V \times V^*.$$

We are interested in the case when the above is an equality, hence we consider the corresponding –possibly multivalued– *self-dual vector field* $\bar{\partial} L : V \to 2^{V^*}$ defined for each $u \in V$ as the –possibly empty– subset $\bar{\partial} L(u)$ of $V^*$ given by

$$\bar{\partial} L(u) = \{p \in V^*;\ L(u,p) - \langle u,p\rangle = 0\} = \{p \in V^*;\ (p,u) \in \partial L(u,p)\},$$

where $\partial L$ is the subdifferential of the convex function $L$.

## 2.1 Self-dual Lagrangians as potentials for monotone vector fields

Self-dual vector fields are natural extensions of subdifferentials of convex lower semi-continuous functions. Indeed, the most basic self-dual Lagrangians are of the form $L(u,p) = \varphi(u) + \varphi^*(p)$ where $\varphi$ is a convex function on $V$, and $\varphi^*$ is its Fenchel dual on $V^*$ (i.e., $\varphi^*(p) = \sup\{\langle u,p\rangle - \varphi(u), u \in V\}$) for which

$$\bar{\partial} L(u) = \partial\varphi(u).$$

Other examples of self-dual Lagrangians are of the form $L(u,p) = \varphi(u) + \varphi^*(-\Gamma u + p)$ where $\Gamma : V \to V^*$ is a skew-adjoint operator. The corresponding self-dual vector field is then

$$\bar{\partial} L(u) = \partial\varphi(u) + \Gamma u.$$

Actually, both $\partial\varphi$ and $\partial\varphi + \Gamma$ are particular examples of the so-called maximal monotone operators, which are set-valued maps $A : V \to 2^{V^*}$ whose graphs in $V \times V^*$ are maximal (for set inclusion) among all monotone subsets $G$ of $V \times V^*$. In fact, it turned out that maximal monotone operators and self-dual vector fields are essentially the same. The following was first noted by Fitzpatrick [13] (with a weaker notion of (sub) self-duality), and re-discovered and strengthened later by various authors. See [19] for details.

**Theorem 2.1.** *If $A : D(A) \subset V \to 2^{V^*}$ is a maximal monotone operator with a non-empty domain, then there exists a self-dual Lagrangian $L$ on $V \times V^*$ such that $A = \bar{\partial} L$. Conversely, if $L$ is a proper self-dual Lagrangian on a reflexive Banach space $V \times V^*$, then the vector field $u \mapsto \bar{\partial} L(u)$ is maximal monotone.*

Another needed property of the class of self-dual Lagrangians is its stability under convolution.

**Lemma 2.2.** ([19] Proposition 3.4) *If $L$ and $N$ are two self-dual Lagrangians on a reflexive Banach space $X \times X^*$ such that $Dom_1(L) - Dom_1(N)$ contains a neighborhood of the origin, then the Lagrangian defined by*

$$(L \oplus N)(u,p) = \inf_{r \in X^*} \{L(u,r) + N(u,p-r)\}$$

*is also self-dual on $X \times X^*$.*

As in deterministic evolution equations, one often aim for more regular solutions that are valued in suitable Sobolev spaces, as opposed to just $L^2$. Moreover, the required coercivity condition (on the underlying Hilbert space) is quite restrictive and is not satisfied by most Lagrangians of interest. A natural setting is the so-called evolution triple of Gelfand, which consists of having a Hilbert space sandwiched between a reflexive Banach space $V$ and its dual $V^*$, i.e.,

$$V \subset H \cong H^* \subset V^*,$$

where the injections are continuous and with dense range, in such a way that if $v \in V$ and $h \in H$, then $\langle v,h \rangle_H = \langle v,h \rangle_{V,V^*}$. A typical evolution triple is $V := H_0^1(D) \subset H := L^2(D) \subset V^* := H^{-1}(D)$, where $D$ is a bounded domain in $\mathbb{R}^n$. The following lemma explains the connection between the self-duality on $H$ and $V$.

**Lemma 2.3.** ([19] Lemma 3.4) *Let $V \subset H \subset V^*$ be an evolution triple, and suppose $L : V \times V^* \to \mathbb{R} \cup \{+\infty\}$ is a self-dual Lagrangian on the Banach space $V$, that satisfies for some $C_1, C_2 > 0$ and $r_1 \geq r_2 > 1$,*

$$C_2(\|u\|_V^{r_2} - 1) \leq L(u,0) \leq C_1(1 + \|u\|_V^{r_1}) \qquad \text{for all } u \in V.$$

*Then, the Lagrangian defined on $H \times H$ by*

$$\bar{L}(u,p) := \begin{cases} L(u,p) & u \in V \\ +\infty & u \in H \backslash V \end{cases}$$

*is self-dual on the Hilbert space $H \times H$.*

### 2.2 Two self-dual variational principles

The basic premise of self-dual variational calculus is that several differential systems can be written in the form $0 \in \bar{\partial} L(u)$, where $L$ is a self-dual Lagrangian on phase space $V \times V^*$. These are the *completely self-dual systems.* A solution to these systems can be obtained as a minimizer of a *completely self-dual functional* $I(u) = L(u,0)$ for which the minimum value is 0. The following is the basic minimization principle for self-dual energy functionals.

**Theorem 2.4.** ([14]) *Suppose $X$ is a reflexive Banach space, and let $L$ be a self-dual Lagrangian on $X \times X^*$ such that the mapping $u \to L(u,0)$ is coercive in the sense that $\lim_{\|u\| \to \infty} \frac{L(u,0)}{\|u\|} = +\infty$. Then, there exists $\bar{u} \in X$ such that $I(\bar{u}) = \inf_{u \in X} L(u,0) = 0$.*

As noted in [14], it actually suffices that $L$ be *partially self-dual*, that is if

$$L^*(0,u) = L(u,0) \quad \text{for every } u \in X.$$

We shall also need the Hamiltonian associated to a self-dual Lagrangian, that is the functional on $X \times X$ defined as $H_L : X \times X \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$

$$H_L(u,v) = \sup_{p \in V^*} \{\langle v,p \rangle - L(u,p)\},$$

which is the Legendre transform in the second variable. It is easy to see that if $L$ is a self-dual Lagrangian on $X \times X^*$, then its Hamiltonian on $X \times X$ satisfies the following properties:

- $H_L$ is concave in $u$ and convex lower semi-continuous in $v$.

- $H_L(v,u) \leq -H_L(u,v)$ for all $u, v \in X$.

As established in [17], the Hamiltonian formulation allows for the minimization of direct sums of self-dual functionals. The following variational principle is useful in the case when non-linear and unbounded operators are involved.

**Theorem 2.5.** ([19]) *Consider three reflexive Banach spaces $Z, X_1, X_2$ and operators $A_1 : D(A_1) \subset Z \to X_1$ , $\Gamma_1 : D(\Gamma_1) \subset Z \to X_1^*$, $A_2 : D(A_2) \subset Z \to X_2$, and $\Gamma_2 : D(\Gamma_2) \subset Z \to X_2^*$, such that $A_1$ and $A_2$ are linear, while $\Gamma_1$ and $\Gamma_2$ –not necessarily linear– are weak-to-weak continuous. Suppose $G$ is a closed linear subspace of $Z$ such that $G \subset D(A_1) \cap D(A_2) \cap D(\Gamma_1) \cap D(\Gamma_2)$, while the following properties are satisfied:*

1. *The image of $G_0 := \mathrm{Ker}(A_2) \cap G$ by $A_1$ is dense in $X_1$.*

2. *The image of $G$ by $A_2$ is dense in $X_2$.*

3. *$u \mapsto \langle A_1 u, \Gamma_1 u\rangle + \langle A_2 u, \Gamma_2 u\rangle$ is weakly upper semi-continuous on $G$.*

*Let $L_i, i = 1,2$ be self-dual Lagrangians on $X_i \times X_i^*$ such that the Hamiltonians $H_{L_i}$ are continuous in the first variable on $X_i$. Under the following coercivity condition,*

$$\lim_{\substack{\|u\|\to\infty \\ u\in G}} H_{L_1}(0, A_1 u) - \langle A_1 u, \Gamma_1 u\rangle + H_{L_2}(0, A_2 u) - \langle A_2 u, \Gamma_2 u\rangle = +\infty, \tag{2.2}$$

*the functional*

$$I(u) = L_1(A_1 u, \Gamma_1 u) - \langle A_1 u, \Gamma_1 u\rangle + L_2(A_2 u, \Gamma_2 u) - \langle A_2 u, \Gamma_2 u\rangle$$

*attains its minimum at a point $v \in G$ such that $I(v) = 0$, and*

$$\begin{aligned} \Gamma_1(v) &\in \bar{\partial} L_1(A_1 v), \\ \Gamma_2(v) &\in \bar{\partial} L_2(A_2 v). \end{aligned} \tag{2.3}$$

# 3 Lifting random self-dual Lagrangians to Itô path spaces

Let $V$ be a reflexive Banach space, and $T \in [0,\infty)$ be fixed. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $\mathcal{F}_t,\ t \in [0,T]$, and let $L^\alpha(\Omega \times [0,T]; V)$ be the space of Bochner integrable functions from $\Omega_T := \Omega \times [0,T]$ into $V$ with the norm $\|u\|^\alpha_{L^\alpha_V} := \mathbb{E}\int_0^T \|u(t)\|^\alpha_V\, dt$. We may use the shorter notation $L^\alpha_V(\Omega_T) := L^\alpha(\Omega \times [0,T]; V)$ in the sequel.

**Definition 3.1.** *A self-dual $\Omega_T$-dependent convex Lagrangian on $V \times V^*$ is a function $L : \Omega_T \times V \times V^* \to \mathbb{R} \cup \{+\infty\}$ such that:*

1. *$L$ is progressively measurable with respect to the $\sigma$-field generated by the products of $\mathcal{F}_t$ and Borel sets in $[0,t]$ and $V \times V^*$, i.e. for every $t \in [0,T]$, $L(t,\cdot,\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t]) \otimes \mathcal{B}(V) \otimes \mathcal{B}(V^*)$-measurable.*

2. *For each $t \in [0,T]$, $\mathbb{P}$-a.s. the function $L(t,\cdot,\cdot)$ is convex and lower semi-continuous on $V \times V^*$.*

3. *For any $t \in [0,T]$, we have $\mathbb{P}$-a.s. $L^*(t,p,u) = L(t,u,p)$ for all $(u,p) \in V \times V^*$, where $L^*$ is the Legendre transform of $L$ in the last two variables.*

To each $\Omega_T$-dependent Lagrangian $L$ on $\Omega_T \times V \times V^*$, one can associate the corresponding Lagrangian $\mathcal{L}$ on the path space $L^\alpha_V(\Omega_T) \times L^\beta_{V^*}(\Omega_T)$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, to be

$$\mathcal{L}(u,p) := \mathbb{E}\int_0^T L(t, u(t), p(t))\ dt,$$

with the duality between $L^\alpha_V(\Omega_T)$ and $L^\beta_{V^*}(\Omega_T)$ given by $\langle u, p\rangle = \mathbb{E}\int_0^T \langle u(t), p(t)\rangle_{V,V^*} dt$. The associated Hamiltonian on $L^\alpha_V(\Omega_T) \times L^\alpha_V(\Omega_T)$ will then be

$$H_{\mathcal{L}}(u,v) = \sup\Big\{\mathbb{E}\int_0^T \{\langle v(t), p(t)\rangle - L(t, u(t), p(t))\}dt\ ;\ p \in L^\beta_{V^*}(\Omega_T)\Big\}.$$

The Legendre dual of a "lifted" Lagrangian in both variables naturally lifts to the space of paths $L^\alpha_V(\Omega_T) \times L^\beta_{V^*}(\Omega_T)$ via

$$\mathcal{L}^*(q,v) = \sup_{\substack{u\in L^\alpha_V(\Omega_T)\\ p\in L^\beta_{V^*}(\Omega_T)}} \Big\{\mathbb{E}\int_0^T \{\langle q(t), u(t)\rangle + \langle v(t), p(t)\rangle - L(t, u(t), p(t))\}\, dt\Big\}.$$

The following proposition is standard. See for example [11].

**Proposition 3.2.** *Suppose that $L$ is an $\Omega_T$-dependent Lagrangian on $V \times V^*$, and $\mathcal{L}$ is the corresponding Lagrangian on the path space $L^\alpha_V(\Omega_T) \times L^\beta_{V^*}(\Omega_T)$. Then,*

1. $\mathcal{L}^*(p,u) = \mathbb{E}\int_0^T L^*(t, p(t), u(t))dt.$
2. $H_{\mathcal{L}}(u,v) = \mathbb{E}\int_0^T H_L(t, u(t), v(t))dt.$

## 3.1 Self-dual Lagrangians associated to progressively measurable monotone fields

Consider now a progressively measurable –possibly set-valued– maximal monotone map that is a map $A : \Omega_T \times V \to 2^{V^*}$ that is measurable for each $t$, with respect to the product $\sigma$-field $\mathcal{F}_t \otimes \mathcal{B}([0,t]) \otimes \mathcal{B}(V)$, and such that for each $t \in [0,T]$, $\mathbb{P}$-a.s., the vector field $A_{\omega,t} := A(t,\omega,\cdot,\cdot)$ is maximal monotone on $V$. By Theorem 2.1, one can associate to the maximal monotone maps $A_{\omega,t}$, self-dual Lagrangians $L_{A_{\omega,t}}$ on $V \times V^*$, in such a way that

$$A_{\omega,t} = \bar\partial L_{A_{\omega,t}} \quad \text{for every } t \in [0,T], \text{ and } \mathbb{P}\text{-a.s.}$$

This correspondence can be done measurably in such a way that if $A$ is progressively measurable, then the same holds for the corresponding $\Omega_T$-dependent Lagrangian $L$. We can then lift the random Lagrangian to the space $L^\alpha_V(\Omega_T) \times L^\beta_{V^*}(\Omega_T)$ via

$$\mathcal{L}_A(u,p) = \mathbb{E}\int_0^T L_{A_{\omega,t}}(u(\omega,t), p(\omega,t))dt.$$

Boundedness and coercivity conditions on $A$ translate into corresponding conditions on the representing Lagrangians as follows. For simplicity, we shall assume throughout that the monotone operators are single-valued, though the results apply for general vector fields.

**Lemma 3.3.** ([21]) Let $A_{\omega,t}$ be the maximal monotone operator as above with the corresponding potential Lagrangian $L_{A_{\omega,t}}$. Assume that for all $u \in V, dt \otimes \mathbb{P}\, a.s.$, $A_{\omega,t}$ satisfies

$$\langle A_{\omega,t}u, u\rangle \geq \max\Big\{c_1(\omega,t)\|u\|^\alpha_V - m_1(\omega,t),\ c_2(\omega,t)\|A_{\omega,t}u\|^\beta_{V^*} - m_2(\omega,t)\Big\}, \tag{3.1}$$

where $c_1, c_2 \in L^\infty(\Omega_T, dt \otimes \mathbb{P})$ and $m_1, m_2 \in L^1(\Omega_T, dt \otimes \mathbb{P})$. Then the corresponding Lagrangians satisfy the following:

$$C_1(\omega,t)(\|u\|_V^\alpha + \|p\|_{V^*}^\beta - n_1(\omega,t) \le L_{A_{w,t}}(u,p) \le C_2(\omega,t)(\|u\|_V^\alpha + \|p\|_{V^*}^\beta + n_2(\omega,t)),$$

for some $C_1, C_2 \in L^\infty(\Omega_T)$ and $n_1, n_2 \in L^1(\Omega_T)$.
The lifted Lagrangian on the $L^\alpha$-spaces then satisfy for some $C_1, C_2 > 0$,

$$C_1(\|u\|^\alpha_{L_V^\alpha(\Omega_T)} + \|p\|^\beta_{L^\beta_{V^*}(\Omega_T)} - 1) \le \mathcal{L}_A(u,p) \le C_2(1 + \|u\|^\alpha_{\mathcal{L}_V^\alpha(\Omega_T)} + \|p\|^\beta_{L^\beta_{V^*}(\Omega_T)}).$$

## 3.2 Itô path spaces over a Hilbert space

Suppose now that $U$ is a Hilbert space. For $t \in [0,T]$, a cylindrical Wiener process $W(t)$ in $U$ can be represented by

$$W(t) = \sum_{k\in\mathbb{N}} \beta_k(t)\, e_k, \qquad t \ge 0,$$

where $\{\beta_k\}$ is a sequence of mutually independent Brownian motions on the filtered probability space and $\{e_k\}$ is an orthonormal basis in $U$. For simplicity, we shall assume in the sequel that $W$ is a real-valued Wiener process i.e. $U = \mathbb{R}$. We now recall Itô's formula.

**Proposition 3.4.** ([28], [29]) *Let $H$ be a Hilbert space with $\langle\, ,\,\rangle_H$ as its scalar product. Fix $x_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, and let $y \in L^2(\Omega_T; H)$, $Z \in L^2(\Omega_T; H)$ be two progressively measurable processes. Define the $H$-valued process $u$ as*

$$u(t) := x_0 + \int_0^t y(s)ds + \int_0^t Z(s)dW(s). \tag{3.2}$$

*Then, the following holds:*

1. *$u$ is a continuous $H$-valued adapted process such that $\mathbb{E}\left(\sup_{t\in[0,T]} \|u(t)\|_H^2\right) < \infty$.*
2. (Itô's formula) *For all $t \in [0,T]$,*

$$\|u(t)\|_H^2 = \|x_0\|_H^2 + 2\int_0^t \langle y(s), u(s)\rangle_H ds + \int_0^t \|Z(s)\|_H^2 ds + 2\int_0^t \langle u(s), Z(s)\rangle_H dW(s),$$

*and consequently*

$$\mathbb{E}(\|u(t)\|_H^2) = \mathbb{E}(\|x_0\|_H^2) + \mathbb{E}\int_0^t \Big(2\langle y(s), u(s)\rangle_H + \|Z(s)\|_H^2\Big)ds.$$

More generally, the following *integration by parts* formula holds. For two processes $u$ and $v$ of the form:

$$u(t) = u(0) + \int_0^t \tilde{u}(s)ds + \int_0^t F_u(s)dW(s), \quad v(t) = v(0) + \int_0^t \tilde{v}(s)ds + \int_0^t G_v(s)dW(s),$$

we have

$$\begin{aligned}\mathbb{E}\int_0^T \langle u(t), \tilde{v}(t)\rangle dt = &- \mathbb{E}\int_0^T \langle v(t), \tilde{u}(t)\rangle dt - \mathbb{E}\int_0^T \langle F_u(t), G_v(t)\rangle dt \\ &+ \mathbb{E}\langle u(T), v(T)\rangle_H - \mathbb{E}\langle u(0), v(0)\rangle_H.\end{aligned} \tag{3.3}$$

Now we define the *Itô space* $\mathcal{A}_H^2$ consisting of all $H$-valued processes of the following form:

$$\begin{aligned}\mathcal{A}_H^2 = \Big\{u :\Omega_T \to H;\ u(t) = u_0 + \int_0^t \tilde{u}(s)ds + \int_0^t F_u(s)dW(s),& \\ \text{for } u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H),\ \tilde{u} \in L^2(\Omega_T; H),\ F_u \in L^2(\Omega_T; H)\Big\},&\end{aligned} \tag{3.4}$$

where $\tilde{u}$ and $F_u$ are both progressively measurable. We equip $\mathcal{A}_H^2$ with the norm

$$\|u\|_{\mathcal{A}_H^2}^2 = \mathbb{E}\left(\|u(0)\|_H^2 + \int_0^T \|\tilde{u}(t)\|_H^2\, dt + \int_0^T \|F_u(t)\|_H^2\, dt\right),$$

so that it becomes a Hilbert space. Indeed, the following correspondence

$$\begin{aligned}(x_0, y, Z) &\in L^2(\Omega; H) \times L^2(\Omega_T; H) \times L^2(\Omega_T; H)\\ &\qquad \mapsto x_0 + \int_0^t y(s)ds + \int_0^t Z(s)dW(s) \in \mathcal{A}_H^2,\\ u \in \mathcal{A}_H^2 &\mapsto (u(0), \tilde{u}, F_u) \in L^2(\Omega; H) \times L^2(\Omega_T; H) \times L^2(\Omega_T; H),\end{aligned} \tag{3.5}$$

induces an isometry, since Itô's formula applied to two processes $u, v \in \mathcal{A}_H^2$ yields

$$\begin{aligned}\|u(t) - v(t)\|_H^2 = \|u(0) - v(0)\|_H^2 &+ 2\int_0^t \langle \tilde{u}(s) - \tilde{v}(s), u(s) - v(s)\rangle_H ds\\ &+ \int_0^t \|F_u(s) - F_v(s)\|_H^2 ds + 2\int_0^t \langle u(s) - v(s), F_u(s) - F_v(s)\rangle_H dW_s,\end{aligned}$$

which means that $u = v$ if and only if $u(0) = v(0)$, $F_u = F_v$ and $\tilde{u} = \tilde{v}$. We therefore can and shall identify the Itô space $\mathcal{A}_H^2$ with the product space $L^2(\Omega; H) \times L^2(\Omega_T; H) \times L^2(\Omega_T; H)$. The dual space $(\mathcal{A}_H^2)^*$ can also be identified with $L^2(\Omega; H) \times L^2(\Omega_T; H) \times L^2(\Omega_T; H)$. In other words, each $p \in (\mathcal{A}_H^2)^*$ can be represented by the triplet

$$p = (p_0, p_1(t), P(t)) \in L^2(\Omega; H) \times L^2(\Omega_T; H) \times L^2(\Omega_T; H),$$

in such a way that the duality can be written as:

$$\langle u, p\rangle_{\mathcal{A}_H^2 \times (\mathcal{A}_H^2)^*} = \mathbb{E}\Big\{\langle p_0, u(0)\rangle_H + \int_0^T \langle p_1(t), \tilde{u}(t)\rangle_H\, dt + \frac{1}{2}\int_0^T \langle P(t), F_u(t)\rangle_H\, dt\Big\}. \tag{3.6}$$

## 3.3 Self-dual Lagrangians on Itô spaces of random processes

We now prove the following.

**Theorem 3.5.** *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete probability space with normal filtration, and let $L$ and $M$ be two $\Omega_T$-dependent self-dual Lagrangians on $H \times H$, Assume $\ell$ is an $\Omega$-dependent function on $H \times H$, such that $\mathbb{P}$-a.s.*

$$\ell(\omega, a, b) = \ell^*(\omega, -a, b), \qquad (a, b) \in H \times H. \tag{3.7}$$

*The Lagrangian on $\mathcal{A}_H^2 \times (\mathcal{A}_H^2)^*$ defined by*

$$\begin{aligned}\mathcal{L}(u, p) = \mathbb{E}\Big\{&\int_0^T L(t, u(t) - p_1(t), -\tilde{u}(t))\, dt + \ell(u(0) - p_0, u(T))\\ &+ \frac{1}{2}\int_0^T M(F_u(t) - P(t), -F_u(t))\, dt\Big\},\end{aligned} \tag{3.8}$$

*is then partially self-dual. Actually, it is self-dual on the subset $\mathcal{A}_H^2 \times \mathcal{D}$ of $\mathcal{A}_H^2 \times (\mathcal{A}_H^2)^*$, where $\mathcal{D} := (\{0\} \times L_H^2 \times L_H^2)$.*

*Proof.* Take $(q,v) \in (\mathcal{A}_H^2)^* \times \mathcal{A}_H^2$ with $q$ an element in the dual space identified with the triple $(0, q_1(t), Q(t))$, then

$$\begin{aligned}
\mathcal{L}^*(q,v) &= \sup_{\substack{u\in\mathcal{A}_H^2 \\ p\in(\mathcal{A}_H^2)^*}} \{\langle q,u\rangle + \langle v,p\rangle - \mathcal{L}(u,p)\} \\
&= \sup_{u\in\mathcal{A}_H^2} \sup_{\substack{p_0\in L_H^2(\Omega) \\ p_1\in L_H^2(\Omega_T)}} \sup_{P\in L_H^2(\Omega_T)} \mathbb{E}\Big\{\langle p_0, v(0)\rangle + \int_0^T \Big(\langle q_1(t),\tilde{u}(t)\rangle + \langle p_1(t),\tilde{v}(t)\rangle\Big)\,dt \\
&\qquad + \frac{1}{2}\int_0^T \Big(\langle Q(t),F_u(t)\rangle + \langle P(t),G_v(t)\rangle\Big)\,dt \\
&\qquad - \int_0^T L(t,u(t)-p_1(t),-\tilde{u}(t))\,dt - \ell(u(0)-p_0,u(T)) \\
&\qquad - \frac{1}{2}\int_0^T M(F_u(t)-P(t),-F_u(t))\,dt\Big\}.
\end{aligned}$$

Make the following substitutions:

$$\begin{aligned}
u(t)-p_1(t) &= y(t) \in L_H^2(\Omega_T) \\
u(0)-p_0 &= a \in L_H^2(\Omega) \\
F_u(t)-P(t) &= J(t) \in L_H^2(\Omega_T),
\end{aligned}$$

to obtain

$$\begin{aligned}
\mathcal{L}^*(q,v) = \sup_{u\in\mathcal{A}_H^2} \sup_{a\in L_H^2(\Omega)} \sup_{y\in L_H^2(\Omega_T)} \sup_{J\in L_H^2(\Omega_T)} \mathbb{E}\Big\{&\langle u(0)-a, v(0)\rangle - \ell(a,u(T)) \\
&+ \int_0^T \Big(\langle q_1(t),\tilde{u}(t)\rangle + \langle u(t)-y(t),\tilde{v}(t)\rangle - L(t,y(t),-\tilde{u}(t))\Big)\,dt \\
&+ \frac{1}{2}\int_0^T \langle Q(t),F_u(t)\rangle + \langle F_u(t)-J(t),G_v(t)\rangle - M(J(t),-F_u(t))\,dt\Big\}.
\end{aligned}$$

Use Itô's formula (3.3) for the processes $u$ and $v$ in $\mathcal{A}_H^2$, to get

$$\begin{aligned}
\mathcal{L}^*(q,v) = \sup_{u\in\mathcal{A}_H^2} \sup_{a\in L_H^2(\Omega)} \sup_{y\in L_H^2(\Omega_T)} \sup_{J\in L_H^2(\Omega_T)} \mathbb{E}\Big\{&\langle a,-v(0)\rangle + \langle u(T),v(T)\rangle - \ell(a,u(T)) \\
&+ \int_0^T \langle v(t)-q_1(t),-\tilde{u}(t)\rangle + \langle y(t),-\tilde{v}(t)\rangle - L(t,y(t),-\tilde{u}(t))\,dt \\
&+ \frac{1}{2}\int_0^T \langle G_v(t)-Q(t),-F_u(t)\rangle + \langle J(t),-G_v(t)\rangle - M(J(t),-F_u(t))\,dt\Big\}.
\end{aligned}$$

In view of the correspondence

$$\begin{aligned}
&(b,r,Z) \in L^2(\Omega;H)\times L^2(\Omega_T;H)\times L^2(\Omega_T;H) \\
&\qquad \mapsto b + \int_0^t r(s)ds + \int_0^t Z(s)dW(s) \in \mathcal{A}_H^2.
\end{aligned}$$

$$u \in \mathcal{A}_H^2 \mapsto (u(T),-\tilde{u},-F_u) \in L^2(\Omega;H)\times L^2(\Omega_T;H)\times L^2(\Omega_T;H),$$

it follows that

$$\mathcal{L}^*(q,v) = \sup_{(a,b)\in L^2_H(\Omega)\times L^2_H(\Omega)} \mathbb{E}\left\{\langle a, -v(0)\rangle + \langle b, v(T)\rangle - \ell(a,b)\right\}$$

$$+ \sup_{(y,r)\in L^2_H(\Omega_T)\times L^2_H(\Omega_T)} \mathbb{E}\left\{ \int_0^T \langle v(t)-q_1(t), r(t)\rangle + \langle y(t), -\tilde{v}(t)\rangle - L(t,y(t),r(t))\,dt\right\}$$

$$+ \frac{1}{2}\sup_{\substack{J\in L^2_H(\Omega_T)\\ Z\in L^2_H(\Omega_T)}} \mathbb{E}\left\{\int_0^T \langle G_v(t)-Q(t), Z(t)\rangle + \langle J(t), -G_v(t)\rangle - M(J(t),Z(t))\,dt\right\},$$

and therefore taking into account Proposition 3.2 gives

$$\mathcal{L}^*(q,v) = \mathbb{E}\ \ell^*(-v(0), v(T)) + \mathbb{E}\int_0^T L^*(t, -\tilde{v}(t), v(t)-q_1(t))\,dt$$
$$+\frac{1}{2}\,\mathbb{E}\int_0^T M^*(-G_v(t), G_v(t)-Q(t))\,dt.$$

Now with the self-duality assumptions on $L$ and $M$, and the condition on $\ell$, we have $\mathcal{L}^*(0,v) = \mathcal{L}(v,0)$, for every $v\in\mathcal{A}^2_H$, which means that $\mathcal{L}$ is partially self-dual on $\mathcal{A}^2_H\times(\mathcal{A}^2_H)^*$. □

# 4 Variational resolution of stochastic equations driven by additive noise

For simplicity, we shall work in an $L^2$-setting in $w$ and in time.

## 4.1 A variational principle on Itô space

The following is now a direct consequence of Theorem 3.5 and Theorem 2.4.

**Proposition 4.1.** *Let $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ be a complete probability space with normal filtration and let $H$ be a Hilbert space. Suppose $L$ and $M$ are $\Omega_T$-dependent self-dual Lagrangians on $H\times H$, and $\ell$ is an $\Omega$-dependent time-boundary Lagrangian on $H\times H$. Assume that for some positive $C_1, C_2$ and $C_3$, we have*

$$\begin{aligned} \mathbb{E}\int_0^T L(t,v(t),0)\ dt &\le C_1(1+\|v\|^2_{L^2_H(\Omega_T)}) && \text{for } v\in L^2_H(\Omega_T),\\ \mathbb{E}\ \ell(a,0) &\le C_2(1+\|a\|^2_{L^2_H(\Omega)}) && \text{for } a\in L^2_H(\Omega),\\ \mathbb{E}\int_0^T M(\sigma(t),0)\ dt &\le C_3(1+\|\sigma\|^2_{L^2_H(\Omega_T)}) && \text{for } \sigma\in L^2_H(\Omega_T). \end{aligned} \tag{4.1}$$

*Consider on $\mathcal{A}^2_H$ the functional*

$$I(u) = \mathbb{E}\Big\{\int_0^T L(t,u(t),-\tilde{u}(t))\,dt + \ell(u(0),u(T)) + \frac{1}{2}\int_0^T M(F_u(t), -F_u(t))\,dt\Big\}.$$

*Then, there exists $v\in\mathcal{A}^2_H$ such that $I(v) = \inf_{u\in\mathcal{A}^2_H} I(u) = 0$, and consequently, $\mathbb{P}$-a.s. and for almost all $t\in[0,T]$, we have*

$$\begin{aligned} -\tilde{v}(t) &\in \bar{\partial}L(t,v(t)) \\ (-v(0),v(T)) &\in \partial\ell(v(0),v(T)) \\ -F_v(t) &\in \bar{\partial}M(F_v(t)). \end{aligned} \tag{4.2}$$

*Moreover, if $L$ is strictly convex, then $v$ is unique.*

*Proof.* The functional $I$ can be written as $I(u) = \mathcal{L}(u, 0)$, where $\mathcal{L}$ is the partially self-dual Lagrangian defined by (3.8).
In order to apply Theorem 2.4, we need to verify the coercivity condition. To this end, we use Conditions (4.1) to show that the map $p \to \mathcal{L}(0, p)$ is bounded on the bounded sets of $(\mathcal{A}^2_H)^*$. Indeed,

$$\begin{aligned}\mathcal{L}(0,p) &= \mathbb{E}\Big\{ \int_0^T L(t, p_1(t), 0)\, dt + \ell(-p_0, 0) + \frac{1}{2}\int_0^T M(-P(t), 0)\, dt \Big\} \\ &\leq C\left(3 + \|p_1\|^2_{L^2_H(\Omega)} + \|p_0\|^2_{L^2_H(\Omega_T)} + \|P\|^2_{L^2_H(\Omega_T)}\right),\end{aligned}$$

and by duality, $\lim_{\|u\|\to\infty} \dfrac{\mathcal{L}(u,0)}{\|u\|} = +\infty$. By Theorem 2.4, there exists $v \in \mathcal{A}^2_H$ such that $I(v) = 0$. We now rewrite $I$ as follows:

$$\begin{aligned}0 = I(v) = \mathbb{E}\,\Big\{ &\int_0^T L(t, v(t), -\tilde{v}(t)) + \langle v(t), \tilde{v}(t)\rangle\, dt - \int_0^T \langle v(t), \tilde{v}(t)\rangle\, dt \\ &+ \ell(v(0), v(T)) + \frac{1}{2}\int_0^T M(F_v(t), -F_v(t))\, dt \Big\}.\end{aligned}$$

By Itô's formula

$$\mathbb{E}\int_0^T \langle v(t), \tilde{v}(t)\rangle = \frac{1}{2}\,\mathbb{E}\|v(T)\|^2_H - \frac{1}{2}\,\mathbb{E}\|v(0)\|^2_H - \frac{1}{2}\,\mathbb{E}\int_0^T \|F_v(t)\|^2_H\, dt,$$

which yields

$$\begin{aligned}0 \;=\; I(v) = \mathbb{E}\,\Big\{ &\int_0^T \Big(L(t, v(t), -\tilde{v}(t)) + \langle v(t), \tilde{v}(t)\rangle\Big)\, dt \Big\} \\ &+ \mathbb{E}\Big\{\ell(v(0), v(T)) - \frac{1}{2}\|v(T)\|^2_H + \frac{1}{2}\|v(0)\|^2_H\Big\} \\ &+ \frac{1}{2}\,\mathbb{E}\Big\{ \int_0^T \Big(\|F_v\|^2_H + M(F_v(t), -F_v(t))\Big)\, dt \Big\}.\end{aligned}$$

The self-duality of the Lagrangians $L$ and $M$ and the hypothesis on the boundary Lagrangian, yield that for a.e. $t \in [0, T]$ and $\mathbb{P}$-a.s. each of the integrands inside the curly-brackets are non-negative, thus

$$\begin{aligned}L(t, v(t), -\tilde{v}(t)) + \langle v(t), \tilde{v}(t)\rangle &= 0, \\ \ell(v(0), v(T)) - \frac{1}{2}\|v(T)\|^2_H + \frac{1}{2}\|v(0)\|^2_H &= 0, \\ M(F_v(t), -F_v(t)) + \langle F_v, F_v\rangle &= 0,\end{aligned}$$

which translate into the three assertions in (4.2).
Finally, if $L$ is strictly convex, then the functional $I$ is strictly convex and the minimum is attained uniquely. □

## 4.2 Regularization via inf-involution

The boundedness Condition (4.1) is quite restrictive and not satisfied by most Lagrangians of interest. One way to deal with such a difficulty is to assume similar bounds on $L$ but in stronger Banach norms. Moreover, we need to find more regular solutions that are valued in more suitable Banach spaces than $H$. To this end, we consider an evolution triple $V \subset H \subset V^*$, where $V$ is a reflexive Banach space and $V^*$ is its dual. We recall the following easy lemma from [19].

**Lemma 4.2.** *Let $L$ be a self-dual Lagrangian on $V \times V^*$.*

1. *If for some $r > 1$ and $C > 0$, we have $L(u,0) \le C(1 + \|u\|_V^r)$ for all $u \in V$, then there exists $D > 0$ such that $L(u,p) \ge D(\|p\|_{V^*}^s - 1)$ for all $(u,p) \in V \times V^*$, where $\frac{1}{r} + \frac{1}{s} = 1$.*

2. *If for $C_1, C_2 > 0$ and $r_1 \ge r_2 > 1$, we have*
$$C_2(\|u\|_V^{r_2} - 1) \le L(u,0) \le C_1(1 + \|u\|_V^{r_1}) \qquad \text{for all } u \in V,$$
*then, there exists $D_1, D_2 > 0$ such that*
$$D_2(\|p\|_{V^*}^{s_1} + \|u\|_V^{r_2} - 1) \le L(u,p) \le D_1(1 + \|u\|_V^{r_1} + \|p\|_{V^*}^{s_2}). \tag{4.3}$$
*where $\frac{1}{r_i} + \frac{1}{s_i} = 1$ for $i = 1,2$, and therefore $L$ is continuous in both variables.*

**Proposition 4.3.** *Consider a Gelfand triple $V \subset H \subset V^*$ and let $L$ be an $\Omega_T$-dependent self-dual Lagrangian on $V \times V^*$. Let $M$ be an $\Omega_T$-dependent self-dual Lagrangian on $H \times H$, and $\ell$ an $\Omega$-dependent boundary Lagrangian on $H \times H$ satisfying $\ell^*(a,b) = \ell(-a,b)$. Assume the following conditions hold:*

$(A_1)$ *For some $m, n > 1$, $C_1, C_2 > 0$,*
$$C_2(\|v\|_{L_V^2(\Omega_T)}^m - 1) \le \mathbb{E} \int_0^T L(t, v(t), 0)\, dt \le C_1(1 + \|v\|_{L_V^2(\Omega_T)}^n) \quad \text{for all } v \in L^2(\Omega_T; V).$$

$(A_2)$ *For some $C_3 > 0$,*
$$\mathbb{E}\ \ell(a,b) \le C_3(1 + \|a\|_{L_H^2(\Omega)}^2 + \|b\|_{L_H^2(\Omega)}^2) \quad \text{for all } a, b \in L^2(\Omega; H).$$

$(A_3)$ *For some $C_4 > 0$,*
$$\mathbb{E} \int_0^T M(G_1(t), G_2(t))dt \le C_4(1 + \|G_1\|_{L_H^2(\Omega_T)}^2 + \|G_2\|_{L_H^2(\Omega_T)}^2) \quad \text{for all } G_1, G_2 \in L_H^2(\Omega_T).$$

*Then, there exists $v \in \mathcal{A}_H^2$ with trajectories in $L^2(\Omega_T; V)$ such that $\tilde{v} \in L^2(\Omega_T; V^*)$, at which the minimum of the following functional is attained and is equal to $0$.*
$$I(u) = \mathbb{E}\Big\{ \int_0^T L(t, u(t), -\tilde{u}(t)\, dt + \ell(u(0), u(T)) + \frac{1}{2} \int_0^T M(F_u(t), -F_u(t))\, dt \Big\}.$$
*Consequently, $\mathbb{P}$-a.s. and for almost all $t \in [0,T]$, we have*
$$\begin{aligned} -\tilde{v}(t) &\in \bar{\partial} L(t, v(t)) \\ (-v(0), v(T)) &\in \partial \ell(v(0), v(T)) \\ -F_v(t) &\in \bar{\partial} M(F_v(t)). \end{aligned} \tag{4.4}$$

*Proof.* First, apply Lemma 2.3 to lift $L$ to an $\Omega_T$-dependent self-dual Lagrangian on $H \times H$, then consider for $t \in [0,T]$ and $\mathbb{P}$-a.s., the $\lambda$-regularization of $L$, that is
$$L_\lambda(t, u, p) = \inf_{z \in H} \Big\{ L(t, z, p) + \frac{\|u - z\|_H^2}{2\lambda} + \frac{\lambda}{2} \|p\|_H^2 \Big\}.$$
By Lemma 2.2, $L_\lambda$ is also an $\Omega_T$-dependent self-dual Lagrangian on $H \times H$ in such a way that the conditions (4.1) of Proposition 4.1 hold. Hence, there exists $v_\lambda \in \mathcal{A}_H^2$ such that
$$0 = \mathbb{E}\ \Big\{ \int_0^T L_\lambda(t, v_\lambda(t), -\tilde{v}_\lambda(t))\, dt + \ell(v_\lambda(0), v_\lambda(T)) + \frac{1}{2} \int_0^T M(F_{v_\lambda}(t), -F_{v_\lambda}(t))\, dt \Big\}.$$

Since $L$ is convex and lower semi-continuous, then $dt \otimes \mathbb{P}$ a.s, there exists $J_\lambda(v_\lambda) \in H$ so that

$$L_\lambda(t, v_\lambda(t), -\tilde{v}_\lambda(t)) = L(t, J_\lambda(v_\lambda)(t), -\tilde{v}_\lambda(t)) + \frac{\|v_\lambda(t) - J_\lambda(v_\lambda)(t)\|_H^2}{2\lambda} + \frac{\lambda}{2}\|\tilde{v}_\lambda(t)\|_H^2,$$

and hence

$$\begin{aligned} 0 = \mathbb{E}\Big\{ \int_0^T \Big( L(t, J_\lambda(v_\lambda)(t), -\tilde{v}_\lambda(t)) + \frac{\|v_\lambda(t) - J_\lambda(v_\lambda)(t)\|_H^2}{2\lambda} + \frac{\lambda}{2}\|\tilde{v}_\lambda(t)\|_H^2 \Big)\, dt \\ + \ell(v_\lambda(0), v_\lambda(T)) + \frac{1}{2}\int_0^T M(F_{v_\lambda}(t), -F_{v_\lambda}(t))\, dt \Big\}. \end{aligned} \tag{4.5}$$

From (4.5), condition $(A_1)$ and the assertion of part (2) of Lemma 4.2, we can deduce that $J_\lambda(v_\lambda)$ is bounded in $L^2(\Omega_T; V)$ and $\tilde{v}_\lambda$ is bounded in $L^2(\Omega_T; V^*)$. Also from condition $(A_2)$ and $(A_3)$, we can deduce the following estimates:

$$\mathbb{E}\int_0^T M(G, H)\, dt \geq C(\|G\|^2_{L^2_H(\Omega_T)} - 1) \quad \text{and} \quad \mathbb{E}\,\ell(a, b) \geq C(\|b\|^2_{L^2_H(\Omega)} - 1).$$

These coercivity properties, together with (4.5), imply that $v_\lambda(0)$ and $v_\lambda(T)$ are bounded in $L^2(\Omega; H)$, and that $F_{v_\lambda}$ is bounded in $L^2(\Omega_T; H)$. Moreover, since all other terms in (4.5) are bounded below, it follows that

$$\mathbb{E}\int_0^T \|v_\lambda(t) - J_\lambda(v_\lambda)(t)\|^2 dt \leq 2\lambda C \quad \text{for some } C > 0.$$

Hence $v_\lambda$ is bounded in $\mathcal{A}^2_H$ and there exists a subsequence $v_{\lambda_j}$ that converges weakly to a path $v \in L^2(\Omega_T; V)$ such that $\tilde{v} \in L^2(\Omega_T; V^*)$, and

$$\begin{aligned} J_{\lambda_j}(v_{\lambda_j}) &\rightharpoonup v \quad \text{in} \quad L^2(\Omega_T; V) \\ \tilde{v}_{\lambda_j} &\rightharpoonup \tilde{v} \quad \text{in} \quad L^2(\Omega_T; V^*) \\ v_{\lambda_j} &\rightharpoonup v \quad \text{in} \quad L^2(\Omega_T; H) \\ v_{\lambda_j}(0) \rightharpoonup v(0), \quad v_\lambda(T) &\rightharpoonup v(T) \quad \text{in} \quad L^2(\Omega; H) \\ F_{v_{\lambda_j}} &\rightharpoonup F_v \quad \text{in} \quad L^2(\Omega_T; H). \end{aligned}$$

Since $L, \ell$ and $M$ are lower semi-continuous, we have

$$\begin{aligned} I(v) \leq \liminf_j \mathbb{E}\Big\{ \int_0^T \Big( L(t, J_{\lambda_j}(v_{\lambda_j})(t), -\tilde{v}_{\lambda_j}(t)) + \frac{\|v_{\lambda_j}(t) - J_{\lambda_j}(v_{\lambda_j})(t)\|^2}{2\lambda_j} + \frac{\lambda_j}{2}\|\tilde{v}_{\lambda_j}(t)\|^2 \Big)\, dt \\ + \ell(v_{\lambda_j}(0), v_{\lambda_j}(T)) + \frac{1}{2}\int_0^T M(F_{v_{\lambda_j}}(t), -F_{v_{\lambda_j}}(t))\, dt \Big\} = 0. \end{aligned}$$

For the reverse inequality, we use the self-duality of $L$ and $M$ and the fact that $\ell(-a, b) = \ell^*(a, b)$ to deduce that

$$\begin{aligned} I(v) = \mathbb{E}\Big\{ &\int_0^T \Big( L(t, v(t), -\tilde{v}(t)) + \langle v(t), \tilde{v}(t)\rangle \Big)\, dt \Big\} \\ &+ \mathbb{E}\Big\{ \ell(v(0), v(T)) - \frac{1}{2}\|v(T)\|_H^2 + \frac{1}{2}\|v(0)\|_H^2 \Big\} \\ &+ \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \Big( \|F_v\|_H^2 + M(F_v(t), -F_v(t)) \Big)\, dt \Big\} \geq 0. \end{aligned}$$

Therefore, $I(v) = 0$ and the rest of the proof is similar to the last part of the proof in Proposition 4.1. $\square$

We now deduce the following.

**Theorem 4.4.** *Consider a Gelfand triple $V \subset H \subset V^*$, and let $A : D(A) \subset V \to V^*$ be an $\Omega_T$-dependent progressively measurable maximal monotone operator satisfying*

$$\langle A_{w,t}u, u\rangle \geq \max\{c_1(\omega,t)\|u\|_V^\alpha - m_1(\omega,t),\ c_2(\omega,t)\|Au\|_{V^*}^\beta - m_2(\omega,t)\},$$

*where $c_1, c_2 \in L^\infty(\Omega_T, dt \otimes \mathbb{P})$ and $m_1, m_2 \in L^1(\Omega_T, dt \otimes \mathbb{P})$. Let $B$ be a given $H$-valued progressively measurable process in $L^2(\Omega_T; H)$, and $u_0$ a given random variable in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. Then, the equation*

$$\begin{cases} du(t) = -A(t,u(t))dt + B(t)dW(t) \\ u(0) = u_0, \end{cases} \tag{4.6}$$

*has a solution $u \in \mathcal{A}_H^2$ that is valued in $V$. It can be obtained by minimizing the functional*

$$\begin{aligned} I(u) = {} & \mathbb{E}\int_0^T L(t, u(t), -\tilde{u}(t))\, dt \\ & + \mathbb{E}\left(\frac{1}{2}\|u(0)\|_H^2 + \frac{1}{2}\|u(T)\|_H^2 - 2\langle u_0, u(0)\rangle_H + \|u_0\|_H^2\right) \\ & + \mathbb{E}\int_0^T \left(\frac{1}{2}\|F_u(t) - 2B(t)\|_H^2 + \frac{1}{2}\|F_u(t)\|_H^2 - 2\langle F_u(t), B(t)\rangle_H\right) dt, \end{aligned}$$

*where $L$ is a self-dual Lagrangian such that $\bar{\partial}L(t,\cdot) = A(t,\cdot)$, $\mathbb{P}$-almost surely.*

*Proof.* It suffices to apply Proposition 4.3 with the self-dual Lagrangian $L$ associated with $A$, the time boundary $\Omega$-dependent Lagrangian $\ell_{u_0}$ on $H \times H$ given by

$$\ell_{u_0}(a,b) = \frac{1}{2}\|a\|_H^2 + \frac{1}{2}\|b\|_H^2 - 2\langle u_0(w), a\rangle_H + \|u_0(w)\|_H^2,$$

and the $\Omega_T$-dependent self-dual Lagrangian $M$ on $L_H^2(\Omega_T)$, given by

$$M_B(G_1, G_2) = \Psi_{B(w,t)}(G_1) + \Psi^*_{B(w,t)}(G_2),$$

where $\Psi_{B(w,t)} : H \to \mathbb{R} \cup \{+\infty\}$ is the convex function $\Psi_{B(w,t)}(G) = \frac{1}{2}\|G - 2B(w,t)\|_H^2$. □

# 5 Applications to various SPDEs with additive noise

In the following examples, we shall assume $D$ is a smooth bounded domain in $\mathbb{R}^n$, $W$ is a real Brownian motion, and $B : \Omega \times [0,T] \to L^2(D)$ is a fixed progressively measurable stochastic process.

## 5.1 Stochastic evolution driven by diffusion and transport

Consider the following stochastic transport equation:

$$\begin{cases} du = (\Delta u + \mathbf{a}(x)\cdot\nabla u)dt + B(t)dW & \text{on } [0,T]\times D \\ u(0) = u_0 & \text{on } D, \end{cases} \tag{5.1}$$

where $\mathbf{a} : D \to \mathbb{R}^n$ is a smooth vector field with compact support in $D$, such that $\operatorname{div}(\mathbf{a}) \geq 0$. Assume $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_0^1(D))$ such that $\Delta u_0 \in L^2(D)$, $\mathbb{P}$-a.s.
Consider the operator $\Gamma u = \mathbf{a}\cdot\nabla u + \frac{1}{2}(\operatorname{div}\mathbf{a})\, u$, which, by Green's formula, is skew-adjoint on $H_0^1(D)$. Also consider the convex function

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_D |\nabla u|^2 dx + \frac{1}{4}\int_D (\operatorname{div}\mathbf{a})|u|^2 dx & u \in H_0^1(D) \\ +\infty & \text{otherwise,} \end{cases}$$

which is clearly coercive on $H_0^1(D)$. Consider the Gelfand triple $H_0^1(D) \subset L^2(D) \subset H^{-1}(D)$, and the self-dual Lagrangian on $H_0^1(D) \times H^{-1}(D)$, defined by

$$L(u,p) = \varphi(u) + \varphi^*(\Gamma u + p).$$

The corresponding functional on Itô space is then,

$$\begin{aligned}
I(u) = \ & \mathbb{E}\left\{\int_0^T \left(\left(\frac{1}{2}\int_D (|\nabla u|^2 dx + \frac{1}{4}\int_D (\operatorname{div}\mathbf{a})|u|^2)dx\right) + \varphi^*\Big(-\tilde{u}(t,\cdot) + \Gamma(u(t,\cdot))\Big)\right)\, dt\right\} \\
& + \mathbb{E}\left\{\frac{1}{2}\int_0^T \left(\int_D \Big(|F_u(t,x)|^2 + 2|B(t,x)|^2 - 4F_u(t,x)B(t,x)\Big)dx\right)\, dt\right\} \\
& + \mathbb{E}\left\{\int_D \left(\frac{1}{2}|u(0,x)|^2 + \frac{1}{2}|u(T,x)|^2 - 2u_0(x)u(0,x) + \frac{1}{2}|u_0(x)|^2\right) dx\right\}.
\end{aligned}$$

Apply Theorem 4.4 to find a path $v \in \mathcal{A}^2_{L^2(D)}$, valued in $H_0^1(\Omega)$, that minimizes $I$ in such a way that $I(v) = 0$, to obtain

$$-\tilde{v} + \mathbf{a}\cdot\nabla v + \frac{1}{2}(\operatorname{div}\mathbf{a})v \in \partial\varphi(v) = -\Delta v + \frac{1}{2}(\operatorname{div}\mathbf{a})v,$$
$$v(0) = u_0, \ \ F_v = B.$$

The process $v(t) = v_0 + \int_0^t \Delta v(s)ds + \int_0^t \mathbf{a}\cdot\nabla v(s)ds + \int_0^t B(s)dW(s)$ is therefore a solution to (5.1).

## 5.2 Stochastic porous media

Consider the following SPDE,

$$\begin{cases} du(t) = \Delta u^p(t)dt + B(t)dW(t) & \text{on } D\times[0,T] \\ u(0) = u_0 & \text{on } D, \end{cases} \tag{5.2}$$

where $p \geq \frac{n-2}{n+2}$, and $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H^{-1}(D))$.
Equip the Hilbert space $H = H^{-1}(D)$ with the inner product

$$\langle u, v\rangle_{H^{-1}} = \langle u, (-\Delta)^{-1}v\rangle = \int_D u(x)(-\Delta)^{-1}v(x)\, dx.$$

Since $p \geq \frac{n-2}{n+2}$, $L^{p+1}(D) \subset H^{-1}(D) \subset L^{\frac{p+1}{p}}(D)$ is an evolution triple.
We consider the convex functional

$$\varphi(u) = \begin{cases} \frac{1}{p+1}\int_D |u(x)|^{p+1}dx & \text{on } L^{p+1}(D) \\ +\infty & \text{elsewhere,} \end{cases}$$

whose Legendre conjugate is given by

$$\varphi^*(u^*) = \frac{p}{p+1}\int_D |(-\Delta)^{-1}u^*|^{\frac{p+1}{p}}\, dx.$$

Now, minimize the following self-dual functional on $\mathcal{A}^2_H$,

$$\begin{aligned}
I(u) = & \mathbb{E}\left\{\frac{1}{p+1}\int_0^T \int_D \left(|u(x)|^{p+1} + p\left|(-\Delta)^{-1}(-\tilde{u}(t))\right|^{\frac{p+1}{p}}\right)dx\, dt\right\} \\
& + \mathbb{E}\left\{\frac{1}{2}\|u(0)\|^2_{H^{-1}} + \frac{1}{2}\|u(T)\|^2_{H^{-1}} + \|u_0\|^2_{H^{-1}} - 2\langle u_0, u(0,\cdot)\rangle_{H^{-1}}\right\} \\
& + \mathbb{E}\left\{\int_0^T \left(\frac{1}{2}\Big(\|F_u(t)\|^2_{H^{-1}} + 2\|B(t)\|^2_{H^{-1}} - 4\langle F_u(t), B(t)\rangle_{H^{-1}}\Big)dt\right\}.
\end{aligned}$$

Apply Theorem 4.4 to find a process $v \in \mathcal{A}_H^2$ with values in $L^{p+1}(D)$ such that

$$(-\Delta)^{-1}(-\tilde{v}(t)) \in \partial\varphi(v(t)) = v^p, F_v = B, \text{ and } v(0) = v_0.$$

It follows that $v(t) = v_0 + \int_0^t \Delta v^p(s)ds + \int_0^t B(s)dW(s)$, provides a solution for (5.2).

### 5.3 Stochastic PDE involving the p-Laplacian

Consider the equation

$$\begin{cases} du = (\Delta_p u - u|u|^{p-2})dt + B(t)dW & \text{on } D \times [0,T] \\ u(0) = u_0 & \text{on } \partial D, \end{cases}$$

where $p \in [2, +\infty)$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator, and $u_0$ is given such that $u_0 \in W_0^{1,p}(D) \cap \{u; \Delta_p u \in L^p(D)\}$. It is clear that $W_0^{1,p}(D) \subset L^p(D)$ continuously and densely, which ensures that the functional

$$\varphi(u) = \frac{1}{p}\int_D |\nabla u(x)|^p dx + \frac{1}{p}\int_D |u(x)|^p dx,$$

is convex, lower semi-coninuous and coercive on $W_0^{1,p}(D)$ with respect to the evolution triple

$$W_0^{1,p}(D) \subset L^p(D) \subset L^2(D) \subset W_0^{1,p}(D)^* \subset L^q(D),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Theorem 4.4 applies to the self-dual functional

$$\begin{aligned} I(u) = \ & \mathbb{E}\int_0^T \Big(\varphi(t,u) + \varphi^*(t,-\tilde{u})\Big)\, dt \\ & + \mathbb{E}\left(\frac{1}{2}\|u(0)\|^2_{L^2(D)} + \frac{1}{2}\|u(T)\|^2_{L^2(D)} - 2\langle u_0, u(0)\rangle + \|u_0\|^2_{L^2(D)}\right) \\ & + \mathbb{E}\int_0^T \left(\frac{1}{2}\|F_u(t)\|^2_{L^2(D)} + \|B(t)\|^2_{L^2(D)} - 2\langle F_u(t), B(t)\rangle\right) dt. \end{aligned}$$

to yield a $W_0^{1,p}(D)$-valued process $v \in \mathcal{A}^2_{L^2(D)}$, where the null infimum is attained. It follows that

$$-\tilde{v} \in \partial\varphi(v) = -\Delta_p v + v|v|^{p-2},$$
$$v(0) = u_0, \ F_v = B,$$

and hence $v(t) - u_0 - \int_0^t B(s)dW(s) = \int_0^t \tilde{v}(s)ds = \int_0^t \Delta_p v(s)ds - \int_0^t v(s)|v(s)|^{p-2}ds$.

## 6 Non-additive noise driven by self-dual Lagrangians

In this section, we give a variational resolution for stochastic equations of the form

$$\begin{cases} du = -\bar{\partial}\mathcal{L}(u)(t)\, dt + B(t, u(t))dW \\ u(0) = u_0, \end{cases} \tag{6.1}$$

where $\mathcal{L}$ is a self-dual Lagrangian on $L^\alpha(\Omega_T; V) \times L^\beta(\Omega_T; V^*)$, $1 < \alpha < +\infty$ and $\beta$ is its conjugate, and where $V \subset H \subset V^*$ is a given Gelfand triple.
We shall assume that $\mathcal{L}$ satisfies the following conditions:

$$C_2(\|u\|^\alpha_{L^\alpha_V(\Omega_T)} + \|p\|^\beta_{L^\beta_{V^*}(\Omega_T)} - 1) \le \mathcal{L}(u,p) \le C_1(1 + \|u\|^\alpha_{L^\alpha_V(\Omega_T)} + \|p\|^\beta_{L^\beta_{V^*}(\Omega_T)}). \tag{6.2}$$

and

$$\|\bar{\partial}\mathcal{L}(u)\|_{L^{\beta}_{V^*}(\Omega_T)} \leq C_3(1 + \|u\|_{L^{\alpha}_{V}(\Omega_T)}). \tag{6.3}$$

Note that in the last section, we worked in a Hilbertian setting, then used inf-convolution to find a solution that is valued in the Sobolev space $V$. This approach does not work in the non-additive case, since we need to work with stronger topologies on the space of Itô processes that will give the operator $B$ a chance to be completely continuous. We shall therefore strengthen the norm on the Itô space over a Gelfand triple, at the cost of losing coercivity, that we shall recover through perturbation methods.
More precisely, we are searching for a solution $u$ of the form

$$u(t) = u(0) + \int_0^t \tilde{u}(s)ds + \int_0^t F_u(s)dW(s), \tag{6.4}$$

where $u \in L^{\alpha}(\Omega_T; V)$, $\tilde{u} \in L^{\beta}(\Omega_T; V^*)$ and $F_u \in L^2(\Omega_T; H)$ are progressively measurable. The space of such processes, will be denoted $\mathcal{Y}_V^{\alpha}$, and will be equipped with the norm,

$$\|u\|_{\mathcal{Y}_V^{\alpha}} = \|u(t)\|_{L^{\alpha}_V(\Omega_T)} + \|\tilde{u}(t)\|_{L^{\beta}_{V^*}(\Omega_T)} + \|F_u(t)\|_{L^2_H(\Omega_T)}.$$

As shown in [29], any such a process $u \in \mathcal{Y}_V^{\alpha}$ has a $dt \otimes \mathbb{P}$-equivalent version $\hat{u}$ that is a $V$-valued progressively measurable process that satisfies the following Itô formula:
$\mathbb{P}$-a.s. and for all $t \in [0, T]$,

$$\|u(t)\|_H^2 = \|u(0)\|_H^2 + 2\int_0^t \langle \tilde{u}(s), \hat{u}(s)\rangle_{V^*,V}\, ds + \int_0^t \|F_u(s)\|_H^2 ds + 2\int_0^t \langle u(s), F_u(s)\rangle_H dW(s), \tag{6.5}$$

In particular, we have for all $t \in [0, T]$,

$$\mathbb{E}(\|u(t)\|_H^2) = \mathbb{E}(\|u(0)\|_H^2) + \mathbb{E}\int_0^t \Big(2\langle \tilde{u}(s), \hat{u}(s)\rangle_{V^*,V} + \|F_u(s)\|_H^2\Big)ds.$$

Furthermore, we have $u \in C([0, T]; H)$. In fact, one can deduce that for any $u \in \mathcal{Y}_V^{\alpha}$, $u \in C([0, T]; V^*)$ and $u \in L^{\infty}(0, T; H)$ $\mathbb{P}$-a.s ([28] and [29]). From now on, a process $u$ in $\mathcal{Y}_V^{\alpha}$ will always be identified with its $dt \otimes \mathbb{P}$-equivalent $V$-valued version $\hat{u}$.

**Theorem 6.1.** *Consider a self-dual Lagrangian $\mathcal{L}$ on $L^{\alpha}(\Omega_T; V) \times L^{\beta}(\Omega_T; V^*)$ satisfying (6.2) and (6.3), and let $B : \mathcal{Y}_V^{\alpha} \to L^2(\Omega_T; H)$ be a –not-necessarily linear– weak-to-norm continuous map such that for some $C > 0$ and $0 < \delta < \frac{\alpha+1}{2}$,*

$$\|Bu\|_{L^2_H(\Omega_T)} \leq C\|u\|^{\delta}_{L^{\alpha}(\Omega_T)} \quad \textit{for any } u \in \mathcal{Y}_V^{\alpha}. \tag{6.6}$$

*Let $u_0$ be a given random variable in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. Equation (6.1) has then a solution $u$ in $\mathcal{Y}_V^{\alpha}$, that is a stochastic process satisfying*

$$u(t) = u_0 - \int_0^t \bar{\partial}\mathcal{L}(u)(s)ds + \int_0^t Bu(s)dW(s). \tag{6.7}$$

We would like to apply Theorem 2.5 to $\mathcal{L}$ on $L^{\alpha}(\Omega_T; V) \times L^{\beta}(\Omega_T; V^*)$ and to the following operators acting on $G = \{u \in \mathcal{Y}_V^{\alpha}; u(0) = u_0\}$,

$$A_1 : G \subset \mathcal{Y}_V^{\alpha} \to L^{\alpha}(\Omega_T; V), \qquad \Gamma_1 : G \subset \mathcal{Y}_V^{\alpha} \to L^{\beta}(\Omega_T; V^*)$$
$$A_1(u) = u, \qquad \Gamma_1(u) = -\tilde{u}$$

$$A_2 : G \subset \mathcal{Y}_V^{\alpha} \to L^2(\Omega_T; H), \qquad \Gamma_2 : G \subset \mathcal{Y}_V^{\alpha} \to L^2(\Omega_T; H)$$
$$A_2(u) = \frac{1}{2}F_u, \qquad \Gamma_2(u) = -F_u + \frac{3}{2}Bu.$$

Unfortunately, the coercivity condition (2.2) required to conclude is not satisfied. We have to therefore perturb the Lagrangian $\mathcal{L}$ (i.e., essentially perform a stochastic elliptic regularization) as well as the operator $\Gamma_1$ in order to ensure coercivity. We will then let the perturbations go to zero to conclude.

### 6.1 Stochastic elliptic regularization

To do that, we consider the convex lower semi-continuous function on $L^\alpha(\Omega_T, V)$

$$\psi(u) = \begin{cases} \frac{1}{\beta}\mathbb{E}\int_0^T \|\tilde{u}(t)\|_{V^*}^\beta dt & \text{if } u \in \mathcal{Y}_V^\alpha \\ +\infty & \text{if } u \in L_V^\alpha(\Omega_T)\backslash\mathcal{Y}_V^\alpha, \end{cases} \tag{6.8}$$

and for any $\mu > 0$, its associated self-dual Lagrangian on $L_V^\alpha(\Omega_T) \times L_{V^*}^\beta(\Omega_T)$ given by

$$\Psi_\mu(u,p) = \mu\psi(u) + \mu\psi^*(\frac{p}{\mu}). \tag{6.9}$$

We also consider a perturbation operator

$$Ku := (\|u\|_{L_V^\alpha(\Omega_T)}^{\alpha-1})Du,$$

where $D$ is the duality map between $V$ and $V^*$. Note that by definition, $K$ is a weak-to-weak continuous operator from $\mathcal{Y}_V^\alpha$ to $L_{V^*}^\beta(\Omega_T)$.

**Lemma 6.2.** *Under the above hypothesis on $\mathcal{L}$ and $B$, there exists a process $u_\mu \in \mathcal{Y}_V^\alpha$ such that $u(0) = u_0$, $\tilde{u}(T) = \tilde{u}(0) = 0$, and satisying*

$$\tilde{u}_\mu + Ku_\mu + \mu\,\partial\psi(u_\mu) \in -\bar{\partial}\mathcal{L}(u_\mu)$$
$$F_{u_\mu} = Bu_\mu.$$

*Proof.* Apply Theorem 2.5 as follow: Let $Z = \mathcal{Y}_V^\alpha, X_1 = L^\alpha(\Omega_T;V), X_2 = L^2(\Omega_T;H)$ with $G = \{u \in \mathcal{Y}_V^\alpha; u(0) = u_0\}$ which is a closed linear subspace of $\mathcal{Y}_V^\alpha$, and consider the operators

$$A_1 : G \subset \mathcal{Y}_V^\alpha \to L^\alpha(\Omega_T;V), \qquad \Gamma_1 : G \subset \mathcal{Y}_V^\alpha \to L^\beta(\Omega_T;V^*)$$
$$A_1(u) = u, \qquad \Gamma_1(u) = -\tilde{u} - Ku$$

$$A_2 : G \subset \mathcal{Y}_V^\alpha \to L^2(\Omega_T;H), \qquad \Gamma_2 : G \subset \mathcal{Y}_V^\alpha \to L^2(\Omega_T;H)$$
$$A_2(u) = \frac{1}{2}F_u, \qquad \Gamma_2(u) = -F_u + \frac{3}{2}Bu \tag{6.10}$$

where their domain is $G$, $A_1, A_2$ are linear, and $\Gamma_1, \Gamma_2$ are weak-weak continuous.
As to the Lagrangians, we take on $L_V^\alpha(\Omega_T) \times L_{V^*}^\beta(\Omega_T)$, the Lagrangian

$$L_1(u,p) = \mathcal{L} \oplus \Psi_\mu(u,p),$$

while on $L_H^2(\Omega_T) \times L_H^2(\Omega_T)$, we take

$$L_2(P,Q) = \mathbb{E}\int_0^T M(P(t,w), Q(t,w))\,dt,$$

where $M(P,Q) = \frac{1}{2}\|P\|_H^2 + \frac{1}{2}\|Q\|_H^2$.
In other words, we are considering the functional

$$\begin{aligned} I_\mu(u) &= \mathcal{L} \oplus \Psi_\mu(A_1u, \Gamma_1u) - \mathbb{E}\int_0^T \langle A_1u, \Gamma_1u\rangle dt + \mathbb{E}\int_0^T M(\Gamma_2u, A_2u) - \langle A_2u, \Gamma_2u\rangle\,dt \\ &= \mathcal{L} \oplus \Psi_\mu(u, -\tilde{u} - Ku) - \mathbb{E}\int_0^T \langle u, -\tilde{u} - Ku\rangle\,dt \\ &\qquad + \mathbb{E}\int_0^T M(F_u/2, -F_u + 3Bu/2) - \langle F_u/2, -F_u + 3Bu/2\rangle\,dt. \end{aligned}$$

We now verify the conditions of Theorem 2.5.

$$G_0 = \mathrm{Ker}(A_2) \cap G = \Big\{ u \in \mathcal{Y}_V^\alpha;\ u(t) = u_0 + \int_0^t \tilde{u}(s)ds,\ \text{for some } \tilde{u} \in L^\beta_{V^*}(\Omega_T) \Big\}.$$

It is clear that $A_1(G_0)$ is dense in $L^\alpha(\Omega_T; V)$. Moreover, $A_2(G)$ is dense in $L^2(\Omega_T; H)$. To check the upper semi-continuity of

$$u \to \mathbb{E} \int_0^T \langle A_1 u, \Gamma_1 u\rangle + \langle A_2 u, \Gamma_2 u\rangle \, dt$$

on $\mathcal{Y}_V^\alpha$ equipped with the weak topology, we apply Itô's formula to obtain that

$$\begin{aligned}\mathbb{E} \int_0^T \langle A_1 u, \Gamma_1 u\rangle + \langle A_2 u, \Gamma_2 u\rangle \, dt &= \mathbb{E} \int_0^T \langle u, -\tilde{u} - Ku\rangle + \langle F_u/2, -F_u + 3Bu/2\rangle \, dt \\ &= \frac{1}{2}\, \mathbb{E}\, \|u_0\|_H^2 - \frac{1}{2}\, \mathbb{E}\, \|u(T)\|_H^2 - \|u\|^{\alpha+1}_{L^\alpha_V(\Omega_T)} \\ &\quad + \frac{3}{4}\, \mathbb{E} \int_0^T \langle F_u(t), Bu(t)\rangle dt.\end{aligned}$$

Upper semi-continuity then follows from the compactness of the maps $\mathcal{Y}_V^\alpha \to L^2(\Omega; H)$ given by $u \mapsto (u(0), u(T))$, as well as the weak to norm continuity of $B$, which makes the functional $u \mapsto \mathbb{E} \int_0^T \langle F_u, Bu\rangle dt$ weakly continuous.

To verify the coercivity, we note first that condition (6.2) implies that for some (different) $C_1 > 0$,

$$H_{\mathcal{L}}(0, u) \geq C_1 \Big( \|u\|^\alpha_{L^\alpha_V(\Omega_T)} - 1 \Big).$$

By also taking into account condition (6.6) on $B$, with the fact that $\delta < \frac{\alpha+1}{2}$, we get that

$$\begin{aligned}&H_{\mathcal{L}}(0, u) + \mu\, \psi(u) + \mathbb{E} \int_0^T \langle u, \tilde{u} + Ku\rangle \, dt + \mathbb{E} \int_0^T H_M(0, F_u/2) - \langle F_u/2, -F_u + 3Bu/2\rangle \, dt \\ &\quad = H_{\mathcal{L}}(0, u) + \frac{\mu}{\beta} \|\tilde{u}\|^\beta_{L^\beta_{V^*}(\Omega_T)} - \frac{1}{2}\, \|u_0\|^2_{L^2(\Omega;H)} + \frac{1}{2}\, \|u(T)\|^2_{L^2(\Omega;H)} + \|u\|^{\alpha+1}_{L^\alpha_V(\Omega_T)} \\ &\qquad + \frac{1}{8} \|F_u(t)\|^2_{L^2(\Omega_T;H)} - \frac{3}{4}\, \mathbb{E} \int_0^T \langle F_u(t), Bu(t)\rangle \, dt \\ &\quad \geq C_1 \Big( \|u\|^\alpha_{L^\alpha_V(\Omega_T)} - 1 \Big) + \frac{\mu}{\beta}\, \|\tilde{u}\|^\beta_{L^\beta_{V^*}(\Omega_T)} + \|u\|^{\alpha+1}_{L^\alpha_V(\Omega_T)} \\ &\qquad + C_2 \Big( \|F_u(t)\|^2_{L^2_H(\Omega_T)} - \|F_u\|_{L^2_H(\Omega_T)} \|Bu\|_{L^2_H(\Omega_T)} \Big) + C \\ &\quad \geq C_1 \Big( \|u\|^\alpha_{L^\alpha_V(\Omega_T)} - 1 \Big) + \frac{\mu}{\beta}\, \|\tilde{u}\|^\beta_{L^\beta_{V^*}(\Omega_T)} + \|u\|^{\alpha+1}_{L^\alpha_V(\Omega_T)} \\ &\qquad + C_2 \Big( \|F_u(t)\|^2_{L^2_H(\Omega_T)} - \|F_u\|_{L^2_H(\Omega_T)} \|u\|^\delta_{L^2_H(\Omega_T)} \Big) + C \\ &\quad \geq \frac{\mu}{\beta}\, \|\tilde{u}\|^\beta_{L^\beta_{V^*}(\Omega_T)} + \|u\|^{\alpha+1}_{L^\alpha_V(\Omega_T)} \Big( 1 + o(\|u\|_{L^\alpha_V(\Omega_T)}) \Big) + C_2 \|F_u(t)\|^2_{L^2_H(\Omega_T)}.\end{aligned}$$

Therefore, by Theorem 2.5, there exists $u_\mu \in G \subset \mathcal{Y}_V^\alpha$ such that $I_\mu(u_\mu) = 0$, i.e.

$$\begin{aligned}0 &= \mathcal{L} \oplus \Psi_\mu(u_\mu, -\tilde{u}_\mu - Ku_\mu) - \mathbb{E} \int_0^T \langle u_\mu, -\tilde{u}_\mu - Ku_\mu\rangle \, dt \\ &\quad + \mathbb{E} \int_0^T M(\frac{1}{2} F_{u_\mu}, -F_{u_\mu} + \frac{3}{2} Bu_\mu) - \langle \frac{1}{2} F_{u_\mu}, -F_{u_\mu} + \frac{3}{2} Bu_\mu\rangle \, dt.\end{aligned}$$

Since $\mathcal{L} \oplus \Psi_\mu$ is convex and coercive in the second variable, there exists $\bar{r} \in L^\beta_{V^*}(\Omega_T)$ such that

$$\mathcal{L} \oplus \Psi_\mu(u_\mu, -\tilde{u}_\mu - Ku_\mu) = \mathcal{L}(u_\mu, \bar{r}) + \Psi_\mu(u_\mu, -\tilde{u}_\mu - Ku_\mu - \bar{r}),$$

hence

$$\begin{aligned}0 = \mathcal{L}(u_\mu, \bar{r}) - \langle u_\mu, \bar{r}\rangle + \Psi_\mu(u_\mu, -\tilde{u}_\mu - Ku_\mu - \bar{r}) + \mathbb{E}\int_0^T \langle u_\mu, \tilde{u}_\mu + Ku_\mu + \bar{r}\rangle\, dt \\ + \mathbb{E}\int_0^T M(\frac{1}{2}F_{u_\mu}, -F_{u_\mu} + \frac{3}{2}Bu_\mu) - \langle \frac{1}{2}F_{u_\mu}, -F_{u_\mu} + \frac{3}{2}Bu_\mu\rangle\, dt.\end{aligned}$$

Due to the self-duality of $\mathcal{L}$, $\Psi_\mu$ and $M$, this becomes the sum of three non-negative terms, and therefore

$$\begin{aligned}\mathcal{L}(u_\mu, \bar{r}) - \mathbb{E}\int_0^T \langle u_\mu(t), \bar{r}(t)\rangle dt = 0, \\ \Psi_\mu(u_\mu, -\tilde{u}_\mu - Ku_\mu - \bar{r}) + \mathbb{E}\int_0^T \langle u_\mu(t), \tilde{u}_\mu(t) + Ku_\mu(t) + \bar{r}(t)\rangle\, dt = 0, \\ \mathbb{E}\int_0^T M(\frac{1}{2}F_{u_\mu}(t), -F_{u_\mu}(t) + \frac{3}{2}Bu_\mu(t)) - \langle \frac{1}{2}F_{u_\mu}(t), -F_{u_\mu}(t) + \frac{3}{2}Bu_\mu(t)\rangle\, dt = 0.\end{aligned}$$

By the limiting case of Legendre duality, this yields

$$\begin{aligned}\tilde{u}_\mu + Ku_\mu + \mu\, \partial\psi(u_\mu) \in -\bar{\partial}\mathcal{L}(u_\mu) \qquad (6.11) \\ -F_{u_\mu}(t) + \frac{3}{2}Bu_\mu(t) \in \bar{\partial}M(t, \frac{1}{2}F_{u_\mu}(t)) = \frac{1}{2}F_{u_\mu}(t).\end{aligned}$$

The second line implies that for a.e. $t \in [0, T]$ we have $\mathbb{P}$-a.s. $F_{u_\mu} = Bu_\mu$. Moreover, from (6.11) we have that $\partial\psi(u_\mu) \in L^\beta_{V^*}(\Omega_T)$.
Now for an arbitrary process $v \in \mathcal{Y}^\alpha_V$ of the form $v(t) = v(0) + \int_0^t \tilde{v}(s)ds + \int_0^t F_v(s)dW(s)$, we have $\langle \partial\psi(u_\mu(t)), v\rangle = \langle \|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}, \tilde{v}\rangle$. Applying Ito's formula with the progressively measurable process $X(t) := \|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}$, we obtain

$$\begin{aligned}\mathbb{E}\int_0^T \langle \|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu(t), \tilde{v}(t)\rangle = -\mathbb{E}\int_0^T \langle \frac{d}{dt}(\|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu), v(t)\rangle \\ + \mathbb{E}\,\langle \|\tilde{u}_\mu(T)\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu(T), v(T)\rangle \\ - \mathbb{E}\,\langle \|\tilde{u}_\mu(0)\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu(0), v(0)\rangle, \qquad (6.12)\end{aligned}$$

which, in view of (6.11), implies that

$$\begin{aligned}0 &= \mathbb{E}\int_0^T \Big[\langle \tilde{u}_\mu(t) + Ku_\mu(t) + \bar{\partial}\mathcal{L}(u_\mu), v\rangle + \mu\langle \|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu, \tilde{v}\rangle\Big]\, dt \\ &= \mathbb{E}\int_0^T \Big\langle \tilde{u}_\mu(t) + Ku_\mu(t) + \bar{\partial}\mathcal{L}(u_\mu) - \mu\frac{d}{dt}(\|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu), v\Big\rangle\, dt \\ &\quad + \mu\,\mathbb{E}\,\langle \|\tilde{u}_\mu(T)\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu(T), v(T)\rangle - \mu\,\mathbb{E}\,\langle \|\tilde{u}_\mu(0)\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu(0), v(0)\rangle,\end{aligned}$$

hence $\tilde{u}_\mu(T) = \tilde{u}_\mu(0) = 0$ and $\tilde{u}_\mu + Ku_\mu - \mu\frac{d}{dt}(\|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu) \in -\bar{\partial}\mathcal{L}(u_\mu)$. □

In the following lemma, we shall remove the regularizing term $\mu\partial\psi$.

**Lemma 6.3.** *Under the above assumptions on $\mathcal{L}$ and $B$, there exists $u \in \mathcal{Y}^\alpha_V$ with $u(0) = u_0$, such that*

$$\begin{aligned}\mathcal{L}(u, -\tilde{u} - Ku) + \mathbb{E}\int_0^T \langle u(t), \tilde{u}(t) + Ku(t)\rangle\, dt = 0, \\ F_u = Bu.\end{aligned}$$

*Proof.* Lemma 6.2 yields that for every $\mu > 0$ there exist $u_\mu \in \mathcal{Y}_V^\alpha$ such that $u_\mu(0) = u_0$, $\tilde{u}_\mu(T) = \tilde{u}_\mu(0) = 0$, and satisfying

$$\begin{aligned} \tilde{u}_\mu + K u_\mu + \mu\, \partial\psi(u_\mu) &\in -\bar{\partial}\mathcal{L}(u_\mu) \\ F_{u_\mu}(t) &= B u_\mu(t). \end{aligned} \tag{6.13}$$

Now we show that $u_\mu$ is bounded in $\mathcal{Y}_V^\alpha$ with bounds independent of $\mu$. Indeed, multiplying (6.13) by $u_\mu$ and integrating over $\Omega \times [0, T]$, we obtain

$$\mathbb{E}\int_0^T \Big\langle \tilde{u}_\mu(t) + K u_\mu(t) + \mu\, \partial\psi(u_\mu(t)), u_\mu \Big\rangle = -\mathbb{E}\int_0^T \langle \bar{\partial}\mathcal{L}(u_\mu), u_\mu\rangle dt.$$

Apply Itô's formula and use the fact that $\mathbb{E}\int_0^T \langle \mu\, \partial\psi(u_\mu(t)), u_\mu\rangle\, dt \geq 0$ to get

$$\begin{aligned} -\frac{1}{2}\, \|u_{\mu,0}\|^2_{L^2(\Omega;H)} + \frac{1}{2}\, \|u_\mu(T)\|^2_{L^2(\Omega;H)} - \frac{1}{2}\|F_{u_\mu}\|^2_{L^2_H(\Omega_T)} + \|u_\mu\|^{\alpha+1}_{L^\alpha_V(\Omega_T)} \\ = -\mathbb{E}\int_0^T \langle \mu\, \partial\psi(u_\mu) + \bar{\partial}\mathcal{L}(u_\mu), u_\mu\rangle\, dt \\ \leq -\mathbb{E}\int_0^T \langle \bar{\partial}\mathcal{L}(u_\mu), u_\mu\rangle\, dt. \end{aligned}$$

Since for $u_\mu \in \mathcal{Y}_V^\alpha$ we have $u_\mu \in L^\infty(0, T; H)$, then in view of (6.3), we get

$$\begin{aligned} C_1 + \|u_\mu\|^{\alpha+1}_{L^\alpha_V(\Omega_T)} &\leq \|\bar{\partial}\mathcal{L}(u_\mu)\|_{L^\beta_{V^*}(\Omega_T)} \|u_\mu\|_{L^\alpha_V(\Omega_T)} \\ &\leq C\, \|u_\mu\|^2_{L^\alpha_V(\Omega_T)}. \end{aligned}$$

The above inequality implies that $\|u_\mu\|_{L^\alpha_V(\Omega_T)}$ is bounded.
Next, we multiply (6.13) by $D^{-1}\tilde{u}_\mu$ and integrate over $\Omega_T$ to get that

$$0 = \mathbb{E}\int_0^T \Big\langle \tilde{u}_\mu(t) + K u_\mu(t) + \mu\, \partial\psi(u_\mu(t)) + \bar{\partial}\mathcal{L}(t, u_\mu), D^{-1}\tilde{u}_\mu \Big\rangle\, dt$$

From (6.12), and choosing $v = \|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu$ with $\tilde{v} = \frac{d}{dt}(\|\tilde{u}_\mu\|^{\beta-2}_{V^*} D^{-1}\tilde{u}_\mu)$ and $F_v = 0$, we get that $\mathbb{E}\int_0^T \langle \partial\psi(u_\mu(t)), D^{-1}\tilde{u}_\mu\rangle\, dt = 0$, which together with condition(6.3) imply that

$$\|\tilde{u}_\mu\|^2_{L^\beta_{V^*}(\Omega_T)} \leq \|K u_\mu\|_{L^\beta_{V^*}(\Omega_T)} \|\tilde{u}_\mu\|_{L^\beta_{V^*}(\Omega_T)} + C\, \|u_\mu\|_{L^\alpha_V(\Omega_T)} \|\tilde{u}_\mu\|_{L^\beta_{V^*}(\Omega_T)},$$

hence

$$\|\tilde{u}_\mu\|_{L^\beta_{V^*}(\Omega_T)} \leq \|K u_\mu\|_{L^\beta_{V^*}(\Omega_T)} + C\, \|u_\mu\|_{L^\alpha_V(\Omega_T)}$$

which means that $\|\tilde{u}_\mu\|_{L^\beta_{V^*}(\Omega_T)}$ is bounded. From (6.6) and since $F_{u_\mu} = B u_\mu$ we deduce that $\|F_{u_\mu}\|_{L^2_H(\Omega_T)}$ is also bounded. Now since $(u_\mu)_\mu$ is bounded in $\mathcal{Y}_V^\alpha$, there exists $u \in \mathcal{Y}_V^\alpha$ such that $u_\mu \rightharpoonup u$ weakly in $\mathcal{Y}_V^\alpha$, which means that $u_\mu \rightharpoonup u$ weakly in $L^\alpha_V(\Omega_T)$, $\tilde{u}_\mu \rightharpoonup \tilde{u}$ weakly in $L^\beta_{V^*}(\Omega_T)$, and $F_{u_\mu} \rightharpoonup F_u$ weakly in $L^2_H(\Omega_T)$. From (6.13) and since $B$ is weak-norm continuous we have $F_u = Bu$. Then, by (6.11) we obtain

$$\begin{aligned} 0 &= \mathcal{L}(u_\mu, -\tilde{u}_\mu - K u_\mu - \mu\, \partial\psi(u_\mu)) \\ &\quad + \mathbb{E}\int_0^T \Big\langle u_\mu(t), \tilde{u}_\mu(t) + K u_\mu(t) + \mu\, \partial\psi(u_\mu(t) \Big\rangle\, dt \\ &\geq \mathcal{L}(u_\mu, -\tilde{u}_\mu - K u_\mu - \mu\, \partial\psi(u_\mu)) + \mathbb{E}\int_0^T \langle u_\mu(t), \tilde{u}_\mu(t) + K u_\mu(t)\rangle\, dt. \end{aligned}$$

Since $K$ is weak-to-weak continuous, $\langle \partial\psi(u_\mu)), u_\mu\rangle = \|\tilde{u}_\mu\|^\beta_{L^\beta_{V^*}}$ is uniformly bounded, and $\mathcal{L}$ is weakly lower semi-continuous on $L^\alpha_V \times L^\beta_{V^*}$, we get

$$\begin{aligned}0 &\geq \liminf_{\mu\to 0} \mathcal{L}(u_\mu, -\tilde{u}_\mu - Ku_\mu - \mu\,\partial\psi(u_\mu)) + \mathbb{E}\int_0^T \langle u_\mu(t), \tilde{u}_\mu(t) + Ku_\mu(t)\rangle\, dt \\ &\geq \mathcal{L}(u, -\tilde{u} - Ku) + \mathbb{E}\int_0^T \langle u(t), \tilde{u}(t) + Ku(t)\rangle\, dt.\end{aligned}$$

Since $\mathcal{L}$ is a self-dual Lagrangian on $L^\alpha_V \times L^\beta_{V^*}$, the reverse inequality is always true, and therefore

$$\mathcal{L}(u, -\tilde{u} - Ku) + \mathbb{E}\int_0^T \langle u(t), \tilde{u}(t) + Ku(t)\rangle\, dt = 0.$$

□

## 6.2 A general existence result

We shall work toward eliminating the perturbation $K$. By Lemma 6.3, for each $\varepsilon > 0$, there exists a $u_\varepsilon \in G$ such that $F_{u_\varepsilon} = Bu_\varepsilon$ and

$$\mathcal{L}(u_\varepsilon, -\tilde{u}_\varepsilon - \varepsilon Ku_\varepsilon) + \mathbb{E}\int_0^T \langle u_\varepsilon(t), \tilde{u}_\varepsilon(t) + \varepsilon Ku_\varepsilon(t)\rangle\, dt = 0, \tag{6.14}$$

or equivalently

$$\tilde{u}_\varepsilon + \varepsilon Ku_\varepsilon \in -\bar{\partial}\mathcal{L}(u_\varepsilon). \tag{6.15}$$

Similar to the argument in Lemma 6.3 we show that $u_\varepsilon$ is bounded in $\mathcal{Y}^\alpha_V$ with bounds independent of $\varepsilon$. First, we multiply (6.15) by $u_\varepsilon$ and integrate over $\Omega_T$ to obtain

$$\begin{aligned}\mathbb{E}\int_0^T \langle \tilde{u}_\varepsilon(t) + \varepsilon Ku_\varepsilon(t), u_\varepsilon(t)\rangle\, dt &= -\mathbb{E}\int_0^T \langle \bar{\partial}\mathcal{L}(u_\varepsilon), u_\varepsilon\rangle\, dt \\ &\leq \|\bar{\partial}\mathcal{L}(u_\varepsilon)\|_{L^\beta_{V^*}(\Omega_T)} \|u_\varepsilon\|_{L^\alpha_V(\Omega_T)} \\ &\leq C\,\|u_\varepsilon\|^2_{L^\alpha_V(\Omega_T)},\end{aligned}$$

where we used (6.3). In view of (6.14) and (6.2), this implies that

$$C(\|u_\varepsilon\|^\alpha_{L^\alpha_V(\Omega_T)} - 1) \leq \mathcal{L}(u_\varepsilon, -\tilde{u}_\varepsilon - \varepsilon Ku_\varepsilon) \leq C\,\|u_\varepsilon\|^2_{L^\alpha_V(\Omega_T)},$$

from which we deduce that $u_\varepsilon$ is bounded in $L^\alpha_V(\Omega_T)$. Next, we multiply (6.15) by $D^{-1}\tilde{u}_\varepsilon$ to obtain

$$\mathbb{E}\int_0^T \langle \tilde{u}_\varepsilon(t) + \varepsilon Ku_\varepsilon(t), D^{-1}\tilde{u}_\varepsilon(t)\rangle = -\mathbb{E}\int_0^T \langle \bar{\partial}\mathcal{L}(u_\varepsilon), D^{-1}\tilde{u}_\varepsilon(t)\rangle\, dt,$$

and therefore similar to the reasoning as in Lemma 6.3 we deduce that

$$\|\tilde{u}_\varepsilon\|_{L^\beta_{V^*}(\Omega_T)} \leq \|Ku_\mu\|_{L^\beta_{V^*}(\Omega_T)} + C\,\|u_\mu\|_{L^\alpha_V(\Omega_T)}.$$

Hence $\tilde{u}_\varepsilon$ is bounded in $L^\beta_{V^*}(\Omega_T)$, and there exists $u \in \mathcal{Y}^\alpha_V$ such that $u_\varepsilon \rightharpoonup u$ weakly in $L^\alpha_V(\Omega_T)$, and $\tilde{u}_\varepsilon \rightharpoonup \tilde{u}$ weakly in $L^\beta_{V^*}(\Omega_T)$, and $F_{u_\varepsilon} \rightharpoonup F_u$ weakly in $L^2_H(\Omega_T)$. Moreover,

$$\begin{aligned}0 &= \mathcal{L}(u_\varepsilon, -\tilde{u}_\varepsilon - \varepsilon Ku_\varepsilon)) + \mathbb{E}\int_0^T \langle u_\varepsilon(t), \tilde{u}_\varepsilon(t) + Ku_\varepsilon(t)\rangle\, dt \\ &\geq \mathcal{L}(u_\varepsilon, -\tilde{u}_\varepsilon - \varepsilon Ku_\varepsilon) + \mathbb{E}\int_0^T \langle u_\varepsilon(t), \tilde{u}_\varepsilon(t)\rangle\, dt.\end{aligned}$$

Again, $\mathcal{L}$ is weakly lower semi-continuous on $L^\alpha_V \times L^\beta_{V^*}$, therefore by letting $\varepsilon \to 0$ we get

$$0 \geq \mathcal{L}(u, -\tilde{u}) + \mathbb{E}\int_0^T \langle u(t), \tilde{u}(t)\rangle \, dt.$$

Since the reverse inequality is always true we have

$$\mathcal{L}(u, -\tilde{u}) + \mathbb{E}\int_0^T \langle u(t), \tilde{u}(t)\rangle \, dt = 0,$$

and also $F_u(t) = Bu(t)$. By the limiting case of Legendre duality, we now have for a.e. $t \in [0,T]$, $\mathbb{P}$-a.s. $\tilde{u} \in -\bar{\partial}\mathcal{L}(u)$, integrating over $[0,t]$ with the fact that $\int_0^t \tilde{u}(s)ds = u(t) - u_0 - \int_0^t F_u(s)dW(s)$, and $F_u(t) = Bu(t)$ we obtain

$$u(t) = u_0 - \int_0^t \bar{\partial}\mathcal{L}(u)(s)ds + \int_0^t B(u(s))dW(s).$$

# 7 Non-additive noise driven by monotone vector fields

## 7.1 Non-additive noise driven by gradients of convex energies

The first immediate application is the following case when the equation is driven by the gradient of a convex function.

**Theorem 7.1.** *Let $V \subset H \subset V^*$ be a Gelfand triple, and let $\phi : V \to \mathbb{R} \cup \{+\infty\}$ be an $\Omega_T$-dependent convex lower semi-continuous function on $V$ such that for $\alpha > 1$ and some constants $C_1, C_2 > 0$, for every $t \in [0,T]$, $\mathbb{P}$-a.s. we have*

$$C_2(\|u\|^\alpha_{L^\alpha_V(\Omega_T)} - 1) \leq \mathbb{E}\int_0^T \phi(t, u(t))\, dt \leq C_1(1 + \|u\|^\alpha_{L^\alpha_V(\Omega_T)}).$$

*Consider the equation*

$$\begin{cases} du(t) = -\partial\phi(t, u(t)dt + B(u(t))\, dW(t) \\ u(0) = u_0, \end{cases} \tag{7.1}$$

*where $B : \mathcal{Y}^\alpha_V \to L^2(\Omega_T; H)$ is a weak-to-norm continuous map satisfying for some $C > 0$ and $0 < \delta < \frac{\alpha+1}{2}$,*

$$\|Bu\|_{L^2_H(\Omega_T)} \leq C\|u\|^\delta_{L^\alpha(\Omega_T)} \quad \text{for any } u \in \mathcal{Y}^\alpha_V.$$

*Let $u_0$ be a random variable in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, then Equation (7.1) has a solution $u$ in $\mathcal{Y}^\alpha_V$.*

*Proof.* It suffices to apply Theorem 6.1 to the self-dual Lagrangian

$$\mathcal{L}(u,p) = \mathbb{E}\int_0^T \phi(t, u(t,w)) + \phi^*(t, p(t,w))\, dt.$$

□

**Example 7.2.** Let $D \subset \mathbb{R}^n$ be an open bounded domain, then the SPDE

$$\begin{cases} du(t) = \Delta u\, dt + |u|^{q-1}u\, dW \\ u(0) = u_0. \end{cases} \tag{7.2}$$

has a solution provided $\frac{1}{2} \leq q < \frac{n}{n-2}$.

*Proof.* Applying Theorem 6.1 with $\alpha = 2$, $V = H_0^1(D), H = L^2(D)$, $\varphi(u) = \frac{1}{2}\int_D |\nabla u|^2\,dx$ and $Bu = |u|^{q-1}u$, we see that $B$ is weak-to-norm continuous from $\mathcal{Y}_V^2$ to $L^2(\Omega_T; L^2(D))$, as long as $2q < 2^*$, that is $q < \frac{n}{n-2}$. As to the second condition on $B$, one notes that

$$\|Bu\|_{L^2_H(\Omega_T)} = \left(\mathbb{E}\int_0^T \|u^q\|^2_{L^2(D)}\,dt\right)^{\frac{1}{2}} \leq C\|u\|^{\frac{1}{2}}_{L^2_V}\ \|u\|^{q-\frac{1}{2}}_{L_V^{4q-2}},$$

which means that if $\frac{1}{2} \leq q \leq 1$, then $0 \leq 4q-2 \leq 2$ and

$$\|Bu\|_{L^2_H(\Omega_T)} \leq C\|u\|^{\frac{1}{2}}_{L^2_V}\ \|u\|^{q-\frac{1}{2}}_{L_V^{4q-2}} \leq C\|u\|^{\frac{1}{2}}_{L^2_V}\ \|u\|^{q-\frac{1}{2}}_{L^2_V} \leq C\|u\|^q_{L^2_V},$$

which is the condition required by the above theorem. Note that here, $\delta = q < \frac{3}{2} = \frac{\alpha+1}{2}$. On the other hand, if $1 < q$, then we apply the theorem with $\alpha = 4q-2$, then the above computation gives that

$$\|Bu\|_{L^2_H(\Omega_T)} \leq C\|u\|^q_{L_V^{4q-2}},$$

since $2 < 4q-2$. Note also that $q < 2q - \frac{1}{2} = \frac{\alpha+1}{2}$. However, the Lagrangian (here the convex function $\varphi$) is not coercive on the space $\mathcal{A}_V^\alpha = \mathcal{A}_V^{4q-2}$. To remedy this, we add a perturbation that makes the Lagrangian coercive on this space by considering the convex function

$$\varphi_\epsilon(u) = \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{\epsilon}{4q-2}\int_D |\nabla u|^{4q-2}\,dx.$$

By applying Theorem 6.1 with $\alpha = 4q-2$, $V = H_0^1(D), H = L^2(D)$, and $\varphi_\epsilon$, we get a solution $u_\epsilon$ for the equation

$$\begin{cases} du(t) = (\Delta u + \epsilon\Delta_{4q-2}u)\,dt + |u|^{q-1}u\,dW \\ u(0) = u_0. \end{cases} \tag{7.3}$$

An argument like what we have already done (twice) above, then allows us to let $\epsilon$ go to zero and get a solution for (7.2). $\square$

## 7.2 Non-additive noise driven by general monotone vector fields

More generally, consider the following type of equations

$$\begin{cases} du(t) = -A(t,u(t))dt + B(t,u(t))dW(t) \\ u(0) = u_0, \end{cases} \tag{7.4}$$

where $V \subset H \subset V^*$ is a Gelfand triple, and $A : \Omega\times[0,T]\times V \to V^*$, and $B : \Omega\times[0,T]\times V \to H$, are progressively measurable.

**Theorem 7.3.** *Assume $A : D(A) \subset V \to V^*$ is a progressively measurable $\Omega_T$-dependent maximal monotone operator satisfying condition (3.1) with $\alpha > 1$ and its conjugate $\beta$, as well as*

$$\|A_{w,t}u\|_{V^*} \leq k(\omega,t)(1+\|u\|_V) \quad \textit{for all } u \in V,\ dt\otimes\mathbb{P} \textit{ a.s.} \tag{7.5}$$

*for some $k \in L^\infty(\Omega_T)$.*
*Let $B : \mathcal{Y}_V^\alpha \to L^2(\Omega_T; H)$ be a weak-to-norm continuous map such that for some $C > 0$ and $0 < \delta < \frac{\alpha+1}{2}$,*

$$\|Bu\|_{L^2_H(\Omega_T)} \leq C\|u\|^\delta_{L^\alpha(\Omega_T)} \quad \textit{for any } u \in \mathcal{Y}_V^\alpha.$$

*Let $u_0$ be a given random variable in $L^2_H(\Omega,\mathcal{F}_0,\mathbb{P};H)$, then equation (7.4) has a variational solution in $\mathcal{Y}_V^\alpha$.*

*Proof.* Associate again to $A_{\omega,t}$ an $\Omega_T$-dependent self-dual Lagrangian $L_{A_{\omega,t}}(u,p)$ on $V \times V^*$ in such a way that for almost every $t \in [0,T]$, $\mathbb{P}$-a.s. we have $A_{\omega,t} = \bar{\partial} L_{A_{\omega,t}}$. Then by Lemma 3.3, the Lagrangian

$$\mathcal{L}_A(u,p) = \mathbb{E}\int_0^T L_{A_{\omega,t}}(u(\omega,t), p(\omega,t))dt$$

is self-dual on $L^\alpha(\Omega_T;V) \times L^\beta(\Omega_T;V^*)$, and satisfies

$$C_1(\|u\|^\alpha_{L^\alpha_V(\Omega_T)} + \|p\|^\beta_{L^\beta_{V^*}(\Omega_T)} - 1) \leq \mathcal{L}(u,p) \leq C_2(1 + \|u\|^\alpha_{L^\alpha_V(\Omega_T)} + \|p\|^\beta_{L^\beta_{V^*}(\Omega_T)}).$$

(7.5) also implies that for some $C_3 > 0$,

$$\|\bar{\partial}\mathcal{L}_{\mathcal{A}}(u)\|_{L^\beta_{V^*}(\Omega_T)} \leq C_3(1 + \|u\|_{L^\alpha_V(\Omega_T)}).$$

The rest follows from Theorem 6.1. □

### 7.3 Non-additive noise driven by monotone vector fields in divergence form

We now show the existence of a variational solution to the following equation:

$$\begin{cases} du = \operatorname{div}(\beta(\nabla u(t,x)))dt + B(u(t))dW(t) & \text{in } [0,T] \times D \\ u(0,x) = u_0 & \text{on } \partial D, \end{cases} \tag{7.6}$$

where $D$ is a bounded domain in $\mathbb{R}^n$, and where the initial position $u_0$ belongs to $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(D))$. We assume that

1. The $\Omega_T$-dependent vector field $\beta : \mathbb{R}^n \to \mathbb{R}^n$ is progressively measurable and maximal monotone such that for functions $c_1, c_2, c_3 \in L^\infty(\Omega_T)$, and $m_1, m_2 \in L^1(\Omega_T)$, it satisfies $dt \otimes \mathbb{P}$-a.s.

   $$\langle \beta(x), x\rangle \geq \max\{c_1\|x\|^2_{\mathbb{R}^n} - m_1,\ c_2\|\beta(x)\|^2_{\mathbb{R}^n} - m_2\} \quad \text{for all } x \in \mathbb{R}^n, \tag{7.7}$$

   and

   $$\|\beta(x)\|_{\mathbb{R}^n} \leq c_3(1 + \|x\|_{\mathbb{R}^n}) \quad \text{for all } x \in \mathbb{R}^n, \tag{7.8}$$

2. The operator $B : \mathcal{Y}^2_{H^1_0(D)} \to L^2(\Omega_T; L^2(D))$ is a weak-to-norm continuous map such that for some $C > 0$ and $0 < \delta < \frac{\alpha+1}{2}$,

   $$\|Bu\|_{L^2_{L^2}(\Omega_T)} \leq C\|u\|^\delta_{L^\alpha_{H^1_0}(\Omega_T)} \quad \text{for any } u \in \mathcal{Y}^2_{H^1_0(D)}.$$

**Theorem 7.4.** *Under the above conditions on $\beta$ and $B$, Equation (7.6) has a variational solution.*

We shall need the following lemma, which associates to an $\Omega_T$-dependent self-dual Lagrangian on $\mathbb{R}^n \times \mathbb{R}^n$, a self-dual Lagrangian on $L^2(\Omega_T; H^1_0(D)) \times L^2(\Omega_T; H^{-1}(D))$.

**Lemma 7.5.** *Let $L$ be a $\Omega_T$-dependent self-dual Lagrangian on $\mathbb{R}^n \times \mathbb{R}^n$, then the Lagrangian defined by*

$$\mathscr{L}(u,p) = \inf\left\{\mathbb{E}\int_0^T \int_D L(t, \nabla u(t,x), f(t,x))\,dx\,dt;\ f \in L^2(\Omega_T; L^2_{\mathbb{R}^n}(D)), -\operatorname{div}(f) = p\right\}$$

*is self-dual on $L^2(\Omega_T; H^1_0(D)) \times L^2(\Omega_T; H^{-1}(D))$.*

We shall need the following general lemma.

**Lemma 7.6.** *Let $L$ be a self-dual Lagrangian on a Hilbert space $\mathcal{H} \times \mathcal{H}$, and let $\Pi : \mathcal{V} \to \mathcal{H}$ be a bounded linear operator from a reflexive Banach space $\mathcal{V}$ into $\mathcal{H}$ such that the operator $\Pi^*\Pi$ is an isomorphism from $\mathcal{V}$ into $\mathcal{V}^*$. Then, the Lagrangian*

$$\mathcal{L}(u,p) = \inf \{L(\Pi u, f);\ f \in \mathcal{H}, \Pi^*(f) = p\},$$

*is self-dual on $\mathcal{V} \times \mathcal{V}^*$.*

*Proof.* For a fixed $(q,v) \in \mathcal{V}^* \times \mathcal{V}$, write

$$\begin{aligned}
\mathcal{L}^*(q,v) &= \sup\Big\{\langle q,u\rangle + \langle v,p\rangle - \mathcal{L}(u,p);\ u \in \mathcal{V},\, p \in \mathcal{V}^*\Big\} \\
&= \sup\Big\{\langle q,u\rangle + \langle v,p\rangle - L(\Pi u,f);\ u \in \mathcal{V},\, p \in \mathcal{V}^*, f \in \mathcal{H}, \Pi^*(f) = p\Big\} \\
&= \sup\Big\{\langle q,u\rangle + \langle v,-\Pi^* f\rangle - L(\Pi u,f);\ u \in \mathcal{V}, f \in \mathcal{H}\Big\} \\
&= \sup\Big\{\langle q,u\rangle + \langle \Pi v,f\rangle - L(\Pi u,f);\ u \in \mathcal{V}, f \in \mathcal{H}\Big\}.
\end{aligned}$$

Since $\Pi^*\Pi$ is an isomorphism, for $q \in \mathcal{V}^*$ there exists a fixed $f_0 \in \mathcal{H}$ such that $\Pi^* f_0 = q$. Moreover, the space

$$\mathcal{E} = \{g \in \mathcal{H};\ g = \Pi u, \text{ for some } u \in \mathcal{V}\},$$

is closed in $\mathcal{H}$ in such a way that its indicator function $\chi_{\mathcal{E}}$ on $\mathcal{H}$

$$\chi_{\mathcal{E}}(g) = \begin{cases} 0 & g \in \mathcal{E} \\ +\infty & \text{elsewhere,} \end{cases}$$

is convex and lower semi-continuous. Its Legendre transform is then given for each $f \in \mathcal{H}$ by

$$\chi^*_{\mathcal{E}}(f) = \begin{cases} 0 & \Pi^* f = 0 \\ +\infty & \text{elsewhere.} \end{cases}$$

It follows that

$$\begin{aligned}
\mathcal{L}^*(q,v) &= \sup\Big\{\langle f_0,\Pi u\rangle + \langle \Pi v,f\rangle - L(\Pi u,f);\ u \in \mathcal{V}, f \in \mathcal{H}\Big\} \\
&= \sup\Big\{\langle f_0,g\rangle + \langle \Pi v,f\rangle - L(g,f) - \chi_{\mathcal{E}}(g);\ g \in \mathcal{H}, f \in \mathcal{H}\Big\} \\
&= (L + \chi_{\mathcal{E}})^*(f_0, \Pi v) \\
&= \inf\Big\{L^*(f_0 - r, \Pi v) + \chi^*_{\mathcal{E}}(r);\, r \in \mathcal{H}\Big\}
\end{aligned}$$

where we have used that the Legendre dual of the sum is inf-convolution. Finally taking into account the expression for $\chi^*_{\mathcal{E}}$ we obtain

$$\begin{aligned}
\mathcal{L}^*(q,v) &= \inf\Big\{L^*(f_0 - r, \Pi v);\, r \in \mathcal{H}, \Pi^* r = 0\Big\} \\
&= \inf\Big\{L(\Pi v, f_0 - r);\, r \in \mathcal{H}, \Pi^* r = 0\Big\} \\
&= \inf\Big\{L(\Pi v, f);\ f \in \mathcal{H}, \Pi^* f = q\Big\} \\
&= \mathcal{L}(v,q).
\end{aligned}$$

□

**Proof of Lemma 7.5:** This is now a direct application of Lemma 7.6. First, lift the random Lagrangian to define a self-dual Lagrangian on $L^2(\Omega_T; L^2(D;\mathbb{R}^n)) \times L^2(\Omega_T; L^2(D;\mathbb{R}^n))$, via

$$\mathcal{L}(u,p) = \mathbb{E}\int_0^T \int_D L(t, u(t,x), p(t,x))\, dx\, dt,$$

then use Lemma 7.6 with this Lagrangian and the operators

$$L^2(\Omega_T; H_0^1(D)) \xrightarrow{\Pi=\nabla} L^2(\Omega_T; L^2(D;\mathbb{R}^n)) \xrightarrow{\Pi^*=\nabla^*} L^2(\Omega_T; H^{-1}(D)),$$

to get that $\mathscr{L}$ is a self-dual Lagrangian on $L^2(\Omega_T; H_0^1(D)) \times L^2(\Omega_T; H^{-1}(D))$. Note that $\Pi^*\Pi = \nabla^*\nabla = -\Delta$ induces an isomorphism from $L^2(\Omega_T; H_0^1(D))$ to $L^2(\Omega_T; H^{-1}(D))$.

**Proof of Theorem 7.4:** Again, by Theorem 2.1 and the discussion in Section 3.1, one can associate to the maximal monotone map $\beta_{\omega,t}$, an $\Omega_T$-dependent self-dual Lagrangian $L_{\beta_{\omega,t}}(u,p)$ on $\mathbb{R}^n \times \mathbb{R}^n$ in such a way that

$$\beta_{\omega,t} = \bar{\partial} L_{\beta_{\omega,t}}.$$

If $\beta$ satisfies (7.7), then the $\Omega_T$-dependent self-dual Lagrangian $L_{\beta_{\omega,t}}$ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfy for almost every $t \in [0,T]$, $\mathbb{P}$-a.s.

$$C_1(\|x\|^2_{\mathbb{R}^n} + \|p\|^2_{\mathbb{R}^n} - n_1) \le L_{\beta_{w,t}}(x,p) \le C_2(\|x\|^2_{\mathbb{R}^n} + \|p\|^2_{\mathbb{R}^n} + n_2), \tag{7.9}$$

where $C_1, C_2 \in L^\infty(\Omega_T)$ and $n_1, n_2 \in L^1(\Omega_T)$.
We can then lift it to the space $L^2(\Omega_T; L^2_{\mathbb{R}^n}(D)) \times L^2(\Omega_T; L^2_{\mathbb{R}^n}(D))$ via

$$\mathcal{L}_\beta(u,p) = \mathbb{E}\int_0^T \int_D L_{\beta_{\omega,t}}(u(t,w,x), p(t,w,x))\, dx dt,$$

in such a way that for positive constants $C_1, C_2$ and $C_3$ (different from above)

$$C_2(\|u\|^2_{L^2_H(\Omega_T)} + \|p\|^2_{L^2_H(\Omega_T)} - 1) \le \mathcal{L}_\beta(u,p) \le C_1(1 + \|u\|^2_{L^2_H(\Omega_T)} + \|p\|^2_{L^2_H(\Omega_T)}),$$

where $H := L^2_{\mathbb{R}^n}(D)$. In view of (7.8), we also have

$$\|\bar{\partial}\mathcal{L}_\beta(u)\|_{L^2_H(\Omega_T)} \le C_3(1 + \|u\|_{L^2_H(\Omega_T)}).$$

Use now Lemma 7.5 to lift $\mathcal{L}_\beta$ to a self-dual Lagrangian $\mathscr{L}_\beta$ on $L^2(\Omega_T; H_0^1(D)) \times L^2(\Omega_T; H^{-1}(D))$, via the formula

$$\begin{aligned}
\mathscr{L}_\beta(u,p) &= \inf\left\{\mathbb{E}\int_0^T \int_D L_{\beta_{w,t}}(\nabla u(t,x), f(t,x))\, dx\, dt;\ f \in L^2(\Omega_T; L^2_{\mathbb{R}^n}(D)), -\mathrm{div}(f) = p\right\} \\
&= \inf\left\{\mathcal{L}_\beta(\nabla u, f);\ f \in L^2(\Omega_T; L^2_{\mathbb{R}^n}(D)), -\mathrm{div}(f) = p\right\}, \qquad (7.10)
\end{aligned}$$

Apply now Theorem 6.1 to get a process $v \in \mathcal{Y}^2_{H_0^1(D)}$ such that

$$\begin{aligned}
\mathscr{L}_\beta(v, -\tilde{v}) + \langle v, \tilde{v}\rangle &= 0 \\
F_v &= B \\
v(0) &= u_0.
\end{aligned}$$

Now note that

$$\begin{aligned}
0 &= \mathscr{L}_\beta(v, -\tilde{v}) + \langle v, \tilde{v}\rangle \\
&= \inf_{f \in L^2(\Omega_T; L^2_{\mathbb{R}^n}(D))}\left\{\mathbb{E}\int_0^T \int_D L_{\beta(w,t)}(\nabla v, f)\, dx\, dt; \mathrm{div}(f) = \tilde{v}\right\} + \mathbb{E}\int_0^T \langle v(t), \tilde{v}(t)\rangle_{H_0^1, H^{-1}} dt \\
&= \inf_{f \in L^2(\Omega_T; L^2_{\mathbb{R}^n}(D))}\left\{\mathbb{E}\int_0^T \int_D L_{\beta(w,t)}(\nabla v, f) - \langle \nabla v(x,t), f(x,t)\rangle\, dx\, dt\right\} \\
&= \inf_{f \in L^2(\Omega_T; L^2_{\mathbb{R}^n}(D))} J_v(f),
\end{aligned}$$

where

$$J_v(f) := \mathbb{E}\int_0^T \int_D \{L_{\beta(w,t)}(\nabla v, f) - \langle \nabla v(x,t), f(x,t)\rangle\}\, dx\, dt.$$

Note that condition (7.9) implies that $L(y,0) \le C(1+\|y\|^2_{\mathbb{R}^n})$, which means that $J_v$ is coercive on $L^2(\Omega_T; L^2_{\mathbb{R}^n}(D))$, thus there exists $\bar{f} \in L^2(\Omega_T; L^2_{\mathbb{R}^n}(D))$ with $\operatorname{div}(\bar{f}) = \tilde{v}$ such that

$$\mathbb{E}\int_0^T \int_D L_{\beta(w,t)}(\nabla v, \bar{f}) - \langle \nabla v(x,t), \bar{f}(x,t)\rangle\, dx\, dt = 0.$$

The self-duality of $L$ then implies that $\bar{f}(x,t) = \bar{\partial} L(\nabla v(x,t)) = \beta(\nabla v(x,t))$. Taking divergence leads to $\tilde{v} \in \operatorname{div}(\beta(\nabla v))$. Taking integrals over $[0,t]$ and using the fact that $v \in \mathcal{Y}^2_{H^1_0(D)}$ finally gives

$$\begin{aligned}\int_0^t \operatorname{div}(\beta(\nabla v(s)))\, ds &= \int_0^t \tilde{v}(s) ds = v(t) - v(0) - \int_0^t F_v(s) dW(s)\\ &= v(t) - u_0 - \int_0^t B(v(s)) dW,\end{aligned}$$

which completes the proof.

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