ON THE HARDY-SCHRÖDINGER OPERATOR WITH A BOUNDARY SINGULARITY

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Abstract. We investigate the Hardy-Schrödinger operator $L_\gamma = -\Delta - \frac{\gamma}{|x|^2}$ on smooth domains $\Omega \subset \mathbb{R}^n$, whose boundary contain the singularity 0. The situation is quite different from the well-studied case when 0 is in the interior of Ω. For one, if $0 \in \Omega$, then L_γ is positive if and only if $\gamma < \frac{(n-2)^2}{4}$, while if $0 \in \partial\Omega$ the operator L_γ could be positive for larger value of γ, potentially reaching the maximal constant $\frac{n^2}{4}$ on convex domains.

We prove optimal regularity and a Hopf-type Lemma for variational solutions of corresponding linear Dirichlet boundary value problems of the form $L_\gamma u = a(x)u$, but also for non-linear equations including $L_\gamma u = \frac{|u|^{2^\star(s)-2}u}{|x|^s}$, where $\gamma < \frac{n^2}{4}$, $s \in [0, 2)$ and $2^\star(s) := \frac{2(n-s)}{n-2}$ is the critical Hardy-Sobolev exponent. We also provide a Harnack inequality and a complete description of the profile of all positive solutions –variational or not– of the corresponding linear equation on the punctured domain. The value $\gamma = \frac{n^2-1}{4}$ turned out to be another critical threshold for the operator L_γ, and our analysis yields a corresponding notion of "Hardy singular boundary-mass" $m_\gamma(\Omega)$ of a domain Ω having $0 \in \partial\Omega$, which could be defined whenever $\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$.

As a byproduct, we give a complete answer to problems of existence of extremals for Hardy-Sobolev inequalities of the form

$$C\left(\int_\Omega \frac{u^{2^\star(s)}}{|x|^s}dx\right)^{\frac{2}{2^\star(s)}} \le \int_\Omega |\nabla u|^2 dx - \gamma\int_\Omega \frac{u^2}{|x|^2}dx \quad \text{for all } u \in D^{1,2}(\Omega),$$

whenever $\gamma < \frac{n^2}{4}$, and in particular, for those of Caffarelli-Kohn-Nirenberg. These results extend previous contributions by the authors in the case $\gamma = 0$, and by Chern-Lin for the case $\gamma < \frac{(n-2)^2}{4}$. Namely, if $0 \le \gamma \le \frac{n^2-1}{4}$, then the negativity of the mean curvature of $\partial\Omega$ at 0 is sufficient for the existence of extremals. This is however not sufficient for $\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$, which then requires the positivity of the Hardy singular boundary-mass of the domain under consideration.

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Contents

1. Introduction

Let Ω be a smooth domain of $\mathbb{R}^n$ (i.e. a C^∞ connected open set), define the best constant in the corresponding Hardy inequality by,

$$\gamma_H(\Omega) := \inf\left\{\frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega \frac{u^2}{|x|^2}\,dx};\ u\in D^{1,2}(\Omega)\setminus\{0\}\right\}, \tag{1.1}$$

where $D^{1,2}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm given by $||u||^2=\int_\Omega |\nabla u|^2 dx$. It is well known that $\gamma_H(\Omega)=\frac{(n-2)^2}{4}$ for any domain Ω having 0 in its interior, including $\mathbb{R}^n$, and that it is never attained by a function in $D^{1,2}(\Omega)$. On the other hand, it has been noted by several authors (See for example Pinchover-Tintarev [44] Fall-Musina [19] or the book of Ghoussoub-Moradifam [22]) that the situation is quite different for the half-space $\mathbb{R}^n_+ := \{x\in\mathbb{R}^n/\, x_1>0\}$, in which case, $\gamma_H(\mathbb{R}^n_+)=\frac{n^2}{4}$. More generally, if $0\in\partial\Omega$ the boundary of Ω, then $\gamma_H(\Omega)$ can be anywhere in the interval $\left(\frac{(n-2)^2}{4},\frac{n^2}{4}\right]$ (see Proposition 3.1). Moreover, $\gamma_H(\Omega)$ is attained whenever $\gamma_H(\Omega)<\frac{n^2}{4}$ (See Section 3). This already points to the fact that the Hardy-Schrödinger operator $L_\gamma=-\Delta-\frac{\gamma}{|x|^2}$ behaves differently when the singularity 0 is on the boundary of a domain Ω, than when 0 is in the interior. The latter case has already been extensively covered in the literature. Without being exhaustive, we refer to Ghoussoub-Yuan [26], Guerch-Véron [30], Jaber [34], Kang-Peng [35], Pucci-Servadei [45], Ruiz-Willem [47], Smets [50], and the references therein.

The study of nonlinear singular variational problems when $0\in\partial\Omega$ was initiated by Ghoussoub-Kang [21] and was studied extensively by Ghoussoub-Robert [23–25]. For more recent contributions, we refer to Attar-Merchán-Peral [1], Dávila-Peral [12], and Gmira-Véron [29]. We also learned recently –after a first version of this paper was posted on arxiv– about a paper of Pinchover [43], and a more recent one by Devyver-Fraas-Pinchover [13] that also treat the Hardy potential when $0\in\partial\Omega$.

Our main goal in this paper is to show that the above noted discrepancy – between the case when the singularity 0 belongs to the interior of the domain and when it is on the boundary– is only the tip of the iceberg. The differences manifest themselves in both linear problems of the form

$$\left\{\begin{array}{rcll} -\Delta u-\gamma\frac{u}{|x|^2} &=& a(x)u & \text{on } \Omega\\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.2}$$

and in nonlinear Dirichlet boundary value problems associated to L_γ, such as:

$$\left\{\begin{array}{rcll} -\Delta u-\gamma\frac{u}{|x|^2} &=& \frac{u^{2^\star(s)-1}}{|x|^s} & \text{on } \Omega\\ u &>& 0 & \text{on } \Omega\\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.3}$$

where $0\leq s<2$ and $2^\star(s):=\frac{2(n-s)}{n-2}$. Actually, Ghoussoub-Kang noted in [21] that even when $\gamma=0$, the situation can already be quite different whenever 0 belongs to the boundary of a bounded C^2-smooth domain Ω as long as $s>0$. Ghoussoub and Robert [23, 24] eventually proved that if the mean curvature at 0 of such domains is negative, and provided $s>0$, then minimizers for the functional

$$J_s^\Omega(u):=\frac{\int_\Omega|\nabla u|^2dx}{(\int_\Omega\frac{u^{2^\star}}{|x|^s}dx)^{\frac{2}{2^\star}}} \tag{1.4}$$

exist in $D^{1,2}(\Omega)\setminus\{0\}=H_0^1(\Omega)\setminus\{0\}$ and are solutions to equation (1.3) in the case when $\gamma=0$. While this new phenomenon occured because of the presence of the singularity $|x|^{-s}$ in the nonlinear term, we shall show in this paper, that the differences also appear on the linear level, as soon as $\gamma>0$, but also as one varies γ between 0 and $\frac{n^2}{4}$.

Another motivation for this work came from the recent work of C.S. Lin and his co-authors [9, 10] on the existence of extremals for the Caffarelli-Kohn-Nirenberg (CKN) inequalities [4]. These inequalities state that in dimension $n\geq 3$, there is a constant $C:=C(a,b,n)>0$ such that for all $u\in C_c^\infty(\mathbb{R}^n)$, the following inequality holds:

$$\left(\int_{\mathbb{R}^n}|x|^{-bq}|u|^q\right)^{\frac{2}{q}}\leq C\int_{\mathbb{R}^n}|x|^{-2a}|\nabla u|^2dx, \tag{1.5}$$

where

$$-\infty<a<\frac{n-2}{2},\quad 0\leq b-a\leq 1 \text{ and } q=\frac{2n}{n-2+2(b-a)}. \tag{1.6}$$

A proof and various extensions of (1.5) will be given in section 2.

For a domain Ω in $\mathbb{R}^n$, we let $D_a^{1,2}(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to the norm $||u||_a^2=\int_\Omega|x|^{-2a}|\nabla u|^2dx$. Consider the best constant defined as:

$$S(a,b,\Omega)=\inf\left\{\frac{\int_\Omega|x|^{-2a}|\nabla u|^2dx}{\left(\int_\Omega|x|^{-bq}|u|^q\right)^{\frac{2}{q}}\,dx};u\in D_a^{1,2}(\Omega)\backslash\{0\}\right\}. \tag{1.7}$$

The extremal functions for $S(a,b,\Omega)$ are then the least-energy solutions of the corresponding Euler-Lagrange equations:

$$\left\{\begin{array}{rll} -\mathrm{div}(|x|^{-2a}\nabla u) & = |x|^{-bq}u^{q-1} & \text{on } \Omega \\ u & > 0 & \text{on } \Omega \\ u & = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{1.8}$$

The existence or nonexistence of minimizers for (1.7), when the domain Ω is the whole space $\mathbb{R}^n$, have been extensively studied for the past twenty years, see Catrina-Wang [5], Chou-Chu [6], Dolbeault-Esteban-Loss-Tarantello [14], Lin-Wang [38] and references therein. The result can be briefly summarized in the following:

Theorem A: *Suppose $n \geq 3$ and that a, b and q satisfy condition (1.6). Then minimizers exist for the best constant $S(a;b;\mathbb{R}^n)$ if and only if a, b satisfy*

$$\text{either } a < b < a+1 \text{ or } b = a \geq 0. \tag{1.9}$$

If now Ω is any domain in $\mathbb{R}^n$ that contains 0 in its interior, one can easily see that scale invariance yields for any $\lambda > 0$, that $S(a;b;\lambda\Omega) = S(a;b;\Omega)$ where $\lambda\Omega = \{\lambda x; x \in \Omega\}$. It follows that if $0 \in \Omega$, then $S(a;b;\Omega) = S(a;b;\mathbb{R}^n)$, which means that $S(a;b;\Omega)$ can never be achieved unless $\Omega = \mathbb{R}^n$ (up to a set of capacity zero). However, as mentioned above, if 0 belongs to the boundary of a smooth bounded domain Ω and if the mean curvature at 0 of such domains is negative, then minimizers for the best constant $S(0;b;\Omega)$ were shown to be attained [21, 23, 24]. This result was later extended by Chern and Lin [10], who eventually established existence of minimizers under the same negative mean curvature condition at 0 provided a, b satisfy one of the following conditions:

$$\left\{\begin{array}{ll} \text{(i)} & a < b < a+1 \text{ and } n \geq 3 \\ \text{(ii)} & a = b \text{ and } n \geq 4. \end{array}\right. \tag{1.10}$$

They left open the case when $n = 3$ and $0 < a = b < \frac{n-2}{2}$, a problem that we address in Theorem 1.9 (see also Section 11).

To make the connection, we note that by making the substitution $w(x) = |x|^{-a}u(x)$ for $x \in \Omega$, one can see that if $a < \frac{n-2}{2}$, then $u \in D^{1,2}_a(\Omega)$ if and only if $w \in D^{1,2}(\Omega)$ by the Hardy inequality, and

$$\frac{\int_\Omega |x|^{-2a}|\nabla u|^2 dx}{\left(\int_\Omega |x|^{-bq}|u|^q\right)^{\frac{2}{q}}} = \frac{\int_\Omega |\nabla w|^2 - \gamma \int_\Omega \frac{w^2}{|x|^2} dx}{(\int_\Omega \frac{w^{2^\star}}{|x|^s} dx)^{\frac{2}{2^\star}}},$$

where $\gamma = a(n-2-a)$, $s = (b-a)q$ and $2^\star = \frac{2n}{n-2+2(b-a)}$. This means that u is a solution of (1.8) if and only if $w(x)$ is a solution of equation (1.3) where $0 \leq s < 2$ and $2^\star := 2^\star(s) = \frac{2(n-s)}{n-2}$. Therefore, instead of looking for solutions of (1.8) one can study equation (1.3). To state the result of Chern-Lin in this context, we define the functional

$$J^\Omega_{\gamma,s}(u) := \frac{\int_\Omega |\nabla u|^2 - \gamma \int_\Omega \frac{u^2}{|x|^2} dx}{(\int_\Omega \frac{u^{2^\star}}{|x|^s} dx)^{\frac{2}{2^\star}}}, \tag{1.11}$$

and its infimum on $D^{1,2}(\Omega) \setminus \{0\}$, that is

$$\mu_{\gamma,s}(\Omega) := \inf\left\{J^\Omega_{\gamma,s}(u); u \in D^{1,2}(\Omega) \setminus \{0\}\right\}. \tag{1.12}$$

Theorem 1.1 (Chern-Lin [10]). *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 3)$. Assume $\gamma < \frac{(n-2)^2}{4}$ and $0 \leq s < 2$. If either $\{s > 0\}$ or $\{n \geq 4$ and $\gamma > 0\}$, then there are extremals for $\mu_{\gamma,s}(\Omega)$, provided the mean curvature of $\partial\Omega$ at 0 is negative.*

The case when $n = 3$, $s = 0$ and $\gamma > 0$ was left open. As we shall see in section 4, the infimum $\mu_{\gamma,s}(\Omega)$ is finite for all $\gamma < \frac{n^2}{4}$, whenever $0 \in \partial\Omega$. This means that equation (1.3) may have positive solutions for γ beyond $\frac{(n-2)^2}{4}$ and all the way to $\frac{n^2}{4}$. This turned out to be the case as we shall establish in this paper.

We first note that standard compactness arguments [10, 21, 22] –also described in section 4– yield that for $\mu_{\gamma,s}(\Omega)$ to be attained it is sufficient to have that

$$\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+), \tag{1.13}$$

where the latter is the corresponding best constant on $\mathbb{R}^n_+$. In order to prove the existence of such a gap, one tries to construct test functions for $\mu_{\gamma,s}(\Omega)$ that are based on the extremals of $\mu_{\gamma,s}(\mathbb{R}^n_+)$ provided the latter exist. The cases where this is known are given by the following standard proposition (see for instance Bartsch-Peng-Zhang [3] and Chern-Lin [10]). See Corollary 12.2 in the appendix for a proof.

Proposition 1.2. *Assume $\gamma < \frac{n^2}{4}$, $n \geq 3$ and $0 \leq s < 2$. Then, there are extremals for $\mu_{\gamma,s}(\mathbb{R}^n_+)$ provided either $\{s > 0\}$ or $\{n \geq 4$ and $\gamma > 0\}$.*
On the other hand,

(1) *If $\{s = 0$ and $\gamma \leq 0\}$, then there are no extremals for $\mu_{\gamma,0}(\mathbb{R}^n_+)$ for any $n \geq 3$.*
(2) *Furthermore, whenever $\mu_{\gamma,0}(\mathbb{R}^n_+)$ has no extremals, then necessarily*

$$\mu_{\gamma,0}(\mathbb{R}^n_+) = \inf_{u \in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\star}\, dx\right)^{\frac{2}{2^\star}}} = \frac{1}{K(n,2)^2}, \tag{1.14}$$

where the latter is the best constant in the Sobolev inequality and $2^\star := 2^\star(0) = \frac{2n}{n-2}$.

The only unknown situation is again when $s = 0$, $n = 3$ and $\gamma > 0$, which we address below (see Theorem 1.9) and in full detail in Section 11. For now, we shall discuss the new ingredients that we bring to the discussion.

Assuming first that an extremal for $\mu_{\gamma,s}(\mathbb{R}^n_+)$ exists and that one knows its profile at infinity and at 0, then this information can be used to construct test functions for $\mu_{\gamma,s}(\Omega)$. This classical method has been used by Kang-Ghoussoub [21], Ghoussoub-Robert [23, 24] when $\gamma = 0$, and by Chern-Lin [10] for $0 < \gamma < \frac{(n-2)^2}{4}$ in order to establish (1.13) under the assumption that $\partial\Omega$ has a negative mean curvature at 0. Actually, the estimates of Chern-Lin [10] extend directly to establish an analogue of Theorem 1.1 for all $\gamma < \frac{n^2-1}{4}$ under the same local geometric condition. However, the case where $\gamma = \frac{n^2-1}{4}$ already requires a much more refined analysis of the Hardy-Schrödinger operator $L_\gamma := -\Delta - \frac{\gamma}{|x|^2}$ on the half-space $\mathbb{R}^n_+$.

The remaining range $(\frac{n^2-1}{4}, \frac{n^2}{4})$ for γ turned out to be even more interesting for the operator L_γ. Indeed, the curvature condition at 0 is not sufficient anymore to insure existence, as more global test functions are required. We therefore proceed to isolate a notion of "Hardy boundary-mass" $m_\gamma(\Omega)$ for any bounded domain Ω (with $0 \in \partial\Omega$) that is associated to the operator L_γ. This is stated in Theorem 8.1

below and is reminiscent of the positive mass theorem of Schoen-Yau [49] that was used to complete the solution of the Yamabe problem.

In order to explain the new critical threshold that is $\frac{n^2-1}{4}$, we need first to consider the Hardy-Schrödinger operator $L_\gamma := -\Delta - \frac{\gamma}{|x|^2}$ on $\mathbb{R}^n_+$. The most basic solutions for $L_\gamma u = 0$, with $u = 0$ on $\partial\mathbb{R}^n_+$ are of the form $u(x) = x_1|x|^{-\alpha}$, and a straightforward computation yields $-\Delta(x_1|x|^{-\alpha}) = \frac{\alpha(n-\alpha)}{|x|^2} x_1|x|^{-\alpha}$ on $\mathbb{R}^n_+$, which means that

$$\left(-\Delta - \frac{\gamma}{|x|^2}\right)(x_1|x|^{-\alpha}) = 0 \text{ on } \mathbb{R}^n_+,$$

for $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$ where $\alpha_\pm(\gamma) := \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - \gamma}$. Actually, any non-negative solution of $L_\gamma u = 0$ on $\mathbb{R}^n_+$ with $u = 0$ on $\partial\mathbb{R}^n_+$ is a (positive) linear combination of these two solutions (Proposition 7.4 below).

Note that $\alpha_-(\gamma) < \frac{n}{2} < \alpha_+(\gamma)$, which points to the difference around 0 between the "small" solution, namely $x \mapsto x_1|x|^{-\alpha_-(\gamma)}$, and the "large one" $x \mapsto x_1|x|^{-\alpha_+(\gamma)}$. Indeed, the "small" solution is "variational", i.e. is locally in $D^{1,2}(\mathbb{R}^n_+)$, while the large one is not. This turned out to be a general fact since we shall show that $x \mapsto d(x, \partial\Omega)|x|^{-\alpha_-(\gamma)}$ is essentially the profile at 0 of any variational solution – positive or not– of equations of the form $L_\gamma u = f(x, u)$ on a domain Ω, as long as the nonlinearity f is dominated by $C(|v| + \frac{|v|^{2^\star(s)-1}}{|x|^s})$. Here $d(x, \partial\Omega)$ denotes the distance function to $\partial\Omega$. To state the theorem, we use the following terminology. We say that $u \in D^{1,2}(\Omega)_{loc,0}$ if there exists $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ around 0 and $\eta u \in D^{1,2}(\Omega)$. Say that $u \in D^{1,2}(\Omega)_{loc,0}$ is a weak solution to the equation

$$-\Delta u = F \in \left(D^{1,2}(\Omega)_{loc,0}\right)',$$

if for any $\varphi \in D^{1,2}(\Omega)$ and $\eta \in C_c^\infty(\mathbb{R}^n)$, we have $\int_\Omega (\nabla u, \nabla(\eta\varphi))\, dx = \langle F, \eta\varphi\rangle$.

The following theorem will be established in section 6.

Theorem 1.3 (Optimal regularity and Generalized Hopf's Lemma)**.** *Let Ω be a smooth domain in $\mathbb{R}^n$ such that $0 \in \partial\Omega$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that*

$$|f(x, v)| \leq C|v|\left(1 + \frac{|v|^{2^\star(s)-2}}{|x|^s}\right) \text{ for all } x \in \Omega \text{ and } v \in \mathbb{R}.$$

Assume $\gamma < \frac{n^2}{4}$ and let $u \in D^{1,2}(\Omega)_{loc,0}$ be such that for some $\tau > 0$,

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x, u) \text{ weakly in } D^{1,2}(\Omega)_{loc,0}. \tag{1.15}$$

Then, there exists $K \in \mathbb{R}$ such that

$$\lim_{x \to 0} \frac{u(x)}{d(x, \partial\Omega)|x|^{-\alpha_-(\gamma)}} = K. \tag{1.16}$$

Moreover, if $u \geq 0$ and $u \not\equiv 0$, then $K > 0$.

This theorem can be seen as an extension of Hopf's Lemma [28] in the following sense: when $\gamma = 0$ (and therefore $\alpha_-(\gamma) = 0$), the classical Nash-Moser regularity scheme then yields that $u \in C^1_{loc}$, and when $u \geq 0$, $u \not\equiv 0$, Hopf's comparison principle yields $\partial_\nu u(0) < 0$, which is really a reformulation of (1.16) in the case where $\alpha_-(\gamma) = 0$.

The proof of this theorem is quite interesting since, unlike the regular case (i.e., when $L_\gamma = L_0 = -\Delta$) or the classical situation when the singularity 0 is in the interior of the domain Ω (see Smets [50]), a direct application of the standard Nash-Moser iterative scheme is not sufficient to obtain the required regularity. Indeed, the scheme only yields the existence of p_0, with $1 < p_0 < \frac{n}{\alpha_-(\gamma)-1}$ such that $u \in L^p$ for all $p < p_0$. Unfortunately, p_0 does not reach $\frac{n}{\alpha_-(\gamma)-1}$, which is the optimal rate of integration needed to obtain the profile (1.16) for u. However, the improved order p_0 is enough to allow for the inclusion of the nonlinearity $f(x,u)$ in the linear term of (1.15). We are then reduced to the analysis of the linear equation, that is (1.15) with $f(x,u) \equiv 0$, in which case we get the conclusion by constructing suitable super- and sub- solutions to the linear equation that have the same profile at 0 as (1.16). For details, see Section 6.
As a corollary, one obtains the following description of the profile of variational solutions of (1.3) on $\mathbb{R}^n_+$, which improves on a result of Chern-Lin [10], hence allowing us to construct sharper test functions and to prove existence of solutions for (1.3) when $\gamma = \frac{n^2-1}{4}$.

Theorem 1.4. *Assume $\gamma < \frac{n^2}{4}$ and let $u \in D^{1,2}(\mathbb{R}^n_+)$, $u \geq 0$, $u \not\equiv 0$ be a weak solution to*

$$-\Delta u - \frac{\gamma}{|x|^2}u = \frac{u^{2^\star(s)-1}}{|x|^s} \text{ in } \mathbb{R}^n_+. \tag{1.17}$$

Then, there exist $K_1, K_2 > 0$ such that

$$u(x) \sim_{x\to 0} K_1 \frac{x_1}{|x|^{\alpha_-(\gamma)}} \quad \text{and} \quad u(x) \sim_{|x|\to+\infty} K_2 \frac{x_1}{|x|^{\alpha_+(\gamma)}}.$$

The above theorem yields in particular, the existence of a solution u for (1.17) which satisfies for some $C > 0$, the estimates

$$u(x) \leq C x_1 |x|^{-\alpha_+(\gamma)} \quad \text{and} \quad |\nabla u(x)| \leq C|x|^{-\alpha_+(\gamma)} \text{ for all } x \in \mathbb{R}^n_+. \tag{1.18}$$

Noting that

$$\gamma < \frac{n^2-1}{4} \quad \Leftrightarrow \quad \alpha_+(\gamma) - \alpha_-(\gamma) > 1,$$

it follows from (1.18), that whenever $\gamma < \frac{n^2-1}{4}$, then $|x'|^2|\partial_1 u|^2 = O(|x'|^{2-2\alpha_+(\gamma)})$ as $|x'| \to +\infty$ on $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, from which we could deduce that $x' \mapsto |x'|^2|\partial_1 u(x')|^2$ is in $L^1(\partial\mathbb{R}^n_+)$. This estimate –which does not hold when $\gamma \geq \frac{n^2-1}{4}$ – is key for the construction of test functions for $\mu_{\gamma,s}(\Omega)$ based on the solution u of (1.17), in the case when $\gamma \leq \frac{n^2-1}{4}$.
In order to deal with the remaining cases for γ, that is when $\gamma \in (\frac{n^2-1}{4}, \frac{n^2}{4})$, we prove the following result which describes the general profile of any positive solution of $L_\gamma u = a(x)u$, albeit variational or not.

Theorem 1.5 (Classification of singular solutions)**.** *Assume $\gamma < \frac{n^2}{4}$ and let $u \in C^2(B_\delta(0) \cap (\overline{\Omega} \setminus \{0\}))$ be such that*

$$\begin{cases} -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2}u = 0 & \text{in } \Omega \cap B_\delta(0) \\ u > 0 & \text{in } \Omega \cap B_\delta(0) \\ u = 0 & \text{on } (\partial\Omega \cap B_\delta(0)) \setminus \{0\}. \end{cases} \tag{1.19}$$

Then, there exists $K > 0$ such that

$$\text{either } u(x) \sim_{x\to 0} K\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} \quad \text{or} \quad u(x) \sim_{x\to 0} K\frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}. \tag{1.20}$$

In the first case, the solution u is variational; in the second case, it is not.

This result then allows us to completely classify all positive solutions to $L_\gamma u = 0$ on $\mathbb{R}^n_+$, a fact alluded to in Pinchover-Tintarev ([44], Example 1.5).

Proposition 1.6. *Assume $\gamma < \frac{n^2}{4}$ and let $u \in C^2(\overline{\mathbb{R}^n_+} \setminus \{0\})$ be such that*

$$\begin{cases} -\Delta u - \frac{\gamma}{|x|^2}u = 0 & \text{in } \mathbb{R}^n_+ \\ u > 0 & \text{in } \mathbb{R}^n_+ \\ u = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{1.21}$$

Then, there exists $\lambda_-, \lambda_+ \geq 0$ such that

$$u(x) = \lambda_- x_1|x|^{-\alpha_-(\gamma)} + \lambda_+ x_1|x|^{-\alpha_+(\gamma)} \text{ for all } x \in \mathbb{R}^n_+. \tag{1.22}$$

As mentioned above, the case when $\gamma > \frac{n^2-1}{4}$ is more intricate and requires isolating a new notion of *singular boundary mass* associated to the operator L_γ for domains of $\mathbb{R}^n$ having 0 on their boundary. The following result will be proved in section 8.

Theorem 1.7. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Assume that $\frac{n^2-1}{4} < \gamma < \gamma_H(\Omega)$. Then, up to multiplication by a positive constant, there exists a unique function $H \in C^2(\overline{\Omega} \setminus \{0\})$ such that*

$$-\Delta H - \frac{\gamma}{|x|^2}H = 0 \text{ in } \Omega\ ,\ H > 0 \text{ in } \Omega\ ,\ H = 0 \text{ on } \partial\Omega \setminus \{0\}. \tag{1.23}$$

Moreover, there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$H(x) = c_1\frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}} + c_2\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \qquad \text{as } x \to 0.$$

The quantity $m_\gamma(\Omega) := \frac{c_2}{c_1} \in \mathbb{R}$, which is independent of the choice of H satisfying (1.23)*, will be referred to as the Hardy singular b-mass of Ω.*

Indeed, another interpretation of the threshold is the following. The case $\gamma > \frac{n^2-1}{4}$ is the only situation in which one can write a solution H to (1.23) as the sum of the two profiles given in (1.20) (plus lower-order terms) for any bounded domain Ω. When $\gamma \leq \frac{n^2-1}{4}$, there might be some intermediate terms between the two profiles.

We show that the map $\Omega \to m_\gamma(\Omega)$ is a monotone increasing function on the class of domains having zero on their boundary, once ordered by inclusion. We shall also see below that it is possible to define the mass of some unbounded domains, and that $m_\gamma(\mathbb{R}^n_+) = 0$ for any $\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$, from which follows that the mass of any one of its smooth subsets having zero on its boundary is non-positive. In particular, $m_\gamma(\Omega) < 0$ whenever Ω is convex bounded and $0 \in \partial\Omega$.

We shall however exhibit in section 10 examples of bounded domains Ω in $\mathbb{R}^n$ with $0 \in \partial\Omega$ and with positive mass. Among other things, we provide examples of domains with either positive or negative boundary mass, while satisfying any local behavior at 0 one wishes. In other words, the sign of the Hardy b-mass is totally independent of the local properties of $\partial\Omega$ around 0.

This notion and the preceeding results allow us to establish the following extension of the results of Chern-Lin.

Theorem 1.8. *Let Ω be a bounded smooth domain of $\mathbb{R}^n$ $(n \geq 3)$ such that $0 \in \partial\Omega$, hence $\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \frac{n^2}{4}$. Let $0 \leq s < 2$.*

(1) *If $\gamma_H(\Omega) \leq \gamma < \frac{n^2}{4}$, then there are extremals for $\mu_{\gamma,s}(\Omega)$ for all $n \geq 3$.*
(2) *If $\gamma < \gamma_H(\Omega)$ and either $s > 0$ or $\{s = 0,\ n \geq 4$ and $\gamma > 0\}$, then there are extremals for $\mu_{\gamma,s}(\Omega)$, under either one of the following conditions:*
   - *$\gamma \leq \frac{n^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
   - *$\gamma > \frac{n^2-1}{4}$ and the Hardy b-mass $m_\gamma(\Omega)$ is positive.*
(3) *If $\{s = 0$ and $\gamma \leq 0\}$, then there are no extremals for $\mu_{\gamma,0}(\Omega)$ for any $n \geq 3$.*

Still when Ω is a smooth bounded domain, we shall also address in section 11 the remaining case, i.e., $n = 3$ and $s = 0$ and $\gamma \in (0, \frac{9}{4})$ (note that $n^2/4 = 9/4$). In this situation, there may or may not be extremals for $\mu_{\gamma,0}(\mathbb{R}^3_+)$. If they do exist, we can then argue as before –using the same test functions– to conclude existence of extremals under the same conditions, that is either $\gamma \leq 2 = \frac{3^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative, or $\gamma > 2$ and the mass $m_\gamma(\Omega)$ is positive. However, if no extremal exist for $\mu_{\gamma,0}(\mathbb{R}^3_+)$, then as noted in (1.14), we have that

$$\mu_{\gamma,0}(\mathbb{R}^3_+) = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^3} |u|^{2^\star}\, dx\right)^{\frac{2}{2^\star}}} = \frac{1}{K(3,2)^2},$$

and we are back to the case of the Yamabe problem with no boundary singularity. This means that one needs to resort to a more standard notion of mass $R_\gamma(\Omega, x_0)$ associated to L_γ and an interior point $x_0 \in \Omega$. One can then construct suitable test-functions in the spirit of Schoen [48]. In order to define the "internal mass", we show (see Proposition 11.1) that for a given $\gamma \in (0, \gamma_H(\Omega))$, there exists a solution $G \in C^2(\overline{\Omega} \setminus \{0, x_0\}) \cap D^2_1(\Omega \setminus \{x_0\})_{loc,0}$ of

$$\begin{cases} -\Delta G - \frac{\gamma}{|x|^2} G = 0 & \text{in } \Omega \setminus \{x_0\} \\ G > 0 & \text{in } \Omega \setminus \{x_0\} \\ G = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases}$$

is unique up to multiplication by a constant, and that for any $x_0 \in \Omega$, there exists $R_\gamma(\Omega, x_0) \in \mathbb{R}$ (independent of the choice of G) and $c_G > 0$ such that

$$G(x) = c_G \left(\frac{1}{|x - x_0|} + R_\gamma(\Omega, x_0) \right) + o(1) \quad \text{as } x \to x_0.$$

Withe the uniqueness of Proposition 11.1, the quantity $R_\gamma(\Omega, x_0)$ is well defined, and we prove the following.

Theorem 1.9. *Let Ω be a bounded smooth domain of $\mathbb{R}^3$ such that $0 \in \partial\Omega$. In particular $\frac{1}{4} < \gamma_H(\Omega) \leq \frac{9}{4}$.*

(1) *If $\gamma_H(\Omega) \leq \gamma < \frac{9}{4}$, then there are extremals for $\mu_{\gamma,0}(\Omega)$.*
(2) *If $0 < \gamma < \gamma_H(\Omega)$, and if there exists $x_0 \in \Omega$ such that $R_\gamma(\Omega, x_0) > 0$, then there are extremals for $\mu_{\gamma,0}(\Omega)$, under either one of the following conditions:*
   (a) *$\gamma \leq 2$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
   (b) *$\gamma > 2$ and the mass $m_\gamma(\Omega)$ is positive.*

More precisely, if there are extremals for $\mu_{\gamma,0}(\mathbb{R}^3)$, then conditions (a) and (b) are sufficient to get extremals for $\mu_{\gamma,0}(\Omega)$. If there are no extremals for $\mu_{\gamma,0}(\mathbb{R}^3)$, then the positivity of the internal mass $R_\gamma(\Omega, x_0)$ is sufficient to get extremals for $\mu_{\gamma,0}(\Omega)$. We refer to Theorem 11.3 for a precise statement. The following table summarizes our findings.

Table 1. Singular Sobolev-Critical term: $s > 0$

Hardy term	**Dimension**	**Geometric condition**	**Extremal**
$-\infty < \gamma \leq \frac{n^2-1}{4}$	$n \geq 3$	Negative mean curvature at 0	Yes
$\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$	$n \geq 3$	Positive boundary-mass	Yes

Table 2. Non-singular Sobolev-Critical term: $s = 0$

Hardy term	**Dim.**	**Geometric condition**	**Extr.**
$0 < \gamma \leq \frac{n^2-1}{4}$	$n = 3$	Negative mean curvature at 0 & Positive internal mass	Yes
	$n \geq 4$	Negative mean curvature at 0	Yes
$\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$	$n = 3$	Positive boundary-mass & Positive internal mass	Yes
	$n \geq 4$	Positive boundary mass	Yes
$\gamma \leq 0$	$n \geq 3$	–	No

Notations: in the sequel, $C_i(a, b, ...)$ $(i = 1, 2, ...)$ will denote constants depending on $a, b, ...$ The same notation can be used for different constants, even in the same line. We will always refer to the monograph [28] by Gilbarg and Trudinger for standard elliptic pdes results.

2. Old and new inequalities involving singular weights

The following general form of the Hardy inequality is well known. See for example Cowan [11] or the book of Ghoussoub-Moradifam [22]. We include here a proof for completeness.

Theorem 2.1. *Let Ω be a connected open subset of $\mathbb{R}^n$ and consider $\rho \in C^\infty(\Omega)$ such that $\rho > 0$ and $-\Delta\rho > 0$. Then for any $u \in D^{1,2}(\Omega)$ we have that $\sqrt{\rho^{-1}(-\Delta)\rho}u \in L^2(\Omega)$ and*

$$\int_\Omega \frac{-\Delta\rho}{\rho} u^2\, dx \leq \int_\Omega |\nabla u|^2\, dx. \tag{2.1}$$

Moreover, the case of equality is achieved exactly on $\mathbb{R}\rho \cap D^{1,2}(\Omega)$. In particular, if $\rho \notin D^{1,2}(\Omega)$, there are no nontrivial extremals for (2.1).

Proof of Theorem 2.1: The proof relies of the following integral identity:

$$\int_\Omega |\nabla(\rho v)|^2\, dx - \int_\Omega \frac{-\Delta\rho}{\rho}(\rho v)^2\, dx = \int_\Omega \rho^2|\nabla v|^2\, dx \geq 0 \tag{2.2}$$

for all $v \in C_c^\infty(\Omega)$. This identity is a straightforward integration by parts. Since $\rho, -\Delta\rho > 0$ in Ω, it follows from density arguments that for any $u \in D^{1,2}(\Omega)$, then $\sqrt{\rho^{-1}(-\Delta)\rho} u \in L^2(\Omega)$ and (2.1) holds.

Assume now that there exists $u_0 \in D^{1,2}(\Omega) \setminus \{0\}$ that is an extremal for (2.1). In other words, we have that

$$\int_\Omega \frac{-\Delta\rho}{\rho} u_0^2\, dx = \int_\Omega |\nabla u_0|^2\, dx.$$

Let $(u_i)_i \in C_c^\infty(\Omega)$ be such that $\lim_{i\to+\infty} u_i = u_0$ in $D^{1,2}(\Omega)$ and define $v_i(x) := \frac{u_i(x)}{\rho(x)}$ for all $x \in \Omega$ and all i. This is well defined since u_i has compact support in Ω: therefore $v_i \in C_c^\infty(\Omega)$ for all i. Since $D^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$, Sobolev's embedding theorem yields convergence of u_i to u_0 in $L^{2n/(n-2)}(\Omega)$. Since $\rho > 0$ in Ω, we then get that $(v_i)_i$ is uniformly bounded in $H^2_{1,loc}(\Omega)$. It then follows from reflexivity and a diagonal argument that there exists $v \in H^2_{1,loc}(\Omega)$ such that

$$\lim_{i\to+\infty} v_i = v \text{ in } H^2_{1,loc}(\Omega).$$

Applying (2.2) to $v_i = \rho^{-1} u_i$ yields $\lim_{i\to+\infty} \int_\omega \rho^2 |\nabla v_i|^2\, dx = 0$. Therefore, for any $\omega \subset\subset \Omega$, we have that

$$\int_\omega |\nabla v|^2\, dx \leq \liminf_{i\to+\infty} \int_\omega |\nabla v_i|^2\, dx = 0.$$

Therefore $\int_\omega |\nabla v|^2\, dx = 0$ for all $\omega \subset\subset \Omega$, and then there exists $c \in \mathbb{R}$ such that $v \equiv c$. Up to extracting additional subsequence, we can assume that $u_i(x)$ and $v_i(x)$ converge to $u_0(x)$ and $v(x)$ respectively when $i \to +\infty$ for a.e. $x \in \Omega$. Therefore, $u_0(x) = c \cdot \rho(x)$ for a.e. $x \in \Omega$. Since $u_0 \not\equiv 0$, we have that $c \neq 0$ and then $\rho \in D^{1,2}(\Omega)$. For dimensional reasons, the equality is then achieved exactly on $\mathbb{R}\rho \cap D^{1,2}(\mathbb{R}^n)$. This ends the case of equality in case there is a nontrivial extremal.

Assume now that $\rho \in D^{1,2}(\Omega)$. We let $(\rho_i) \in C_c^\infty(\Omega)$ such that $\lim_{i\to+\infty} \rho_i = \rho$ in $D^{1,2}(\Omega)$. Without loss of generality, we can assume that $\rho_i(x) \to \rho(x)$ as $i \to +\infty$ for a.e. $x \in \Omega$. We define $v_i := \frac{\rho_i}{\rho} \in C_c^\infty(\Omega)$. We have that $v_i(x) \to 1$ as $i \to +\infty$ for a.e. $x \in \Omega$. For any i, j, (2.2) yields

$$\int_\Omega |\nabla(\rho_i - \rho_j)|^2\, dx - \int_\Omega \frac{-\Delta\rho}{\rho} (\rho_i - \rho_j)^2\, dx = \int_\Omega \rho^2 |\nabla(v_i - v_j)|^2\, dx.$$

Therefore $(\rho \nabla v_i)_i$ is a Cauchy sequence in $L^2(\Omega, \mathbb{R}^n)$, and therefore, there exists $\vec{X} \in L^2(\Omega, \mathbb{R}^n)$ such that

$$\lim_{i\to+\infty} \rho \nabla v_i = \vec{X} \text{ in } L^2(\Omega, \mathbb{R}^n). \tag{2.3}$$

Arguing as in the first part of the proof of Theorem 2.1, we get that there exists $v \in H^2_{1,loc}(\Omega)$ such that $\lim_{i\to+\infty} v_i = v$ in $H^2_{1,loc}(\Omega)$. Since $v_i(x) \to 1$ as $i \to +\infty$ for a.e. $x \in \Omega$, we get that $v \equiv 1$ and therefore $\nabla v = 0$, which yields $\vec{X} = 0$. It then follows from (2.3) that $(\rho \nabla v_i)_i$ goes to 0 in $L^2(\Omega, \mathbb{R}^n)$. Using again (2.2) yields

$$\int_\Omega |\nabla \rho_i|^2\, dx - \int_\Omega \frac{-\Delta\rho}{\rho} \rho_i^2\, dx = \int_\Omega \rho^2 |\nabla v_i|^2\, dx. \tag{2.4}$$

Therefore, letting $i \to +\infty$ yields $\int_\Omega |\nabla\rho|^2\, dx = \int_\Omega \frac{-\Delta\rho}{\rho} \rho^2\, dx$, and then ρ is an extremal for (2.1).

Theorem 2.1 follows from the case of the existence of an extremal and the case $\rho \in D^{1,2}(\Omega)$. $\square$

The above theorem applies to various weight functions ρ. See for example the paper of Cowan [11] or the book [22]. For this paper, we need it for the following inequality.

Corollary 2.2. *Fix $1 \leq k \leq n$, we then have the following inequality.*

$$\left(\frac{n+2k-2}{2}\right)^2 = \inf_u \frac{\int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} |\nabla u|^2\, dx}{\int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} \frac{u^2}{|x|^2}\, dx},$$

where the infimum is taken over all u in $D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}) \setminus \{0\}$. Moreover, the infimum is never achieved.

Proof of Corollary 2.2: Take $\rho(x) := x_1 ... x_k |x|^{-\alpha}$ for all $x \in \Omega := \mathbb{R}^k_+ \times \mathbb{R}^{n-k} \setminus \{0\}$. Then $\frac{-\Delta \rho}{\rho} = \frac{\alpha(n+2k-2-\alpha)}{|x|^2}$. We then maximize the constant by taking $\alpha := (n+2k-2)/2$. Since $\rho \notin D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$, Theorem 2.1 applies and we obtain that

$$\left(\frac{n+2k-2}{2}\right)^2 \int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} \frac{u^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} |\nabla u|^2\, dx \tag{2.5}$$

for all $u \in D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$, and that the extremals are trivial.

It remains to prove that the constant in (2.5) is optimal. This will be achieved via the following test-function estimates. Construct a sequence $(\rho_\epsilon)_{\epsilon > 0} \in D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$ as follows. Starting with $\rho(x) = x_1 ... x_k |x|^{-\alpha}$, we fix $\beta > 0$ and define

$$\rho_\epsilon(x) := \begin{cases} \left|\frac{x}{\epsilon}\right|^\beta \rho(x) & \text{if } |x| < \epsilon \\ \rho(x) & \text{if } \epsilon \leq |x| \leq \frac{1}{\epsilon} \\ |\epsilon \cdot x|^{-\beta} \rho(x) & \text{if } |x| > \frac{1}{\epsilon} \end{cases} \tag{2.6}$$

with $\alpha := (n+2k-2)/2$. As one checks, $\rho_\epsilon \in D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$ for all $\epsilon > 0$. The changes of variables $x = \epsilon y$ and $x = \epsilon^{-1} z$ yield

$$\begin{array}{ll} \int_{B_\epsilon(0)} \frac{\rho_\epsilon^2}{|x|^2}\, dx = O(1), & \int_{B_\epsilon(0)} |\nabla \rho_\epsilon|^2\, dx = O(1), \\ \int_{\mathbb{R}^n \setminus \overline{B}_{\epsilon^{-1}}(0)} \frac{\rho_\epsilon^2}{|x|^2}\, dx = O(1), & \int_{\mathbb{R}^n \setminus \overline{B}_{\epsilon^{-1}}(0)} |\nabla \rho_\epsilon|^2\, dx = O(1) \end{array} \tag{2.7}$$

when $\epsilon \to 0$. Integrating by parts yields

$$\begin{aligned} \int_{B_{\epsilon^{-1}}(0) \setminus \overline{B}_\epsilon(0)} |\nabla \rho_\epsilon|^2\, dx &= \int_{B_{\epsilon^{-1}}(0) \setminus \overline{B}_\epsilon(0)} \frac{-\Delta \rho}{\rho} \rho^2\, dx + O(1) \\ &= \left(\frac{n+2k-2}{2}\right)^2 \int_{B_{\epsilon^{-1}}(0) \setminus \overline{B}_\epsilon(0)} \frac{\rho^2}{|x|^2}\, dx + O(1), \end{aligned} \tag{2.8}$$

when $\epsilon \to 0$. Using polar coordinates yields

$$\int_{B_{\epsilon^{-1}}(0) \setminus \overline{B}_\epsilon(0)} \frac{\rho^2}{|x|^2}\, dx = C(2) \ln \frac{1}{\epsilon} \text{ where } C(2) := 2 \int_{\mathbb{S}^{n-1}} \left| \prod_{i=1}^k x_i \right|^2 d\sigma. \tag{2.9}$$

Therefore, (2.7), (2.8) and (2.9) yield

$$\frac{\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k}}|\nabla\rho_\epsilon|^2\,dx}{\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k}}\frac{\rho_\epsilon^2}{|x|^2}\,dx}=\left(\frac{n+2k-2}{2}\right)^2+o(1)$$

as $\epsilon\to 0$, and we are done. Note that the infimum is never achieved since $\rho\notin D^{1,2}(\mathbb{R}_+^k\times\mathbb{R}^{n-k})$. This ends the proof of Corollary 2.2. □

Another approach to prove Corollary 2.2 is to see $\mathbb{R}_+^k\times\mathbb{R}^{n-k}$ as a cone generated by a domain of the unit sphere. Then the Hardy constant is given by the Hardy constant of $\mathbb{R}^n$ plus the first eigenvalue of the Laplacian of the Dirichlet of the above domain of the unit sphere endowed with its canonical metric. This point of view is developed in Pinchover-Tintarev [44] (see also Fall-Musina [19] and Ghoussoub-Moradifam [22] for an exposition in book form).

We get the following generalized Caffarelli-Kohn-Nirenberg inequality.

Proposition 2.3. *Let Ω be an open subset of $\mathbb{R}^n$. Let $\rho,\rho'\in C^\infty(\Omega)$ be such that $\rho,\rho'>0$ and $-\Delta\rho,-\Delta\rho'>0$. Fix $s\in[0,2]$ and assume that there exists $\varepsilon\in(0,1)$ and $\rho_\varepsilon\in C^\infty(\Omega)$ such that*

$$\frac{-\Delta\rho}{\rho}\leq(1-\varepsilon)\frac{-\Delta\rho_\varepsilon}{\rho_\varepsilon}\ \text{ in }\Omega\text{ with }\rho_\varepsilon,-\Delta\rho_\varepsilon>0.$$

Then we have that

$$\left(\int_\Omega\left(\frac{-\Delta\rho'}{\rho'}\right)^{s/2}\rho^{2^\star(s)}|u|^{2^\star(s)}\,dx\right)^{\frac{2}{2^\star(s)}}\leq C\int_\Omega\rho^2|\nabla u|^2\,dx \tag{2.10}$$

for all $u\in C_c^\infty(\Omega)$.

Proof of Proposition 2.3: The Sobolev inequality yields the existence of $C(n)>0$ such that

$$\left(\int_\Omega|u|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\leq C(n)\int_\Omega|\nabla u|^2\,dx$$

for all $u\in C_c^\infty(\Omega)$, where $2^\star=2^\star(0)=\frac{2n}{n-2}$. A Hölder inequality interpolating between this Sobolev inequality and the Hardy inequality (2.1) for ρ' yields the existence of $C>0$ such that

$$\left(\int_\Omega\left(\frac{-\Delta\rho'}{\rho'}\right)^{s/2}|u|^{2^\star(s)}\,dx\right)^{\frac{2}{2^\star(s)}}\leq C\int_\Omega|\nabla u|^2\,dx \tag{2.11}$$

for all $u\in C_c^\infty(\Omega)$. The identity (2.2) for ρ and (2.1) for ρ_ε yield for $v\in C_c^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega\rho^2|\nabla v|^2\,dx &= \int_\Omega|\nabla(\rho v)|^2\,dx-\int_\Omega\frac{-\Delta\rho}{\rho}(\rho v)^2\,dx\\ &\geq \int_\Omega|\nabla(\rho v)|^2\,dx-(1-\varepsilon)\int_\Omega\frac{-\Delta\rho_\varepsilon}{\rho_\varepsilon}(\rho v)^2\,dx\\ &\geq \varepsilon\int_\Omega|\nabla(\rho v)|^2\end{aligned}$$

Taking $u:=\rho v$ in (2.11) and using this latest inequality yield (2.10). This ends the proof of Proposition 2.3. □

Here is an immediate consequence.

Corollary 2.4. *Fix $k \in \{1, \dots, n-1\}$. There exists then a constant $C := C(a,b,n) > 0$ such that for all $u \in C_c^\infty(\mathbb{R}_+^k \times \mathbb{R}^{n-k})$, the following inequality holds:*
(2.12)
$$\left(\int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k}} |x|^{-bq} \left(\Pi_{i=1}^k x_i\right)^q |u|^q\right)^{\frac{2}{q}} \leq C \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k}} \left(\Pi_{i=1}^k x_i\right)^2 |x|^{-2a} |\nabla u|^2 dx,$$
where
$$-\infty < a < \frac{n-2+2k}{2}, \quad 0 \leq b-a \leq 1, \quad q = \frac{2n}{n-2+2(b-a)}. \tag{2.13}$$

Proof of Corollary 2.4: Define $\rho(x) = \rho'(x) = \left(\Pi_{i=1}^k x_i\right)|x|^{-a}$ and $\rho_\varepsilon(x) = \left(\Pi_{i=1}^k x_i\right)|x|^{-\frac{n-2+2k}{2}}$ for all $x \in \mathbb{R}_+^k \times \mathbb{R}^{n-k}$. Here, we have that
$$\frac{\Delta\rho'}{\rho'} = \frac{a(n-2+2k-a)}{|x|^2} \text{ and } \frac{-\Delta\rho_\varepsilon}{\rho_\varepsilon} = \frac{(n-2+2k)^2}{4|x|^2}.$$
Apply Proposition 2.3 with this data, with suitable a, b, q to get Corollary 2.4. $\square$

Remark: Observe that by taking $k = 0$, we recover the classical Caffarelli-Kohn-Nirenberg inequalities (1.5). However, one does not see any improvement in the integrability of the weight functions since $\left(\Pi_{i=1}^k x_i\right)|x|^{-a}$ is of order $k - a > -(n-2)/2$, hence as close as we wish to $(n-2)/2$ with the right choice of a. The relevance here appears when one considers the Hardy inequality of Corollary 2.2.

3. Estimates for the best constant in the Hardy inequality

As mentioned in the introduction, the best constant in the Hardy inequality
$$\gamma_H(\Omega) := \inf\left\{\frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega \frac{u^2}{|x|^2}\, dx} \,/\, u \in D^{1,2}(\Omega) \setminus \{0\}\right\}$$
does not depend on the domain $\Omega \subset \mathbb{R}^n$ if the singularity 0 belongs to the interior of Ω. It is always equal to $\frac{(n-2)^2}{4}$. We have seen, however, in the last section that the situation changes whenever $0 \in \partial\Omega$, since $\gamma_H(\mathbb{R}_+^n) = \frac{n^2}{4}$. Some properties of the best Hardy constants have been studied by Fall-Musina [19] and Fall [18]. In this section, we shall collect whatever information we shall need later on about γ_H.

Proposition 3.1. *Let Ω be a smooth domain of $\mathbb{R}^n$. Then γ_H satisfies the following properties:*

1. *For any smooth domain Ω such that $0 \in \Omega$, we have $\gamma_H(\Omega) = \frac{(n-2)^2}{4}$.*
2. *If $0 \in \partial\Omega$, then $\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \frac{n^2}{4}$.*
3. *$\gamma_H(\Omega) = \frac{n^2}{4}$ for every Ω such that $0 \in \partial\Omega$ and $\Omega \subset \mathbb{R}_+^n$.*
4. *If $\gamma_H(\Omega) < \frac{n^2}{4}$, then it is attained in $D^{1,2}(\Omega)$.*
5. *We have $\inf\{\gamma_H(\Omega);\, 0 \in \partial\Omega\} = \frac{(n-2)^2}{4}$ for $n \geq 3$.*
6. *For every $\epsilon > 0$, there exists a smooth domain $\mathbb{R}_+^n \subsetneq \Omega_\epsilon \subsetneq \mathbb{R}^n$ such that $0 \in \partial\Omega_\epsilon$ and $\frac{n^2}{4} - \epsilon \leq \gamma_H(\Omega_\epsilon) < \frac{n^2}{4}$.*

Proof of Proposition 3.1: Properties (1)-(2)-(3)-(4) are well known (See Fall-Musina [19] and Fall [18]). However, we sketch proofs since we will make frequent use of the test functions involved. Note first that Corollary 2.2 already yields that $\gamma_H(\mathbb{R}_+^n) = \frac{n^2}{4}$.

Proof of (2): Since $\Omega \subset \mathbb{R}^n$, we have that $\gamma_H(\Omega) \geq \gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$. Assume by contradiction that $\gamma_H(\Omega) = \frac{(n-2)^2}{4}$. It then follows from Theorem 4.4 below (applied with $s = 2$) that $\gamma_H(\Omega)$ is achieved by a function in $u_0 \in D^{1,2}(\Omega) \setminus \{0\}$ (note that $\mu_{0,\gamma}(\Omega) = \gamma_H(\Omega) - \gamma$). Therefore, $\gamma_H(\mathbb{R}^n)$ is achieved in $D^{1,2}(\mathbb{R}^n)$. Up to taking $|u_0|$, we can assume that $u_0 \geq 0$. Therefore, the Euler-Lagrange equation and the maximum principle yield $u_0 > 0$ in $\mathbb{R}^n$: this is impossible since $u_0 \in D^{1,2}(\Omega)$. Therefore $\gamma_H(\Omega) > \frac{(n-2)^2}{4}$.

For the other inequality, the standard proof normally uses the fact that the domain contains an interior sphere that is tangent to the boundary at 0. We choose here to perform another proof based on test-functions, which will be used again to prove Proposition 4.1. It goes as follows: since Ω is a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$, there exist U, V open subsets of $\mathbb{R}^n$ such that $0 \in U$, $0 \in V$ and there exists $\varphi \in C^\infty(U, V)$ a diffeomophism such that $\varphi(0) = 0$ and

$$\varphi(U \cap \{x_1 > 0\}) = \varphi(U) \cap \Omega \text{ and } \varphi(U \cap \{x_1 = 0\}) = \varphi(U) \cap \partial\Omega.$$

Moreover, we can and shall assume that $d\varphi_0$ is an isometry. Let $\eta \in C_c^\infty(U)$ such that $\eta(x) = 1$ for $x \in B_\delta(0)$ for some $\delta > 0$ small enough, and consider $(\alpha_\epsilon)_{\epsilon > 0} \in (0, +\infty)$ such that $\alpha_\epsilon = o(\epsilon)$ as $\epsilon \to 0$. For $\epsilon > 0$, define

$$u_\varepsilon(x) := \begin{cases} \eta(y)\alpha_\epsilon^{-\frac{n-2}{2}} \rho_\epsilon\left(\frac{y}{\alpha_\epsilon}\right) & \text{for all } x \in \varphi(U) \cap \Omega,\ x = \varphi(y), \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1}$$

Here ρ_ϵ is constructed as in (2.6) with $k = 1$. Now fix $\sigma \in [0, 2]$, and note that only the case $\sigma = 2$ is needed for the above proposition. We then have as $\epsilon \to 0$,

$$\begin{aligned} \int_\Omega \frac{|u_\varepsilon(y)|^{2^\star(\sigma)}}{|y|^\sigma}\, dy &= \int_{\mathbb{R}^n_+} \frac{u_\varepsilon \circ \varphi(x)^{2^\star(\sigma)}}{|\varphi(x)|^\sigma} |\mathrm{Jac}(\varphi)(x)|\, dx \\ &= \int_{\mathbb{R}^n_+} \frac{u_\varepsilon \circ \varphi(x)^{2^\star(\sigma)}}{|x|^\sigma} |(1 + O(|x|))\, dx \\ &= \int_{B_\delta(0) \cap \mathbb{R}^n_+} \frac{u_\varepsilon \circ \varphi(x)^{2^\star(\sigma)}}{|x|^\sigma} (1 + O(|x|))\, dx + O(1). \end{aligned}$$

Dividing $B_\delta(0) = (B_\delta(0) \setminus B_{\epsilon^{-1}\alpha_\epsilon}(0)) \cup (B_{\epsilon^{-1}\alpha_\epsilon}(0) \setminus B_{\epsilon\alpha_\epsilon}(0)) \cup B_{\epsilon\alpha_\epsilon}(0)$ and arguing as in (2.7) to (2.9), we get as $\epsilon \to 0$,

$$\begin{aligned} \int_\Omega \frac{|u_\varepsilon(y)|^{2^\star(\sigma)}}{|y|^\sigma}\, dy &= \int_{\left[B_{\epsilon^{-1}\alpha_\epsilon}(0) \setminus B_{\epsilon\alpha_\epsilon}(0)\right] \cap \mathbb{R}^n_+} \frac{u_\varepsilon \circ \varphi(x)^{2^\star(\sigma)}}{|x|^\sigma} (1 + O(|x|))\, dx + O(1) \\ &= \int_{\left[B_{\epsilon^{-1}\alpha_\epsilon}(0) \setminus B_{\epsilon\alpha_\epsilon}(0)\right] \cap \mathbb{R}^n_+} \frac{u_\varepsilon \circ \varphi(x)^{2^\star(\sigma)}}{|x|^\sigma}\, dx + O(1) \\ &= \int_{\left[B_{\epsilon^{-1}\alpha_\epsilon}(0) \setminus B_{\epsilon\alpha_\epsilon}(0)\right] \cap \mathbb{R}^n_+} \frac{\rho(x)^{2^\star(\sigma)}}{|x|^\sigma}\, dx + O(1). \end{aligned}$$

Passing to polar coordinates yields

$$\int_\Omega \frac{|u_\varepsilon(y)|^{2^\star(\sigma)}}{|y|^\sigma}\, dy = C(\sigma) \ln \frac{1}{\epsilon} + O(1) \quad \text{as } \epsilon \to 0, \tag{3.2}$$

where $C(\sigma) := 2 \int_{\mathbb{S}^{n-1}} \left|\prod_{i=1}^k x_i\right|^{2^\star(\sigma)} d\sigma$.

Similar arguments yield

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2\,dy &= \int_{B_{\epsilon^{-1}\alpha_\epsilon}(0)\setminus B_{\epsilon\alpha_\epsilon}(0)\cap\mathbb{R}^n_+} |\nabla u_\varepsilon\circ\varphi(x)|^2(1+O(|x|)\,dx+O(1)\\ &= \int_{B_{\epsilon^{-1}\alpha_\epsilon}(0)\setminus B_{\epsilon\alpha_\epsilon}(0)\cap\mathbb{R}^n_+} |\nabla u_\varepsilon\circ\varphi(x)|^2\,dx+O(1)\\ &= \int_{B_{\epsilon^{-1}}(0)\setminus B_{\epsilon}(0)\cap\mathbb{R}^n_+} |\nabla\rho(x)|^2\,dx+O(1)\end{aligned}$$

as $\epsilon\to 0$. Using (2.8) and (2.9) yield

$$\int_\Omega |\nabla u_\varepsilon|^2\,dy = \frac{n^2}{4}C(2)\ln\frac{1}{\epsilon}+O(1)\quad \text{as } \epsilon\to 0. \tag{3.3}$$

As a consequence, we get that

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2\,dx}{\int_\Omega \frac{u_\varepsilon^2}{|x|^2}\,dx} = \frac{n^2}{4}+o(1)\quad \text{as } \epsilon\to 0.$$

In particular, we get that $\gamma_H(\Omega)\le\frac{n^2}{4}$, which proves the upper bound in point (2) of the proposition.

Proof of (3). Assume that $\Omega\subset\mathbb{R}^n_+$, then $D^{1,2}(\Omega)\subset D^{1,2}(\mathbb{R}^n_+)$, and therefore $\gamma_H(\Omega)\ge\gamma_H(\mathbb{R}^n_+)=n^2/4$. With the reverse inequality already given by (2), we get that $\gamma_H(\Omega)=n^2/4$ for all $\Omega\subset\mathbb{R}^n_+$ such that $0\in\partial\Omega$.

Proof of (4). This will be a particular case of Theorem 4.4 when $s=2$.

Proof of (5). Let Ω_0 be a bounded domain of $\mathbb{R}^n$ such that $0\in\Omega_0$ (i.e., it is not on the boundary). Given $\delta>0$, we chop out a ball of radius $\delta/4$ with 0 on its boundary to define

$$\Omega_\delta := \Omega_0\setminus\overline{B}_{\frac{\delta}{4}}\left((\frac{-\delta}{4},0,\dots,0)\right)$$

Note that for $\delta>0$ small enough, Ω is smooth and $0\in\partial\Omega$. We now prove that

$$\lim_{\delta\to 0}\gamma_H(\Omega_\delta)=\frac{(n-2)^2}{4}. \tag{3.4}$$

Define $\eta_1\in C^\infty(\mathbb{R}^n)$ such that

$$\eta_1(x)=\begin{cases}0 & \text{if } |x|<1\\ 1 & \text{if } |x|>2,\end{cases}$$

and let $\eta_\delta(x):=\eta_1(\delta^{-1}x)$ for all $\delta>0$ and $x\in\mathbb{R}^n$. Fix $U\in C^\infty_c(\mathbb{R}^n)$ and consider for any $\delta>0$, an $\varepsilon_\delta>0$ such that

$$\lim_{\delta\to 0}\frac{\delta}{\varepsilon_\delta}=\lim_{\delta\to 0}\varepsilon_\delta=0.$$

For $\delta>0$, we define

$$u_\delta(x):=\eta_\delta(x)\varepsilon_\delta^{-\frac{n-2}{2}}U(\varepsilon_\delta^{-1}x)\text{ for all }x\in\Omega_\delta.$$

For $\delta>0$ small enough, we have that $u_\delta\in C^\infty_c(\Omega_\delta)$. A change of variable yields

$$\int_{\Omega_\delta}\frac{u_\delta^2}{|x|^2}\,dx=\int_{\mathbb{R}^n}\frac{U^2}{|x|^2}\eta_1^2\left(\frac{\varepsilon_\delta x}{\delta}\right)\,dx$$

for all $\delta > 0$ small enough. Since $\delta = o(\varepsilon_\delta)$ as $\delta \to 0$, the dominated convergence theorem yields

$$\lim_{\delta\to 0}\int_{\Omega_\delta}\frac{u_\delta^2}{|x|^2}\,dx = \int_{\mathbb{R}^n}\frac{U^2}{|x|^2}\,dx.$$

For $\delta > 0$ small enough, we have that

$$\begin{aligned}\int_{\Omega_\delta}|\nabla u_\delta|^2\,dx &= \int_{\mathbb{R}^n}|\nabla u_\delta|^2\,dx = \int_{\mathbb{R}^n}|\nabla\left(U\cdot\eta_{\frac{\delta}{\varepsilon_\delta}}\right)|^2\,dx\\ &= \int_{\mathbb{R}^n}|\nabla U|^2\eta_{\frac{\delta}{\varepsilon_\delta}}^2\,dx + \int_{\mathbb{R}^n}\eta_{\frac{\delta}{\varepsilon_\delta}}\left(-\Delta\eta_{\frac{\delta}{\varepsilon_\delta}}\right)U^2\,dx. \qquad (3.5)\end{aligned}$$

Let $R > 0$ be such that U has support in $B_R(0)$. We then have that

$$\begin{aligned}\int_{\mathbb{R}^n}\eta_{\frac{\delta}{\varepsilon_\delta}}\left(-\Delta\eta_{\frac{\delta}{\varepsilon_\delta}}\right)U^2\,dx &= O\left(\left(\frac{\varepsilon_\delta}{\delta}\right)^2\mathrm{Vol}(B_R(0)\cap\mathrm{Supp}\,\left(-\Delta\eta_{\frac{\delta}{\varepsilon_\delta}}\right))\right)\\ &= O\left(\left(\frac{\delta}{\varepsilon_\delta}\right)^{n-2}\right) = o(1)\end{aligned}$$

as $\delta \to 0$ since $n \geq 3$. This latest identity, (3.5) and the dominated convergence theorem yield

$$\lim_{\delta\to 0}\int_{\Omega_\delta}|\nabla u_\delta|^2\,dx = \int_{\mathbb{R}^n}|\nabla U|^2\,dx.$$

Therefore, for $U \in C_c^\infty(\mathbb{R}^n)$, we have

$$\limsup_{\delta\to 0}\gamma_H(\Omega_\delta) \leq \lim_{\delta\to 0}\frac{\int_{\Omega_\delta}|\nabla u_\delta|^2\,dx}{\int_{\Omega_\delta}\frac{u_\delta^2}{|x|^2}\,dx} = \frac{\int_{\mathbb{R}^n}|\nabla U|^2\,dx}{\int_{\mathbb{R}^n}\frac{U^2}{|x|^2}\,dx}.$$

Taking the infimum over all $U \in C_c^\infty(\mathbb{R}^n)$, we get that

$$\limsup_{\delta\to 0}\gamma_H(\Omega_\delta) \leq \inf_{U\in D^{1,2}(\mathbb{R}^n)\setminus\{0\}}\frac{\int_{\mathbb{R}^n}|\nabla U|^2\,dx}{\int_{\mathbb{R}^n}\frac{U^2}{|x|^2}\,dx} = \gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}.$$

Since $\gamma_H(\Omega_\delta) \geq \frac{(n-2)^2}{4}$ for all $\delta > 0$, this completes the proof of (3.4), yielding (5).

Proof of (6). The proof uses the following observation.

Lemma 3.2. *Let $(\Phi_k)_{k\in\mathbb{N}} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be such that*

$$\lim_{k\to+\infty}\left(\|\Phi_k - Id_{\mathbb{R}^n}\|_\infty + \|\nabla(\Phi_k - Id_{\mathbb{R}^n})\|_\infty\right) = 0 \text{ and } \Phi_k(0) = 0. \qquad (3.6)$$

Let $D \subset \mathbb{R}^n$ be an open domain such that $0 \in \partial D$ (the domain is not necessarily bounded nor regular), and set $D_k := \Phi_k(D)$ for all $k \in \mathbb{N}$. Then $0 \in \partial D_k$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to+\infty}\gamma_H(D_k) = \gamma_H(D). \qquad (3.7)$$

Proof of Lemma 3.2: If $u \in C_c^\infty(D_k)$, then $u \circ \Phi_k \in C_c^\infty(D)$ and

$$\begin{aligned}\int_{D_k}|\nabla u|^2\,dx &= \int_{\mathbb{R}^n_+}|\nabla(u\circ\Phi_k)|^2_{\Phi_k^\star\mathrm{Eucl}}|\mathrm{Jac}(\Phi_k)|\,dx,\\ \int_{D_k}\frac{u^2}{|x|^2}\,dx &= \int_{\mathbb{R}^n_+}\frac{(u\circ\Phi_k(x))^2}{|\Phi_k(x)|^2}|\mathrm{Jac}(\Phi_k)|\,dx,\end{aligned}$$

where here and in the sequel $\Phi_k^\star\text{Eucl}$ is the pull-back of the Euclidean metric via the diffeomorphisme Φ_k. Assumption (3.6) yields

$$\lim_{k\to+\infty}\sup_{x\in D}\left(\left|\frac{|\Phi_k(x)|}{|x|}-1\right|+\sup_{i,j}|(\partial_i\Phi_k(x),\partial_j\Phi_k(x))-\delta_{ij})|+|\text{Jac}(\Phi_k)-1|\right)=0,$$

where $\delta_{ij}=1$ if $i=j$ and 0 otherwise. Therefore, for any $\varepsilon>0$, there exists k_0 such that for all $u\in C_c^\infty(D_k)$ and $k\geq k_0$,

$$(1+\varepsilon)\int_D|\nabla(u\circ\Phi_k)|^2\,dx\geq\int_{D_k}|\nabla u|^2\,dx\geq(1-\varepsilon)\int_D|\nabla(u\circ\Phi_k)|^2\,dx,$$

and

$$(1+\varepsilon)\int_D\frac{(u\circ\Phi_k(x))^2}{|x|^2}\,dx\geq\int_{D_k}\frac{u^2}{|x|^2}\,dx\geq(1-\varepsilon)\int_D\frac{(u\circ\Phi_k(x))^2}{|x|^2}\,dx.$$

We can now deduce (3.7) by using a standard density argument. This completes the proof of Lemma 3.2. □

We now prove (6) of Proposition 3.1. Let $\varphi\in C^\infty(\mathbb{R}^{n-1})$ such that $0\leq\varphi\leq 1$, $\varphi(0)=0$, and $\varphi(x')=1$ for all $x'\in\mathbb{R}^{n-1}$ such that $|x'|\geq 1$. For $t\geq 0$, define $\Phi_t(x_1,x'):=(x_1-t\varphi(x'),x')$ for all $(x_1,x')\in\mathbb{R}^n$. Set $\tilde{\Omega}_t:=\Phi_t(\mathbb{R}^n_+)$ and apply Lemma 3.2 to note that $\lim_{\varepsilon\to 0}\gamma_H(\tilde{\Omega}_t)=\gamma_H(\mathbb{R}^n_+)=\frac{n^2}{4}$. Since $\varphi\geq 0$, $\varphi\not\equiv 0$, we have that $\mathbb{R}^n_+\subsetneq\tilde{\Omega}_t$ for all $t>0$. To get (6) it suffices to take $\Omega_\varepsilon:=\tilde{\Omega}_t$ for $t>0$ small enough.

4. Estimates on the best constants in the Hardy-Sobolev inequalities

As in the case of $\gamma_H(\Omega)$, the best Hardy-Sobolev constant

$$\mu_{\gamma,s}(\Omega):=\inf\left\{\frac{\int_\Omega|\nabla u|^2\,dx-\gamma\int_\Omega\frac{u^2}{|x|^2}dx}{(\int_\Omega\frac{u^{2^\star(s)}}{|x|^s}dx)^{\frac{2}{2^\star(s)}}};\,u\in D^{1,2}(\Omega)\setminus\{0\}\right\}$$

will depend on the geometry of Ω whenever $0\in\partial\Omega$. In this section, we collect general facts that will be used throughout the paper.

Proposition 4.1. *Let Ω be a bounded smooth domain such that $0\in\partial\Omega$.*

(1) *If $\gamma<\frac{n^2}{4}$, then $\mu_{\gamma,s}(\Omega)>-\infty$.*
(2) *If $\gamma>\frac{n^2}{4}$, then $\mu_{\gamma,s}(\Omega)=-\infty$.*
Moreover,
(3) *If $\gamma<\gamma_H(\Omega)$, then $\mu_{\gamma,s}(\Omega)>0$.*
(4) *If $\gamma_H(\Omega)<\gamma<\frac{n^2}{4}$, then $0>\mu_{\gamma,s}(\Omega)>-\infty$.*
(5) *If $\gamma=\gamma_H(\Omega)<\frac{n^2}{4}$, then $\mu_{\gamma,s}(\Omega)=0$.*

Proof of Proposition 4.1: We first assume that $\gamma<\frac{n^2}{4}$. Let $\epsilon>0$ be such that $(1+\epsilon)\gamma\leq\frac{n^2}{4}$. It follows from Proposition 4.3 that there exists $C_\epsilon>0$ such that for all $u\in D^{1,2}(\Omega)$,

$$\frac{n^2}{4}\int_\Omega\frac{u^2}{|x|^2}\,dx\leq(1+\epsilon)\int_\Omega|\nabla u|^2\,dx+C_\epsilon\int_\Omega u^2\,dx.$$

For any $u \in D^{1,2}(\Omega) \setminus \{0\}$, we have

$$\begin{aligned} J_{\gamma,s}^{\Omega}(u) &\geq \frac{\left(1-\frac{4\gamma}{n^2}(1+\epsilon)\right)\int_\Omega |\nabla u|^2\,dx - \frac{4\gamma}{n^2}C_\epsilon \int_\Omega u^2\,dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}} \\ &\geq -\frac{4\gamma}{n^2}C_\epsilon \frac{\int_\Omega u^2\,dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}}. \end{aligned}$$

It follows from Hölder's inequality that there exists $C > 0$ independent of u such that $\int_\Omega u^2\,dx \leq C\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}$. It then follows that $J_{\gamma,s}^{\Omega}(u) \geq -\frac{4\gamma}{n^2}C_\epsilon C$ for all $u \in D^{1,2}(\Omega) \setminus \{0\}$. Therefore $\mu_{\gamma,s}(\Omega) > -\infty$ whenever $\gamma < \frac{n^2}{4}$.

Assume now that $\gamma > \frac{n^2}{4}$ and define for every $\varepsilon > 0$ a function $u_\varepsilon \in D^{1,2}(\Omega)$ as in (3.1). It then follows from (3.2) and (3.3) that as $\varepsilon \to 0$,

$$J_{\gamma,s}^{\Omega}(u_\varepsilon) = \frac{\left(\frac{n^2}{4}-\gamma\right)C(2)\ln\frac{1}{\varepsilon} + O(1)}{\left(C(s)\ln\frac{1}{\varepsilon} + O(1)\right)^{\frac{2}{2^\star(s)}}} = \left(\left(\frac{n^2}{4}-\gamma\right)\frac{C(2)}{C(s)^{\frac{2}{2^\star(s)}}} + o(1)\right)\left(\ln\frac{1}{\varepsilon}\right)^{\frac{2-s}{n-s}}.$$

Since $s < 2$ and $\gamma > \frac{n^2}{4}$, we then get that $\lim_{\varepsilon\to 0} J_{\gamma,s}^{\Omega}(u_\varepsilon) = -\infty$, and therefore $\mu_{\gamma,s}(\Omega) = -\infty$.

Now assume that $\gamma < \gamma_H(\Omega)$. Sobolev's embedding theorem yields that $\mu_{0,s}(\Omega) > 0$, hence the result is clear for all $\gamma \leq 0$ since then $\mu_{\gamma,s}(\Omega) \geq \mu_{0,s}(\Omega)$. If now $0 \leq \gamma < \gamma_H(\Omega)$, it follows from the definition of $\gamma_H(\Omega)$ that for all $u \in D^{1,2}(\Omega) \setminus \{0\}$,

$$\begin{aligned} J_{\gamma,s}^{\Omega}(u) = \frac{\int_\Omega |\nabla u|^2 - \gamma\int_\Omega \frac{u^2}{|x|^2}dx}{(\int_\Omega \frac{u^{2^\star(s)}}{|x|^s}dx)^{\frac{2}{2^\star(s)}}} &\geq \left(1-\frac{\gamma}{\gamma_H(\Omega)}\right)\frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}} \\ &\geq \left(1-\frac{\gamma}{\gamma_H(\Omega)}\right)\mu_{0,s}(\Omega). \end{aligned}$$

Therefore $\mu_{\gamma,s}(\Omega) \geq \left(1-\frac{\gamma}{\gamma_H(\Omega)}\right)\mu_{0,s}(\Omega) > 0$ when $\gamma < \gamma_H(\Omega)$.

We now assume that $\gamma_H(\Omega) < \gamma < \frac{n^2}{4}$. It follows from Proposition 3.1 (4), that $\gamma_H(\Omega)$ is attained. We let u_0 be such an extremal. In particular $J_{\gamma_H(\Omega),s}^{\Omega}(u) \geq 0 = J_{\gamma_H(\Omega),s}^{\Omega}(u_0)$, and therefore $\mu_{\gamma_H(\Omega),s}(\Omega) = 0$. Since $\gamma_H(\Omega) < \gamma < \frac{n^2}{4}$, we have that $J_{\gamma,s}^{\Omega}(u_0) < 0$, and therefore $\mu_{\gamma,s}(\Omega) < 0$ when $\gamma_H(\Omega) < \gamma < \frac{n^2}{4}$. This ends the proof of Proposition 4.1. □

Remark 4.2. The case $\gamma = \frac{n^2}{4}$ is unclear and anything can happen at that value of γ. For example, if $\gamma_H(\Omega) < \frac{n^2}{4}$ then $\mu_{\frac{n^2}{4},s}(\Omega) < 0$, while if $\gamma_H(\Omega) = \frac{n^2}{4}$ then $\mu_{\frac{n^2}{4},s}(\Omega) \geq 0$. It is our guess that many examples reflecting different regimes can be constructed.

Proposition 4.3. *Assume $\gamma < \frac{n^2}{4}$ and $s \in [0,2]$. Then, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $u \in D^{1,2}(\Omega)$,*

(4.1)
$$\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \le \left(\frac{1}{\mu_{\gamma,s}(\mathbb{R}^n_+)}+\epsilon\right)\int_\Omega\left(|\nabla u|^2-\gamma\frac{u^2}{|x|^2}\right)dx + C_\varepsilon\int_\Omega u^2\,dx.$$

Proof of Proposition 4.3: Fix $\epsilon > 0$. We first claim that there exists $\delta_\epsilon > 0$ such that for all $u \in C^1_c(\Omega\cap B_{\delta_\epsilon}(0))$,

$$\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \le (\mu_{\gamma,s}(\mathbb{R}^n_+)^{-1}+\epsilon)\int_\Omega\left(|\nabla u|^2-\gamma\frac{u^2}{|x|^2}\right)dx. \tag{4.2}$$

Indeed, for two open subsets of $\mathbb{R}^n$ containing 0, we may define a diffeomorphism $\varphi : U \to V$ such that $\varphi(0) = 0$, $\varphi(U\cap\mathbb{R}^n_+) = \varphi(U)\cap\Omega$ and $\varphi(U\cap\partial\mathbb{R}^n_+) = \varphi(U)\cap\partial\Omega$. Moreover, we can also assume that $d\varphi_0$ is a linear isometry. In particular

$$|\varphi^\star\mathrm{Eucl}-\mathrm{Eucl}|(x) \le C|x| \text{ and } |\varphi(x)| = |x|\cdot(1+O(|x|)) \tag{4.3}$$

for $x \in U$. If now $u \in C^1_c(\varphi(B_\delta(0))\cap\Omega)$, then $v := u\circ\varphi \in C^1_c(B_\delta(0)\cap\mathbb{R}^n_+)$. If $g := \varphi^{-1\star}\mathrm{Eucl}$ denotes the metric induced by φ, then we get from (4.3),

$$\begin{aligned}
\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} &\le \left(\int_{B_\delta(0)\cap\mathbb{R}^n_+}\frac{|v|^{2^\star(s)}}{|\varphi(x)|^s}|\mathrm{Jac}\ \varphi(x)|\,dx\right)^{\frac{2}{2^\star(s)}}\\
&\le (1+C\delta)\left(\int_{B_\delta(0)\cap\mathbb{R}^n_+}\frac{|v|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}\\
&\le (1+C\delta)\mu_{\gamma,s}(\mathbb{R}^n_+)^{-1}\int_{B_\delta(0)\cap\mathbb{R}^n_+}\left(|\nabla v|^2-\gamma\frac{v^2}{|x|^2}\right)dx\\
&\le \frac{1+C\delta}{\mu_{\gamma,s}(\mathbb{R}^n_+)}\int_{\varphi(B_\delta(0))\cap\Omega}\left(|\nabla u|_g^2-\frac{\gamma u^2}{|\varphi^{-1}(x)|^2}\right)|\mathrm{Jac}\ \varphi^{-1}(x)|\,dx\\
&\le (1+C_1\delta)\mu_{\gamma,s}(\mathbb{R}^n_+)^{-1}\int_\Omega\left(|\nabla u|^2-\gamma\frac{u^2}{|x|^2}\right)dx\\
&\quad +C_2\delta\int_\Omega\left(|\nabla u|^2+\frac{u^2}{|x|^2}\right)dx. \qquad (4.4)
\end{aligned}$$

We also have that

$$\begin{aligned}
\int_\Omega \frac{u^2}{|x|^2}\,dx &= \int_{\varphi(B_\delta(0)\cap\mathbb{R}^n_+)}\frac{u^2}{|x|^2}\,dx = \int_{B_\delta(0)\cap\mathbb{R}^n_+}\frac{v^2}{|\varphi(x)|^2}|\mathrm{Jac}(\varphi)(x)|\,dx\\
&= \int_{B_\delta(0)\cap\mathbb{R}^n_+}\frac{v^2}{|x|^2}(1+O(|x|))\,dx \le (1+C_1\delta)\int_{\mathbb{R}^n_+}\frac{v^2}{|x|^2}\,dx
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\nabla u|^2\,dx &= \int_{\varphi(B_\delta(0)\cap\mathbb{R}^n_+)}|\nabla u|^2\,dx = \int_{B_\delta(0)\cap\mathbb{R}^n_+}|\nabla v|^2_{\varphi^\star\mathrm{Eucl}}|\mathrm{Jac}(\varphi)(x)|\,dx\\
&= \int_{B_\delta(0)\cap\mathbb{R}^n_+}|\nabla v|^2(1+O(|x|)\,dx \ge (1-C_2\delta)\int_{\mathbb{R}^n_+}|\nabla v|^2\,dx,
\end{aligned}$$

where $C_1, C_2 > 0$ are independent of δ and v. Hardy's inequality (2.5) then yields for all $u \in C_c^1(\varphi(B_\delta(0) \cap \mathbb{R}^n_+))$,

$$\frac{n^2}{4}\int_\Omega \frac{u^2}{|x|^2}\,dx \le \frac{1+C_1\delta}{1-C_2\delta}\int_\Omega |\nabla u|^2\,dx \le (1+C_3\delta)\int_\Omega |\nabla u|^2\,dx. \tag{4.5}$$

Since $\gamma < \frac{n^2}{4}$, there exists $c > 0$ such that for $\delta > 0$ small enough,

$$c^{-1}\int_\Omega |\nabla u|^2\,dx \le \int_\Omega \left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)dx \le c\int_\Omega |\nabla u|^2\,dx$$

for all $u \in C_c^1(\varphi(B_\delta(0)) \cap \Omega)$. Plugging these latest inequalities in (4.4) yields (4.2) by taking δ_ϵ small enough.

Consider now $\eta \in C^\infty(\mathbb{R}^n)$ such that $\sqrt{\eta}, \sqrt{1-\eta} \in C^2(\mathbb{R}^n)$, such that $\eta(x) = 1$ for $x \in B_{\delta_\varepsilon/2}(0)$ and $\eta(x) = 0$ for $x \notin B_{\delta_\varepsilon}(0)$, where δ_ϵ is chosen such that (4.2) holds. We shall use the notation

$$\|w\|_{p,|x|^{-s}} = \left(\int_\Omega \frac{|w|^p}{|x|^s}\,dx\right)^{1/p}.$$

For $u \in C_c^\infty(\Omega)$, use Hölder's inequality to write

$$\begin{aligned}
\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} &= \|u^2\|_{\frac{2^\star(s)}{2},|x|^{-s}} = \|\eta u^2 + (1-\eta)u^2\|_{\frac{2^\star(s)}{2},|x|^{-s}}\\
&\le \|\eta u^2\|_{\frac{2^\star(s)}{2},|x|^{-s}} + \|(1-\eta)u^2\|_{\frac{2^\star(s)}{2},|x|^{-s}}\\
&\le \|\sqrt{\eta}u\|^2_{2^\star(s),|x|^{-s}} + \|\sqrt{1-\eta}u\|^2_{2^\star(s),|x|^{-s}}.
\end{aligned}$$

Since $\sqrt{\eta}u \in C_c^2(B_{\delta_\varepsilon}(0) \cap \Omega)$, it follows from inequality (4.2) and integrations by parts that

$$\begin{aligned}
\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} &\le (\mu_{\gamma,s}(\mathbb{R}^n_+)^{-1}+\epsilon)\int_\Omega \left(|\nabla(\sqrt{\eta}u)|^2 - \gamma\frac{\eta u^2}{|x|^2}\right)dx\\
&\quad + \|\sqrt{1-\eta}u\|^2_{2^\star(s),|x|^{-s}}\\
&\le (\mu_{\gamma,s}(\mathbb{R}^n_+)^{-1}+\epsilon)\int_\Omega \eta\left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)dx + C\int_\Omega u^2\,dx\\
&\quad + \|\sqrt{1-\eta}u\|^2_{2^\star(s),|x|^{-s}}
\end{aligned} \tag{4.6}$$

Case 1: $s = 0$**.** Then $2^\star(s) = 2^\star$ and it follows from Sobolev's inequality that

$$\begin{aligned}
\|\sqrt{1-\eta}u\|^2_{2^\star(s),|x|^{-s}} &\le K(n,2)^2\int_\Omega |\nabla(\sqrt{1-\eta}u)|^2\,dx\\
&\le K(n,2)^2\int_\Omega (1-\eta)|\nabla u|^2\,dx + C\int_\Omega u^2\,dx,
\end{aligned} \tag{4.7}$$

where $K(n,2)$ is the optimal Sobolev constant defined in (1.14). Since $s = 0$, it follows from the proof of Proposition 9.1 that $K(n,2)^2 \le \mu_{\gamma,s}(\mathbb{R}^n_+)^{-1}$. It then follows from (4.7) that

$$\begin{aligned}
\|\sqrt{1-\eta}u\|^2_{2^\star(s),|x|^{-s}} &\le (\mu_{\gamma,s}(\mathbb{R}^n_+)^{-1}+\epsilon)\int_\Omega (1-\eta)\left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)dx\\
&\quad + C\int_\Omega u^2\,dx.
\end{aligned} \tag{4.8}$$

Plugging together (4.6) and (4.8) yields (4.1) when $s = 0$.

Case 2: $0 < s < 2$**.** We let $\nu > 0$ be a positive number to be fixed later. Since $2 < 2^\star(s) < 2^\star$, the interpolation inequality yields the existence of $C_\nu > 0$ such that

$$\begin{aligned} \|\sqrt{1-\eta}u\|_{2^\star(s),|x|^{-s}}^2 &\leq C\|\sqrt{1-\eta}u\|_{2^\star(s)}^2 \\ &\leq C\left(\nu\|\sqrt{1-\eta}u\|_{2^\star}^2 + C_\nu\|\sqrt{1-\eta}u\|_2^2\right) \\ &\leq C\left(\nu K(n,2)^2\|\nabla(\sqrt{1-\eta}u)\|_2^2 + C_\nu\|\sqrt{1-\eta}u\|_2^2\right). \end{aligned}$$

We choose $\nu > 0$ such that $\nu K(n,2)^2 < \mu_{\gamma,s}(\mathbb{R}^n_+)^{-1} + \epsilon$. Then we get (4.8) and we conclude (4.1) in the case when $2 > s > 0$ by combining it with (4.6).

Case 3: $s = 2$**.** This is the easiest case, since then

$$\|\sqrt{1-\eta}u\|_{2^\star(s),|x|^{-s}}^2 = \int_\Omega \frac{((1-\eta)u)^2}{|x|^2}\,dx \leq C_\delta \int_\Omega u^2\,dx.$$

This completes the proof of (4.1) for all $s \in [0,2]$, and therefore of Proposition 4.3. $\Box$.

Now we prove the following result, which will be central for the sequel. The proof is standard.

Theorem 4.4. *Assume that* $\gamma < \frac{n^2}{4}$*,* $0 \leq s \leq 2$ *and that* $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+)$*. Then there are extremals for* $\mu_{\gamma,s}(\Omega)$*. In particular, there exists a minimizer* u *in* $D^{1,2}(\Omega)\setminus\{0\}$ *that is a positive solution to the equation*

$$\left\{\begin{array}{rcll} -\Delta u - \gamma\frac{u}{|x|^2} &=& \mu_{\gamma,s}(\Omega)\frac{u^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega \\ u &>& 0 & \text{in } \partial\Omega \\ u &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{4.9}$$

Proof of Theorem 4.4: Let $(u_i) \in D^{1,2}(\Omega)\setminus\{0\}$ be a minimizing sequence for $\mu_{\gamma,s}(\Omega)$, that is $J^\Omega_{\gamma,s}(u_i) = \mu_{\gamma,s}(\Omega) + o(1)$ as $i \to +\infty$. Up to multiplying by a constant, we can assume that

$$\int_\Omega \frac{|u_i|^{2^\star(s)}}{|x|^s}\,dx = 1 \quad \text{for all } i, \tag{4.10}$$

$$\int_\Omega \left(|\nabla u_i|^2 - \gamma\frac{u_i^2}{|x|^2}\right)dx = \mu_{\gamma,s}(\Omega) + o(1) \text{ as } i \to +\infty. \tag{4.11}$$

We show that $(u_i)_i$ is bounded in $D^{1,2}(\Omega)$. Indeed, (4.10) yields that

$$\textstyle\int_\Omega u_i^2\,dx \leq C < +\infty \text{ for all } i. \tag{4.12}$$

Fix $\epsilon_0 > 0$ and use Proposition 4.3 and (4.12) to get that

$$\tfrac{n^2}{4}\textstyle\int_\Omega \frac{u_i^2}{|x|^2}\,dx \leq (1+\epsilon_0)\int_\Omega |\nabla u_i|^2\,dx + C \qquad \text{for all } i. \tag{4.13}$$

Since $\gamma < \frac{n^2}{4}$, up to taking $\epsilon_0 > 0$ small enough, this latest inequality combined with (4.11) yield the boundedness of $(u_i)_i$ in $D^{1,2}(\Omega)$. It follows that there exists $u \in D^{1,2}(\Omega)$ such that, up to a subsequence, (u_i) goes to u weakly in $D^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ as $i \to +\infty$.

We now show that $\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx = 1$. For that, define $\theta_i := u_i - u \in D^{1,2}(\Omega)$ for all i. In particular, θ_i goes to 0 weakly in $D^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ as $i \to +\infty$. In particular, we have as $i \to +\infty$,

$$1 = \int_\Omega \frac{|u_i|^{2^\star(s)}}{|x|^s}\,dx = \int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx + \int_\Omega \frac{|\theta_i|^{2^\star(s)}}{|x|^s}\,dx + o(1) \tag{4.14}$$

and

$$\mu_{\gamma,s}(\Omega) = \int_\Omega \left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)dx + \int_\Omega \left(|\nabla \theta_i|^2 - \gamma\frac{\theta_i^2}{|x|^2}\right)dx + o(1), \tag{4.15}$$

For $\epsilon > 0$, it follows from the definition of $\mu_{\gamma,s}(\Omega)$ and from (4.1) that, as $i \to +\infty$

$$\mu_{\gamma,s}(\Omega)\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \le \int_\Omega \left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)dx \tag{4.16}$$

and

$$(\mu_{\gamma,s}(\mathbb{R}^n_+) - \epsilon)\left(\int_\Omega \frac{|\theta_i|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \le \int_\Omega \left(|\nabla \theta_i|^2 - \gamma\frac{\theta_i^2}{|x|^2}\right)dx + o(1). \tag{4.17}$$

Summing these two inequalities and using (4.14) and (4.15) and passing to the limit, as $i \to +\infty$, yields

$$\begin{aligned}\mu_{\gamma,s}(\Omega)\left(1 - \left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}\right) &\ge (\mu_{\gamma,s}(\mathbb{R}^n_+) - \epsilon)\left(1 - \int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \\ &\ge (\mu_{\gamma,s}(\mathbb{R}^n_+) - \epsilon)\left(1 - \left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}\right).\end{aligned}$$

Since $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+)$, then by taking $\epsilon > 0$ small enough, we finally conclude that $\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx = 1$.
It remains to show that u is an extremal for $\mu_{\gamma,s}(\Omega)$. For that, note that since $\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx = 1$, the definition of $\mu_{\gamma,s}(\Omega)$ yields $\int_\Omega \left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)dx \ge \mu_{\gamma,s}(\Omega)$. The second term in the right-hand-side of (4.15) is nonnegative due to (4.17). Therefore, we get that $\int_\Omega \left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)dx = \mu_{\gamma,s}(\Omega)$. This proves the claim and ends the proof of Theorem 4.4. □

5. Sub- and super-solutions for the equation $L_\gamma u = a(x)u$

Here and in the sequel, we shall assume that $0 \in \partial\Omega$, where Ω is a smooth domain. In this section, we shall construct basic sub- and super-solutions for the equation $L_\gamma u = a(x)u$, where $a(x) = O(|x|^{\tau-2})$ for some $\tau > 0$.

First recall from the introduction that two solutions for $L_\gamma u = 0$, with $u = 0$ on $\partial\mathbb{R}^n_+$ are of the form $u_\alpha(x) = x_1|x|^{-\alpha}$, where $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$ with

$$\alpha_-(\gamma) := \tfrac{n}{2} - \sqrt{\tfrac{n^2}{4} - \gamma} \quad \text{and} \quad \alpha_+(\gamma) := \tfrac{n}{2} + \sqrt{\tfrac{n^2}{4} - \gamma}. \tag{5.1}$$

These solutions will be the building blocks for sub- and super-solutions of more general linear equations involving L_γ.

Proposition 5.1. *Let $\gamma < \frac{n^2}{4}$ and $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$. Let $0 < \tau \leq 1$ and $\beta \in \mathbb{R}$ such that $\alpha - \tau < \beta < \alpha$ and $\beta \notin \{\alpha_-(\gamma), \alpha_+(\gamma)\}$. Then, there exist $r > 0$, $u_{\alpha,+}, u_{\alpha,-} \in C^\infty(\overline{\Omega} \setminus \{0\})$ such that*

$$\left\{\begin{array}{ll} u_{\alpha,+}, u_{\alpha,-} > 0 & \text{in } \Omega \cap B_r(0)\\ u_{\alpha,+}, u_{\alpha,-} = 0 & \text{on } \partial\Omega \cap B_r(0)\\ -\Delta u_{\alpha,+} - \frac{\gamma + O(|x|^\tau)}{|x|^2} u_{\alpha,+} > 0 & \text{in } \Omega \cap B_r(0)\\ -\Delta u_{\alpha,-} - \frac{\gamma + O(|x|^\tau)}{|x|^2} u_{\alpha,-} < 0 & \text{in } \Omega \cap B_r(0). \end{array}\right. \tag{5.2}$$

Moreover, we have as $x \to 0$, $x \in \Omega$, that

$$u_{\alpha,+}(x) = \frac{d(x,\partial\Omega)}{|x|^\alpha}(1 + O(|x|^{\alpha-\beta})) \text{ and } u_{\alpha,-}(x) = \frac{d(x,\partial\Omega)}{|x|^\alpha}(1 + O(|x|^{\alpha-\beta})). \tag{5.3}$$

Proof of Proposition 5.1: We first choose an adapted chart to lift the basic solutions from $\mathbb{R}^n_+$. Since $0 \in \partial\Omega$ and Ω is smooth, there exists $\tilde{U}, \tilde{V}$ two bounded domains of $\mathbb{R}^n$ such that $0 \in \tilde{U}$, $0 \in \tilde{V}$, and there exists $c \in C^\infty(\tilde{U}, \tilde{V})$ a $C^\infty-$diffeomorphism such that $c(0) = 0$,

$$c(\tilde{U} \cap \{x_1 > 0\}) = c(\tilde{U}) \cap \Omega \text{ and } c(\tilde{U} \cap \{x_1 = 0\}) = c(\tilde{U}) \cap \partial\Omega.$$

The orientation of $\partial\Omega$ is chosen in such a way that for any $x' \in \tilde{U} \cap \{x_1 = 0\}$,

$$\{\partial_1 c(0,x'), \partial_2 c(0,x'), \ldots, \partial_n c(0,x')\}$$

is a direct basis of $\mathbb{R}^n$ (canonically oriented). For $x' \in \tilde{U} \cap \{x_1 = 0\}$, we define $\nu(x')$ as the unique orthonormal inner vector at the tangent space $T_{c(0,x')}\partial\Omega$ (it is chosen such that $\{\nu(x'), \partial_2 c(0,x'), \ldots, \partial_n c(0,x')\}$ is a direct basis of $\mathbb{R}^n$). In particular, on $\mathbb{R}^n_+ := \{x_1 > 0\}$, $\nu(x') := (1, 0, \ldots, 0)$.

Here and in the sequel, we write for any $r > 0$

$$\tilde{B}_r := (-r, r) \times B_r^{(n-1)}(0) \tag{5.4}$$

where $B_r^{(n-1)}(0)$ denotes the ball of center 0 and radius r in $\mathbb{R}^{n-1}$. It is standard that there exists $\delta > 0$ such that

$$\begin{array}{cccc} \varphi: & \tilde{B}_{2\delta} & \to & \mathbb{R}^n\\ & (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} & \mapsto & c(0,x') + x_1 \nu(x') \end{array} \tag{5.5}$$

is a $C^\infty-$diffeomorphism onto its open image $\varphi(\tilde{B}_{2\delta})$, and

$$\varphi(\tilde{B}_{2\delta} \cap \{x_1 > 0\}) = \varphi(\tilde{B}_{2\delta}) \cap \Omega \text{ and } \varphi(\tilde{B}_{2\delta} \cap \{x_1 = 0\}) = \varphi(\tilde{B}_{2\delta}) \cap \partial\Omega. \tag{5.6}$$

We also have for all $x' \in B_\delta(0)^{(n-1)}$,

$$\nu(x') \text{ is the inner orthonormal unit vector at the tangent space } T_{\varphi(0,x')}\partial\Omega. \tag{5.7}$$

An important remark is that

$$d(\varphi(x_1, x'), \partial\Omega) = |x_1| \text{ for all } (x_1, x') \in \tilde{B}_{2\delta} \text{ close to } 0. \tag{5.8}$$

Consider the metric $g := \varphi^\star\text{Eucl}$ on $\tilde{B}_{2\delta}$, that is the pull-back of the Euclidean metric Eucl via the diffeomorphism φ. Following classical notations, we define

$$g_{ij}(x) := (\partial_i\varphi(x), \partial_j\varphi(x))_{\text{Eucl}} \text{ for all } x \in \tilde{B}_{2\delta} \text{ and } i,j = 1, ..., n. \tag{5.9}$$

Up to a change of coordinates, we can assume that $(\partial_2\varphi(0),...,\partial_n\varphi(0))$ is an orthogonal basis of $T_0\partial\Omega$. In other words, we then have that

$$g_{ij}(0)=\delta_{ij} \text{ for all } i,j=1,...,n. \tag{5.10}$$

We claim that

$$g_{i1}(x)=\delta_{i1} \text{ for all } x\in\tilde{B}_{2\delta} \text{ and } i=1,...,n. \tag{5.11}$$

Indeed, for any $x=(x_1,x')\in\tilde{B}_{2\delta}$, we have that $\partial_1\varphi(x)=\nu(x')$, which is a unitary vector. Therefore $g_{11}(x)=1$. For $i\geq 2$, we have

$$g_{1i}(x)=(\nu(x'),\partial_i\varphi(0,x')+x_1\partial_i\nu(x'))_{\text{Eucl}}=(\nu(x'),\partial_i\varphi(0,x'))_{\text{Eucl}}+x_1\partial_i\left(|\nu(x')|^2\right)/2.$$

Since $\nu(x')$ is orthogonal to the tangent space spanned by $(\partial_2\varphi(0,x'),...,\partial_n\varphi(0,x'))$ and $|\nu(x')|=1$, we get that $g_{1i}(x)=0$, which proves (5.11).

Fix now $\alpha\in\mathbb{R}$ and consider $\Theta\in C^\infty(\tilde{B}_{2\delta})$ such that $\Theta(0)=0$ and which will be constructed later (independently of α) with additional needed properties. Fix $\eta\in C_c^\infty(\tilde{B}_{2\delta})$ such that $\eta(x)=1$ for all $x\in\tilde{B}_\delta$. Define $u_\alpha\in C^\infty(\overline{\Omega}\setminus\{0\})$ as

$$u_\alpha\circ\varphi(x_1,x'):=\eta(x)x_1|x|^{-\alpha}(1+\Theta(x)) \text{ for all } (x_1,x')\in\tilde{B}_{2\delta}\setminus\{0\}. \tag{5.12}$$

In particular, $u_\alpha(x)>0$ for all $x\in\varphi(\tilde{B}_{2\delta})\cap\Omega$ and $u_\alpha(x)=0$ on $\Omega\setminus\varphi(\tilde{B}_{2\delta})$.

We claim that with a good choice of Θ, we have that

$$-\Delta u_\alpha=\tfrac{\alpha(n-\alpha)}{|x|^2}u_\alpha+O\left(\tfrac{u_\alpha(x)}{|x|}\right) \quad \text{ as } x\to 0. \tag{5.13}$$

Indeed, using the chart φ, we have that

$$(-\Delta u_\alpha)\circ\varphi(x_1,x')=-\Delta_g(u_\alpha\circ\varphi)(x_1,x')$$

for all $(x_1,x')\in\tilde{B}_\delta\setminus\{0\}$. Here, $-\Delta_g$ is the Laplace operator associated to the metric g, that is

$$-\Delta_g:=-g^{ij}\left(\partial_{ij}-\Gamma_{ij}^k\partial_k\right),$$

where

$$\Gamma_{ij}^k:=\frac{1}{2}g^{km}\left(\partial_i g_{jm}+\partial_j g_{im}-\partial_m g_{ij}\right),$$

and (g^{ij}) is the inverse of the matrix (g_{ij}). Here and in the sequel, we have adopted Einstein's convention of summation. It follows from (5.11) that

$$\begin{aligned}(-\Delta u_\alpha)\circ\varphi &= -\Delta_{\text{Eucl}}(u_\alpha\circ\varphi)-\sum_{i,j\geq 2}\left(g^{ij}-\delta^{ij}\right)\partial_{ij}(u_\alpha\circ\varphi)\\ &\quad +g^{ij}\Gamma_{ij}^1\partial_1(u_\alpha\circ\varphi)+\sum_{k\geq 2}g^{ij}\Gamma_{ij}^k\partial_k(u_\alpha\circ\varphi).\end{aligned} \tag{5.14}$$

It follows from the definition (5.12) that there exists $C>0$ such that for any $i,j,k\geq 2$, we have that

$$|\partial_{ij}(u_\alpha\circ\varphi)(x_1,x')|\leq C|x_1|\cdot|x|^{-\alpha-2} \text{ and } |\partial_k(u_\alpha\circ\varphi)(x_1,x')|\leq C|x_1|\cdot|x|^{-\alpha-1},$$

for all $(x_1,x')\in\tilde{B}_\delta\setminus\{0\}$. It follows from (5.10) that $g^{ij}-\delta^{ij}=O(|x|)$ as $x\to 0$. Therefore, (5.14) yields that as $x\to 0$,

$$(-\Delta u_\alpha)\circ\varphi = -\Delta_{\text{Eucl}}(u_\alpha\circ\varphi)+g^{ij}\Gamma_{ij}^1\partial_1(u_\alpha\circ\varphi)+O(x_1|x|^{-\alpha-1}) \tag{5.15}$$

The definition of g_{ij} and the expression of $\varphi(x_1,x')$ then yield that as $x\to 0$,

$$\begin{array}{rcl} g^{ij}\Gamma_{ij}^1 & = & -\frac{1}{2}\sum_{i,j\geq 2} g^{ij}\partial_1 g_{ij}\\ & = & -\sum_{i,j\geq 2} g^{ij}(x_1,x')\left((\partial_i\varphi(0,x'),\partial_j\nu(x'))+x_1(\partial_i(x'),\partial_j\nu(x'))\right)\\ & = & -\sum_{i,j\geq 2} g^{ij}(0,x')\left(\partial_i\varphi(0,x'),\partial_j\nu(x')\right)+O(|x_1|)\\ & = & H(x')+O(|x_1|),\end{array}$$

where $H(x')$ is the mean curvature of the $(n-1)-$manifold $\partial\Omega$ at $\varphi(0,x')$ oriented by the outer normal vector $-\nu(x')$. Using the expression (5.12) and using the smoothness of Θ, (5.15) yields

$$\begin{array}{rcl} (-\Delta u_\alpha)\circ\varphi & = & \left(-\Delta_{\text{Eucl}}(x_1|x|^{-\alpha})\right)\cdot(1+\Theta)+|x|^{-\alpha}\left(H(x')(1+\Theta)-2\partial_1\Theta\right)\\ & & +O(x_1|x|^{-\alpha-1})\qquad\text{as } x\to 0.\end{array}$$

We now define

$$\Theta(x_1,x'):=e^{-\frac{1}{2}x_1H(x')}-1 \text{ for all } x=(x_1,x')\in\tilde{B}_{2\delta}.$$

Clearly $\Theta(0)=0$ and $\Theta\in C^\infty(\tilde{B}_{2\delta})$. Noting that

$$-\Delta_{\text{Eucl}}\left(x_1|x|^{-\alpha}\right)=\frac{\alpha(n-\alpha)}{|x|^2}x_1|x|^{-\alpha},$$

we then get that as $x\to 0$,

$$(-\Delta u_\alpha)\circ\varphi \;=\; \frac{\alpha(n-\alpha)}{|x|^2}x_1|x|^{-\alpha}\cdot(1+\Theta)+O(x_1|x|^{-\alpha-1}) \tag{5.16}$$

With the choice that $g_{ij}(0)=\delta_{ij}$, we have that $(\partial_i\varphi(0))_{i=1,\dots,n}$ is an orthonormal basis of $\mathbb{R}^n$, and therefore $|\varphi(x)|=|x|(1+O(|x|))$ as $x\to 0$. It then follows from (5.16) and (5.12) that

$$-\Delta u_\alpha=\frac{\alpha(n-\alpha)}{|x|^2}u_\alpha+O(|x|^{-1}u_\alpha)\qquad\text{as } x\to 0. \tag{5.17}$$

This proves (5.13). We now proceed with the construction of the sub- and super-solutions. Let $\alpha\in\{\alpha_-(\gamma),\alpha_+(\gamma)\}$ in such a way that $\alpha(n-\alpha)=\gamma$ and consider $\beta,\lambda\in\mathbb{R}$ to be chosen later. It follows from (5.13) that

$$\begin{array}{rcl} \left(-\Delta-\frac{\gamma+O(|x|^\tau)}{|x|^2}\right)(u_\alpha+\lambda u_\beta) & = & \frac{\lambda(\beta(n-\beta)-\gamma)}{|x|^2}u_\beta\\ & & +\frac{O(|x|^\tau)}{|x|^2}u_\alpha+O(|x|^{-1}u_\alpha)+O(|x|^{\tau-2}u_\beta)\\ & = & \frac{u_\beta}{|x|^2}\big(\lambda(\beta(n-\beta)-\gamma)\\ & & +O(|x|^\tau)+O(|x|^{\tau+\beta-\alpha})+O(|x|^{1+\beta-\alpha})\big)\end{array}$$

as $x\to 0$. Choose β such that $\alpha-\tau<\beta<\alpha$ in such a way that $\beta\neq\alpha_-(\gamma)$ and $\beta\neq\alpha_+(\gamma)$. In particular, $\beta>\alpha-1$ and $\beta(n-\beta)-\gamma\neq 0$. We then have

$$\left(-\Delta-\frac{\gamma+O(|x|^\tau)}{|x|^2}\right)(u_\alpha+\lambda u_\beta)=\frac{u_\beta}{|x|^2}\left(\lambda(\beta(n-\beta)-\gamma)+O(|x|^{\tau+\beta-\alpha})\right)) \tag{5.18}$$

as $x \to 0$. Choose $\lambda \in \mathbb{R}$ such that $\lambda(\beta(n-\beta)-\gamma) > 0$. Finally, let $u_{\alpha,+} := u_\alpha + \lambda u_\beta$ and $u_{\alpha,-} := u_\alpha - \lambda u_\beta$. They clearly satisfy (5.2) and (5.3), which completes the proof of Proposition 5.1. $\square$

6. Regularity and Hopf-type result for the operator L_γ

This section is devoted to the proof of the following key result.

Theorem 6.1 (Optimal regularity and Generalized Hopf's Lemma)**.** *Fix $\gamma < \frac{n^2}{4}$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that*

$$|f(x,v)| \leq C|v| \left(1 + \frac{|v|^{2^\star(s)-2}}{|x|^s}\right) \text{for all } x \in \Omega \text{ and } v \in \mathbb{R}.$$

Let $u \in D^{1,2}(\Omega)_{loc,0}$ be a weak solution of

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x,u) \text{ in } D^{1,2}(\Omega)_{loc,0} \tag{6.1}$$

for some $\tau > 0$. Then there exists $K \in \mathbb{R}$ such that

$$\lim_{x \to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = K. \tag{6.2}$$

Moreover, if $u \geq 0$ and $u \not\equiv 0$, we have that $K > 0$.

As mentioned in the introduction, this can be viewed as a generalization of Hopf's Lemma in the following sense: when $\gamma = 0$ (and then $\alpha_-(\gamma) = 0$), the classical Nash-Moser regularity scheme yields $u \in C^1_{loc}$, and when $u \geq 0$, $u \not\equiv 0$, Hopf's comparison principle yields $\partial_\nu u(0) < 0$, which is a reformulation of (6.2) when $\alpha_-(\gamma) = 0$.

The remainder of this section is devoted to the proof of Theorem 6.1. In this whole section, by a slight abuse of notation, $u \mapsto -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u$ will denote an operator $u \mapsto -\Delta u - \frac{\gamma + a(x)}{|x|^2} u$ where $a \in C^0(\overline{\Omega} \setminus \{0\})$ such that $a(x) = O(|x|^\tau)$ as $\tau \to 0$.

We shall need the following two lemmas, which will be used frequently throughout the paper.

Lemma 6.2. (Rigidity of solutions) *Let $u \in C^2(\overline{\mathbb{R}^n_+} \setminus \{0\})$ be a nonnegative function such that*

$$-\Delta u - \frac{\gamma}{|x|^2} u = 0 \textit{ in } \mathbb{R}^n_+ \text{ ; } u = 0 \textit{ on } \partial\mathbb{R}^n_+. \tag{6.3}$$

Suppose there exists $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$ such that $u(x) \leq C|x|^{1-\alpha}$, then there exists $\lambda \geq 0$ such that

$$u(x) = \lambda x_1 |x|^{-\alpha} \textit{ for all } x \in \mathbb{R}^n_+.$$

We note that this lemma is only a first step in proving rigidity for solutions of $L_\gamma u = 0$ on $\mathbb{R}^n_+$. Indeed, the pointwise assumption above is not necessary as it will be removed in Proposition 7.4, which will be a consequence of the classification Theorem 7.1.

Proof of Lemma 6.2: We first assume that $\alpha := \alpha_-(\gamma)$ and prove that

$$\text{either } u \equiv 0 \text{ or } \liminf_{|x| \to +\infty} \frac{u(x)}{x_1 |x|^{-\alpha_-(\gamma)}} > 0. \tag{6.4}$$

Indeed suppose that the second situation does not hold, that is there exists $(x_i)_i \in \mathbb{R}^n_+$ such that

$$\lim_{i\to+\infty}|x_i| = +\infty \text{ and } \lim_{i\to+\infty}\frac{u(x_i)}{(x_i)_1|x_i|^{-\alpha_-(\gamma)}} = 0.$$

Define $r_i := |x_i|$, $\theta_i := \frac{x_i}{|x_i|}$ and $u_i(x) := r_i^{\alpha_-(\gamma)-1}u(r_i x)$ for all i and all $x \in \mathbb{R}^n_+$. It follows from the hypothesis of the lemma that for all i,

$$-\Delta u_i - \frac{\gamma}{|x|^2}u_i = 0 \text{ in } \mathbb{R}^n_+ \; ; \; u_i = 0 \text{ on } \partial\mathbb{R}^n_+,$$

and $0 \leq u_i(x) \leq C|x|^{1-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$. It follows from elliptic theory that there exists $\hat{u} \in C^2(\mathbb{R}^n_+)$ such that $u_i \to \hat{u}$ in $C^2_{loc}(\overline{\mathbb{R}^n_+}\setminus\{0\})$. In particular, we have that

$$-\Delta\hat{u} - \frac{\gamma}{|x|^2}\hat{u} = 0 \text{ in } \mathbb{R}^n_+ \; ; \; \hat{u} \geq 0 \text{ in } \mathbb{R}^n_+, \; ; \; \hat{u} = 0 \text{ on } \partial\mathbb{R}^n_+.$$

Let $\theta := \lim_{i\to+\infty}\theta_i$. It follows from the convergence that $\hat{u}(\theta) = 0$ if $\theta \in \mathbb{R}^n_+$, and $\partial_1\hat{u}(\theta) = 0$ if $\theta \in \partial\mathbb{R}^n_+$. Hopf's maximum principle yields that $\hat{u} \equiv 0$. In particular, we get that

$$\lim_{i\to+\infty}\sup_{x\in\partial B_{r_i}(0)}\frac{u(x)}{x_1|x|^{-\alpha_-(\gamma)}} = 0.$$

For $\epsilon > 0$, there exists i_0 such that $u(x) \leq \epsilon x_1|x|^{-\alpha_-(\gamma)}$ for all $x \in \partial B_{r_i}(0)$ and $i \geq i_0$. Since $u - \epsilon x_1|x|^{-\alpha_-(\gamma)}$ is locally in $D^{1,2}$ and $(-\Delta - \frac{\gamma}{|x|^2})(u - \epsilon x_1|x|^{-\alpha_-(\gamma)}) = 0$, it follows from the maximum principle (and coercivity) that $u(x) \leq \epsilon x_1|x|^{-\alpha_-(\gamma)}$ for $x \in B_{r_i}(0)$ with $i \geq i_0$. Letting $i \to +\infty$ yields $u(x) \leq \epsilon x_1|x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$ and all $\epsilon > 0$. Letting $\epsilon \to 0$ yields $u \leq 0$, and therefore $u \equiv 0$. This proves the claim (6.4).

We now assume that $u \not\equiv 0$. It then follows from (6.4) that there exists $\epsilon_0 > 0$ and $R_0 > 0$ such that $u(x) \geq \epsilon_0 x_1|x|^{-\alpha_-(\gamma)}$ for all $|x| \geq B_{R_0}(0)$. Applying again the maximum principle on $B_{R_0}(0)$, we get that $u(x) \geq \epsilon_0 x_1|x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$. We have so far proved that

(6.5) $u \equiv 0$ or there exists $\epsilon_0 > 0$ such that $u(x) \geq \epsilon_0 x_1|x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$.

Let now $\lambda := \max\{k \geq 0$ such that $u(x) \geq kx_1|x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+\}$. Then $\bar{u}(x) := u(x) - \lambda x_1|x|^{-\alpha_-(\gamma)} \geq 0$ satisfies (6.3). It then follows from (6.5) that $\bar{u} \equiv 0$ or $\bar{u}(x) \geq \epsilon_0 x_1|x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$ for some $\epsilon_0 > 0$. This second case cannot happen since it would imply that $u \geq (\lambda + \epsilon_0)x_1|x|^{-\alpha_-(\gamma)}$, which is a contradiction. Therefore $\bar{u} \equiv 0$ and the Proposition is proved when $\alpha = \alpha_-(\gamma)$.

To finish, it remains to consider the case where $\alpha = \alpha_+(\gamma)$. Here we define $\tilde{u}(x) := |x|^{2-n}u(x/|x|^2)$ to be the Kelvin transform of u. The function $\tilde{u}$ then satisfies (6.3) with $\alpha_-(\gamma)$. It then follows from the first part of this proof that $\tilde{u} = \lambda x_1|x|^{-\alpha_-(\gamma)}$. Coming back to the initial function u yields $u = \lambda x_1|x|^{-\alpha_+(\gamma)}$. This completes the proof of Lemma 6.2. $\square$

Lemma 6.3. *Assume that $u \in D^{1,2}(\Omega)_{loc,0}$ is a weak solution of*

$$\left\{\begin{array}{rl} -\Delta u - \frac{\gamma+O(|x|^\tau)}{|x|^2}u = 0 & \text{in } D^{1,2}(\Omega)_{loc,0} \\ u = 0 & \text{on } B_{2\delta}(0)\cap\partial\Omega, \end{array}\right. \tag{6.6}$$

for some $\tau > 0$. We fix $\delta > 0$. Then there exists $C_1 > 0$ such that

$$|u(x)| \leq C_1 d(x, \partial\Omega)|x|^{-\alpha_-(\gamma)} \text{ for } x \in \Omega \cap B_\delta(0). \tag{6.7}$$

Moreover, if $u > 0$ in Ω, then there exists $C_2 > 0$ such that

$$u(x) \geq C_2 d(x, \partial\Omega)|x|^{-\alpha_-(\gamma)} \text{ for } x \in \Omega \cap B_\delta(0). \tag{6.8}$$

Proof of Lemma 6.3: First, we assume that $u \in D^{1,2}(\Omega)_{loc,0}$, satisfies (6.27) and $u > 0$ on $B_\delta(0) \cap \Omega$. We claim that there exists $C_0 > 0$ such that

$$\frac{1}{C_0} \frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}} \leq u(x) \leq C_0 \frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}} \text{ for all } x \in \Omega \cap B_\delta(0). \tag{6.9}$$

Indeed, since u is smooth outside 0, it follows from Hopf's Maximum principle that there exists $C_1, C_2 > 0$ such that

$$C_1 d(x, \partial\Omega) \leq u(x) \leq C_2 d(x, \partial\Omega) \text{ for all } x \in \Omega \cap \partial B_\delta(0). \tag{6.10}$$

Let $u_{\alpha_-(\gamma),+}$ be the super-solution constructed in Proposition 5.1. It follows from (6.10) and the asymptotics (5.3) of $u_{\alpha_-(\gamma),+}$ that there exists $C_3 > 0$ such that

$$u(x) \leq C_3 u_{\alpha_-(\gamma),+}(x) \text{ for all } x \in \partial(B_\delta(0) \cap \Omega).$$

Since u is a solution and $u_{\alpha_-(\gamma),+}$ is a supersolution, both being in $D^{1,2}(\Omega)_{loc,0}$, it follows from the maximum principle (by choosing $\delta > 0$ small enough so that $-\Delta - (\gamma + O(|x|^\tau))|x|^{-2}$ is coercive on $B_\delta(0) \cap \Omega$) that $u(x) \leq C_3 u_{\alpha_-(\gamma),+}(x)$ for all $x \in B_\delta(0) \cap \Omega$. In particular, it follows from the asymptotics (5.3) of $u_{\alpha_-(\gamma),+}$ that there exists $C_4 > 0$ such that

$$u(x) \leq C_4 d(x, \partial\Omega)|x|^{-\alpha_-(\gamma)} \text{ for all } x \in \Omega \cap B_\delta(0).$$

Arguing similarly with the lower-bound in (6.10) and the subsolution $u_{\alpha_-(\gamma),-}$, we get the existence of $C_0 > 0$ such that (6.9) holds. This yields Lemma 6.3 for $u > 0$.

Now we deal with the case when u is a sign-changing solution for (6.6). We then define $u_1, u_2 : B_\delta(0) \cap \Omega \to \mathbb{R}$ be such that

$$\begin{cases} -\Delta u_1 - \frac{\gamma + O(|x|^\tau)}{|x|^2} u_1 = 0 & \text{in } B_\delta(0) \cap \Omega \\ u_1(x) = \max\{u(x), 0\} & \text{on } \partial(B_\delta(0) \cap \Omega). \end{cases}$$

$$\begin{cases} -\Delta u_2 - \frac{\gamma + O(|x|^\tau)}{|x|^2} u_2 = 0 & \text{in } B_\delta(0) \cap \Omega \\ u_2(x) = \max\{-u(x), 0\} & \text{on } \partial(B_\delta(0) \cap \Omega). \end{cases}$$

The existence of such solutions is ensured by choosing $\delta > 0$ small enough so that the operator $-\Delta - (\gamma + O(|x|^\tau))|x|^{-2}$ is coercive on $B_\delta(0) \cap \Omega$. In particular, $u_1, u_2 \in D^{1,2}(\Omega)_{loc,0}$, $u_1, u_2 \geq 0$ and $u = u_1 - u_2$. It follows from the maximum principle that for all i, either $u_i \equiv 0$ or $u_i > 0$. The first part of the proof yields the upper bound for u_1, u_2. Since $u = u_1 - u_2$, we then get (6.7). This ends the proof of Lemma 6.3. □

This lemma allows to construct sub- and super solutions with Dirichlet boundary conditions on any small smooth domain.

Proposition 6.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, and let W be a smooth domain of $\mathbb{R}^n$ such that for some $r>0$ small enough, we have*

$$B_r(0)\cap\Omega\subset W\subset B_{2r}(0)\cap\Omega \text{ and } B_r(0)\cap\partial W=B_r(0)\cap\partial\Omega. \tag{6.11}$$

Fix $\gamma<\frac{n^2}{4}$, $0<\tau\leq 1$ and $\beta\in\mathbb{R}$ such that $\alpha_+(\gamma)-\tau<\beta<\alpha_+(\gamma)$ and $\beta\neq\alpha_-(\gamma)$. Then, up to choosing r small enough, there exists $u^{(d)}_{\alpha_+(\gamma),+}, u^{(d)}_{\alpha_+(\gamma),-}\in C^\infty(\overline{W}\setminus\{0\})$ such that

$$\left\{\begin{array}{rl} u^{(d)}_{\alpha_+(\gamma),+}, u^{(d)}_{\alpha_+(\gamma),+}=0 & \text{in } \partial W\setminus\{0\}\\ -\Delta u^{(d)}_{\alpha_+(\gamma),+}-\frac{\gamma+O(|x|^\tau)}{|x|^2}u^{(d)}_{\alpha_+(\gamma),+}>0 & \text{in } W\\ -\Delta u^{(d)}_{\alpha_+(\gamma),-}-\frac{\gamma+O(|x|^\tau)}{|x|^2}u^{(d)}_{\alpha_+(\gamma),-}<0 & \text{in } W. \end{array}\right. \tag{6.12}$$

Moreover, we have as $x\to 0$, $x\in\Omega$ that

$$u^{(d)}_{\alpha_+(\gamma),+}(x)=\frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}(1+O(|x|^{\alpha-\beta})) \tag{6.13}$$

and

$$u^{(d)}_{\alpha_+(\gamma),-}(x)=\frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}(1+O(|x|^{\alpha-\beta})). \tag{6.14}$$

Proof of Proposition 6.4: Take $\eta\in C^\infty(\mathbb{R}^n)$ such that $\eta(x)=0$ for $x\in B_{\delta/4}(0)$ and $\eta(x)=1$ for $x\in\mathbb{R}^n\setminus B_{\delta/3}(0)$. Define on W the function

$$f(x):=\left(-\Delta-\frac{\gamma+O(|x|^\tau)}{|x|^2}\right)(\eta u_{\alpha_+(\gamma),+}),$$

where $u_{\alpha_+(\gamma),+}$ is given by Proposition 5.1. Note that f vanishes around 0 and that it is in $C^\infty(\overline{W})$. Let $v\in D^{1,2}(W)$ be such that

$$\left\{\begin{array}{rl} -\Delta v-\frac{\gamma+O(|x|^\tau)}{|x|^2}v=f & \text{in } W\\ v=0 & \text{on } \partial W. \end{array}\right.$$

Note that for $r>0$ small enough, $-\Delta-(\gamma+O(|x|^\tau))|x|^{-2}$ is coercive on W, and therefore, the existence of v is ensured for small r. Define

$$u^{(d)}_{\alpha_+(\gamma),+}:=u_{\alpha_+(\gamma),+}-\eta u_{\alpha_+(\gamma),+}+v.$$

The properties of W and the definition of η and v yield

$$\left\{\begin{array}{rl} u^{(d)}_{\alpha_+(\gamma),+}=0 & \text{in } \partial W\setminus\{0\}\\ -\Delta u^{(d)}_{\alpha_+(\gamma),+}-\frac{\gamma+O(|x|^\tau)}{|x|^2}u^{(d)}_{\alpha_+(\gamma),+}>0 & \text{in } W. \end{array}\right.$$

Moreover, since $-\Delta v-(\gamma+O(|x|^\tau))|x|^{-2}v=0$ around 0 and $v\in D^{1,2}(W)$, it follows from Lemma 6.3 that there exists $C>0$ such that $|v(x)|\leq Cd(x,W)|x|^{-\alpha_-(\gamma)}$ for all $x\in W$. Then (6.13) follows from the asymptotics (5.3) of $u_{\alpha_+(\gamma),+}$ and the fact that $\alpha_-(\gamma)<\alpha_+(\gamma)$. We argue similarly for $u^{(d)}_{\alpha_+(\gamma),-}$. This proves Proposition 6.4. ☐

Lemma 6.5. *Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be as in the statement of Theorem 6.1, and consider $u\in D^{1,2}(\Omega)_{loc,0}$ such that (6.1) holds. Assume there exists $C>0$ and $\alpha\in\{\alpha_+(\gamma),\alpha_-(\gamma)\}$ such that*

$$|u(x)|\leq C|x|^{1-\alpha} \text{ for } x\to 0,\ x\in\Omega. \tag{6.15}$$

If $f \not\equiv 0$, we assume that $\alpha = \alpha_-(\gamma)$.

(1) *Then, there exists $C_1 > 0$ such that*

$$|\nabla u(x)| \leq C_1 |x|^{-\alpha} \text{ as } x \to 0,\ x \in \Omega. \tag{6.16}$$

(2) *If $\lim_{x\to 0} |x|^{\alpha-1}u(x) = 0$, then $\lim_{x\to 0} |x|^{\alpha}|\nabla u(x)| = 0$. Moreover, if $u > 0$, then there exists $l \geq 0$ such that*

$$\lim_{x\to 0} \frac{|x|^{\alpha} u(x)}{d(x, \partial\Omega)} = l \text{ and } \lim_{x\to 0,\, x\in\partial\Omega} |x|^{\alpha}|\nabla u(x)| = l \tag{6.17}$$

Proof of Lemma 6.5: Assume that (6.15) holds. As a first remark, we claim that we can assume that for some $\tau > 0$,

$$-\Delta u - \frac{\gamma + O(|x|^{\tau})}{|x|^2} u = 0 \text{ in } D^{1,2}(\Omega)_{loc,0}. \tag{6.18}$$

Indeed, this is clear if $f \equiv 0$. If $f \not\equiv 0$, since $\alpha = \alpha_-(\gamma)$, we have as $x \to 0$,

$$\begin{aligned} |f(x,u)| &\leq C|u| \left(1 + |x|^{-s}|x|^{-(2^\star(s)-2)(\alpha_-(\gamma)-1)}\right) \\ &\leq C\frac{|u|}{|x|^2}\left(|x|^2 + |x|^{(2^\star(s)-2)(\frac{n}{2}-\alpha_-(\gamma))}\right) = O\left(|x|^{\tau'}\frac{u}{|x|^2}\right) \end{aligned}$$

for some $\tau' > 0$. Plugging this inequality into (6.1) and replacing τ by $\min\{\tau, \tau'\}$ yields (6.18).

In the sequel, we shall write $\omega(x) := \frac{|x|^{\alpha}u(x)}{d(x,\partial\Omega)}$ for $x \in \Omega$. Let $(x_i)_i \in \Omega$ be such that

$$\lim_{i\to+\infty} x_i = 0 \tag{6.19}$$

Choose a chart φ as in (5.5) such that $d\varphi_0 = Id_{\mathbb{R}^n}$. For any i, define $X_i \in \mathbb{R}^n_+$ such that $x_i = \varphi(X_i)$, $r_i := |X_i|$ and $\theta_i := \frac{X_i}{|X_i|}$. In particular, $\lim_{i\to+\infty} r_i = 0$ and $|\theta_i|$=1 for all i. Set

$$\tilde{u}_i(x) := r_i^{\alpha-1} u(\varphi(r_i x)) \text{ for all } i \text{ and } x \in B_R(0) \cap \mathbb{R}^n_+ \,;\, x \neq 0.$$

Equation (6.18) then rewrites

$$\left\{ \begin{array}{rl} -\Delta_{g_i}\tilde{u}_i - \frac{\gamma+o(1)}{|x|^2}\tilde{u}_i = 0 & \text{in } B_R(0)\cap\mathbb{R}^n_+ \\ \tilde{u}_i = 0 & \text{in } B_R(0)\cap\partial\mathbb{R}^n_+, \end{array} \right. \tag{6.20}$$

where $g_i(x) := (\varphi^\star \mathrm{Eucl})(r_i x)$ is a metric that goes to Eucl on every compact subset of $\mathbb{R}^n$ as $i \to \infty$. Here, $o(1) \to 0$ in $C^0_{loc}(\overline{\mathbb{R}^n_+} \setminus \{0\})$. It follows from (6.15) and (6.19) that

$$|\tilde{u}_i(x)| \leq C|x|^{1-\alpha} \text{ for all } i \text{ and all } x \in B_R(0) \cap \mathbb{R}^n_+, \tag{6.21}$$

It follows from elliptic theory, that there exists $\tilde{u} \in C^2(\overline{\mathbb{R}^n_+} \setminus \{0\})$ such that $\tilde{u}_i \to \tilde{u}$ in $C^1_{loc}(\overline{\mathbb{R}^n_+} \setminus \{0\})$. By letting $\theta := \lim_{i\to+\infty}\theta_i$ ($|\theta| = 1$), we then have that for any $j = 1, ..., n$, $\partial_j \tilde{u}_i(\theta_i) \to \partial_j \tilde{u}(\theta)$ as $i \to +\infty$, which rewrites

$$\lim_{i\to+\infty} |x_i|^{\alpha} \partial_j u(x_i) = \partial_j \tilde{u}(\theta) \text{ for all } j = 1, ..., n. \tag{6.22}$$

We now prove (6.16). For that, we argue by contradiction and assume that there exists a sequence $(x_i)_i \in \Omega$ that goes to 0 as $i \to +\infty$ and such that $|x_i|^{\alpha}|\nabla u(x_i)| \to +\infty$ as $i \to +\infty$. It then follows from (6.22) that $|x_i|^{\alpha}|\nabla u(x_i)| = O(1)$ as $i \to +\infty$. A contradiction to our assumption, which proves (6.16). The case when $|x|^{\alpha}u(x) \to 0$ as $x \to 0$ goes similarly.

Now we consider the case when $u>0$, which implies that $\tilde{u}_i\geq 0$ and $\tilde{u}\geq 0$. We let $l\in[0,+\infty]$ and $(x_i)_i\in\Omega$ be such that

$$\lim_{i\to+\infty} x_i=0 \text{ and } \lim_{i\to+\infty}\omega(x_i)=l. \tag{6.23}$$

We claim that

$$0\leq l<+\infty \text{ and } \lim_{x\to 0}\omega(x)=l\in[0,+\infty). \tag{6.24}$$

Indeed, using the notations above, we get that

$$\lim_{i\to+\infty}\frac{\tilde{u}_i(\theta_i)}{(\theta_i)_1}=l.$$

The convergence of $\tilde{u}_i$ in $C^1_{loc}(\overline{\mathbb{R}^n_+}\setminus\{0\})$ then yields $l<+\infty$. Passing to the limit as $i\to+\infty$ in (6.20), we get

$$\left\{\begin{array}{rl} -\Delta_{\text{Eucl}}\tilde{u}-\frac{\gamma}{|x|^2}\tilde{u}=0 & \text{in } \mathbb{R}^n_+\\ \tilde{u}\geq 0 & \text{in } \mathbb{R}^n_+\\ \tilde{u}=0 & \text{in } \partial\mathbb{R}^n_+.\end{array}\right.$$

The limit (6.23) can be rewritten as $\tilde{u}(\theta)=l\theta_1$ if $\theta\in\mathbb{R}^n_+$ and $\partial_1\tilde{u}(\theta)=l$ if $\theta\in\partial\mathbb{R}^n_+$. The rigidity Lemma 6.2 then yields

$$\tilde{u}(x)=lx_1|x|^{-\alpha} \text{ for all } x\in\mathbb{R}^n_+.$$

In particular, since the differential of φ at 0 is the identity map, it follows from the convergence of $\tilde{u}_i$ to $\tilde{u}$ locally in C^1 that

$$\lim_{i\to+\infty}\sup_{x\in\Omega\cap\partial B_{r_i}(0)}\frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha}}=\sup_{x\in\mathbb{R}^n_+\cap\partial B_1(0)}\frac{\tilde{u}(x)}{x_1|x|^{-\alpha}}=l \tag{6.25}$$

and

$$\lim_{i\to+\infty}\inf_{x\in\Omega\cap\partial B_{r_i}(0)}\frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha}}=\inf_{x\in\mathbb{R}^n_+\cap\partial B_1(0)}\frac{\tilde{u}(x)}{x_1|x|^{-\alpha}}=l. \tag{6.26}$$

We distinguish two cases:

Case 1: $\alpha=\alpha_+(\gamma)$. Let W and $u^{(d)}_{\alpha_+(\gamma),-}$ be as in Proposition 6.4, and fix $\epsilon>0$. Note that the existence and properties of $u^{(d)}_{\alpha_+(\gamma),-}$ do not use the Lemma that is currently proved. It follows from (6.26) that there exists i_0 such that for $i\geq i_0$, we have that

$$u(x)\geq(l-\epsilon)u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x\in W\cap\partial B_{r_i}(0).$$

Since $(-\Delta-(\gamma+O(|x|^\tau))|x|^{-2})(u-(l-\epsilon)u^{(d)}_{\alpha_+(\gamma),-})\geq 0$ in $W\setminus B_{r_i}(0)$ and since $u_{\alpha_+(\gamma),-}$ vanishes on $\partial W\setminus\{0\}$, it follows from the comparison principle that

$$u(x)\geq(l-\epsilon)u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x\in W\setminus\partial B_{r_i}(0).$$

Letting $i\to+\infty$ yields

$$u(x)\geq(l-\epsilon)u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x\in W\setminus\{0\}.$$

It follows from this inequality and the asymptotics for $u^{(d)}_{\alpha_+(\gamma),-}$ that

$$\liminf_{x\to 0}\omega(x)\geq l.$$

Note that this is valid for any $l\in\mathbb{R}$ satisfying (6.23). By taking $l:=\limsup_{x\to 0}\omega(x)$, we then get that $\lim_{x\to 0}\omega(x)=l$.

Case 2: $\alpha = \alpha_-(\gamma)$**.** Consider the super- and sub-solutions $u_{\alpha_-(\gamma),+}, u_{\alpha_-(\gamma),-}$ constructed in Proposition 5.1. It follows from (6.25) and (6.26) that for $\epsilon > 0$, there exists i_0 such that for $i \geq i_0$, we have

$$(l-\epsilon)u_{\alpha_-(\gamma),-}(x) \leq u(x) \leq (l+\epsilon)u_{\alpha_-(\gamma),+}(x) \text{ for all } x \in \Omega \cap \partial B_{r_i}(0).$$

Since the operator $-\Delta - (\gamma + O(|x|^\tau))|x|^{-2}$ is coercive on $\Omega \cap B_{r_i}(0)$ and that the functions we consider are in $D^{1,2}_{loc,0}(\Omega \cap B_{r_i}(0))$ (i.e., they are variational), it follows from the maximum principle that

$$(l-\epsilon)u_{\alpha_-(\gamma),-}(x) \leq u(x) \leq (l+\epsilon)u_{\alpha_-(\gamma),+}(x) \text{ for all } x \in \Omega \cap B_{r_i}(0).$$

Using the asymptotics (5.3) of the sub- and super-solution, we get that

$$(l-\epsilon) \leq \liminf_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} \leq \limsup_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} \leq (l+\epsilon).$$

Letting $\epsilon \to 0$ yields $\lim_{x\to 0}\omega(x) = l \geq 0$. This ends Case 2 and completes the proof of (6.24).

The case $u > 0$ is a consequence of (6.24) and (6.22) (note that for the second limit, $x_i \in \partial\Omega$ rewrites as $\theta_i \in \partial\mathbb{R}^n_+$ and therefore $(\theta_i)_1 = 0$). This ends the proof of Lemma 6.5. □

The following lemma is essentially Theorem 6.1 in the case of linear equations of the form $L_\gamma u = a(x)u$.

Lemma 6.6. *Assume that $u \in D^{1,2}(\Omega)_{loc,0}$ is a weak solution of*

$$\left\{ \begin{array}{rl} -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = 0 & \text{in } D^{1,2}(\Omega)_{loc,0} \\ u = 0 & \text{on } B_{2\delta}(0) \cap \partial\Omega, \end{array} \right. \tag{6.27}$$

for some $\tau > 0$ with $\gamma < \frac{n^2}{4}$. Then, there exists $\ell \in \mathbb{R}$ such that

$$\lim_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = \ell.$$

Proof of Lemma 6.6: First, we assume that $u \in D^{1,2}(\Omega)_{loc,0}$, satisfies (6.27) and $u > 0$ on $B_\delta(0) \cap \Omega$. It then follows from Lemma 6.3 that there exists $C_0 > 0$ such that

$$\frac{1}{C_0} \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} \leq u(x) \leq C_0 \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} \text{ for all } x \in \Omega \cap B_\delta(0).$$

Since $u > 0$, this estimate coupled with Lemma 6.5 yields Lemma 6.6 for $u > 0$.

Now we deal with the case when u is a sign-changing solution for (6.27). We define $u_1, u_2 : B_\delta(0) \cap \Omega \to \mathbb{R}_{\geq 0}$ as in the proof of Lemma 6.3. The first part of the proof yields that there exist $l_1, l_2 \geq 0$ such that

$$\lim_{x\to 0} \frac{u_1(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = l_1 \text{ and } \lim_{x\to 0} \frac{u_2(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = l_2.$$

Therefore, we get that

$$\lim_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = l_1 - l_2 \in \mathbb{R}.$$

This completes the proof of Lemma 6.6. □

Proof of Theorem 6.1: We let here $u \in D^{1,2}(\Omega)_{loc,0}$ be a solution to (6.1), that is

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x,u) \text{ weakly in } D^{1,2}(\Omega)_{loc,0} \tag{6.28}$$

for some $\tau > 0$. We shall first use the classical DeGiorgi-Nash-Moser iterative scheme (see Gilbarg-Trudinger [28], and Hebey [32] for expositions in book form). We skip most of the computations and refer to Ghoussoub-Robert (Proposition A.1 of [24]) for the details. We fix $\delta_0 > 0$ such that (i) there exists $\tilde{\eta} \in C^\infty(B_{4\delta_0}(0))$ such that $\tilde{\eta}(x) = 1$ for $x \in B_{2\delta_0}(0)$, (ii) $\tilde{\eta} u \in D^{1,2}(\Omega)$ and (iii) u is a weak solution to (6.28) when tested on $\tilde{\eta}\varphi$ with $\varphi \in D^{1,2}(\Omega)$ (see the definition of weak solution given in the introduction).

The proof goes through four steps.

Step 1: Let $\beta \geq 1$ be such that $\frac{4\beta}{(\beta+1)^2} > \frac{4}{n^2}\gamma$. Assume that $u \in L^{\beta+1}(\Omega \cap B_{\delta_0}(0))$. We claim that

$$u \in L^{\frac{n}{n-2}(\beta+1)}(\Omega \cap B_{\delta_0}(0)). \tag{6.29}$$

Indeed, fix $\beta \geq 1$, $L > 0$, and define $G_L, H_L : \mathbb{R} \to \mathbb{R}$ as

$$G_L(t) := \left\{ \begin{array}{ll} |t|^{\beta-1} t & \text{if } |t| \leq L \\ \beta L^{\beta-1}(t-L) + L^\beta & \text{if } t \geq L \\ \beta L^{\beta-1}(t+L) - L^\beta & \text{if } t \leq -L \end{array} \right. \tag{6.30}$$

and

$$H_L(t) := \left\{ \begin{array}{ll} |t|^{\frac{\beta-1}{2}} t & \text{if } |t| \leq L \\ \frac{\beta+1}{2} L^{\frac{\beta-1}{2}}(t-L) + L^{\frac{\beta+1}{2}} & \text{if } t \geq L \\ \frac{\beta+1}{2} L^{\frac{\beta-1}{2}}(t+L) - L^{\frac{\beta+1}{2}} & \text{if } t \leq -L \end{array} \right. \tag{6.31}$$

As easily checked,

$$0 \leq t G_L(t) \leq H_L(t)^2 \text{ and } G_L'(t) = \frac{4\beta}{(\beta+1)^2} (H_L'(t))^2 \tag{6.32}$$

for all $t \in \mathbb{R}$ and all $L > 0$. We fix $\delta > 0$ small that will be chosen later. We let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 1$ for $x \in B_{\delta/2}(0)$ and $\eta(x) = 0$ for $x \in \mathbb{R}^n \setminus B_\delta(0)$. Multiplying equation (6.28) with $\eta^2 G_L(u) \in D^{1,2}(\Omega)$, we get that

$$\begin{aligned} \int_\Omega (\nabla u, \nabla(\eta^2 G_L(u)))\, dx \quad &- \quad \int_\Omega \frac{\gamma + O(|x|^\tau)}{|x|^2} \eta^2 u G_L(u)\, dx \\ &= \quad \int_\Omega f(x,u) \eta^2 G_L(u)\, dx. \end{aligned} \tag{6.33}$$

Integrating by parts, and using formulae (6.30) to (6.32) above (see [24] for details) yield

$$\begin{aligned} \int_\Omega (\nabla u, \nabla(\eta^2 G_L(u)))\, dx \quad &= \quad \frac{4\beta}{(\beta+1)^2} \int_\Omega \left(|\nabla(\eta H_L(u))|^2 - \eta(-\Delta)\eta H_L(u)^2 \right) dx \\ &+ \int_\Omega -\Delta(\eta^2) J_L(u)\, dx \end{aligned} \tag{6.34}$$

where $J_L(t) := \int_0^t G_L(\tau)\,d\tau$. This identity and (6.33) yield

$$
\begin{aligned}
\frac{4\beta}{(\beta+1)^2}\int_\Omega |\nabla(\eta H_L(u))|^2\,dx \quad - & \quad \int_\Omega \frac{\gamma+O(|x|^\tau)}{|x|^2}\eta^2 u G_L(u)\,dx \\
\leq & \quad \int_\Omega |-\Delta(\eta^2)|\cdot|J_L(u)|\,dx \\
& \quad +C(\beta,\delta)\int_{\Omega\cap B_\delta(0)}|H_L(u)|^2\,dx \\
& \quad +C\int_\Omega \frac{|u|^{2^\star(s)-2}}{|x|^s}(\eta H_L(u))^2\,dx. \qquad (6.35)
\end{aligned}
$$

Hölder's inequality and the Sobolev constant given in (1.12) yield

$$
\begin{aligned}
&\int_\Omega \frac{|u|^{2^\star(s)-2}}{|x|^s}(\eta H_L(u))^2\,dx \\
&\leq \left(\int_{\Omega\cap B_\delta(0)}\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2^\star(s)-2}{2^\star(s)}}\left(\int_\Omega \frac{|\eta H_L(u)|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \\
&\leq \left(\int_{\Omega\cap B_\delta(0)}\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2^\star(s)-2}{2^\star(s)}}\cdot\frac{1}{\mu_{0,s}(\Omega)}\int_\Omega|\nabla(\eta H_L(u))|^2\,dx.
\end{aligned}
$$

Plugging this estimate into (6.35) and defining $\gamma_+ := \max\{\gamma,0\}$ yields

$$
\begin{aligned}
\frac{4\beta}{(\beta+1)^2}\int_\Omega |\nabla(\eta H_L(u))|^2\,dx \quad - & \quad (\gamma_+ + C\delta^\tau)\int_\Omega\frac{(\eta H_L(u))^2}{|x|^2}\,dx \\
& \leq C(\beta,\delta)\int_{\Omega\cap B_\delta(0)}\left(|H_L(u)|^2+|J_L(u)|\right)\,dx \\
& +\alpha(\delta)\int_\Omega|\nabla(\eta H_L(u))|^2\,dx,
\end{aligned}
$$

where

$$
\alpha(\delta) := C\left(\int_{\Omega\cap B_\delta(0)}\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2^\star(s)-2}{2^\star(s)}}\cdot\frac{1}{\mu_{0,s}(\Omega)},
$$

so that

$$
\lim_{\delta\to 0}\alpha(\delta)=0.
$$

It follows from (4.5) that

$$
\frac{n^2}{4}\int_\Omega\frac{(\eta H_L(u))^2}{|x|^2}\,dx \leq (1+\epsilon(\delta))\int_\Omega|\nabla(\eta H_L(u))|^2\,dx,
$$

where $\lim_{\delta\to 0}\epsilon(\delta)=0$. Therefore, we get that

$$
\begin{aligned}
&\left(\frac{4\beta}{(\beta+1)^2}-\alpha(\delta)-(\gamma_++C\delta^\tau)\frac{4}{n^2}(1+\epsilon(\delta))\right)\int_\Omega|\nabla(\eta H_L(u))|^2\,dx \\
&\leq C(\beta,\delta)\int_{\Omega\cap B_\delta(0)}\left(|H_L(u)|^2+|J_L(u)|\right)\,dx \leq C(\beta,\delta)\int_{B_\delta(0)\cap\Omega}|u|^{\beta+1}\,dx.
\end{aligned}
$$

Let $\delta\in(0,\delta_0)$ be such that

$$
\frac{4\beta}{(\beta+1)^2}-\alpha(\delta)-(\gamma_++C\delta^\tau)\frac{4}{n^2}(1+\epsilon(\delta))>0.
$$

This is possible since $\frac{4\beta}{(\beta+1)^2} > \frac{4}{n^2}\gamma$. Using Sobolev's embedding, we then get that

$$\begin{aligned}\left(\int_{B_{\delta/2}(0)\cap\Omega}|H_L(u)|^{2^\star}\,dx\right)^{\frac{2}{2^\star}} &\leq \left(\int_{\mathbb{R}^n}|\eta H_L(u)|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\\ &\leq \mu_{0,0}(\Omega)^{-1}\int_\Omega|\nabla(\eta H_L(u))|^2\,dx\\ &\leq C(\beta,\delta,\gamma)\int_{B_\delta(0)\cap\Omega}|u|^{\beta+1}\,dx.\end{aligned}$$

Since $u\in L^{\beta+1}(B_{\delta_0}(0)\cap\Omega)$, let $L\to+\infty$ and use Fatou's Lemma to obtain that $u\in L^{\frac{2^\star}{2}(\beta+1)}(B_{\delta/2}(0)\cap\Omega)$. The standard iterative scheme then yields that $u\in C^1(\overline{\Omega}\cap B_{\delta_0}(0)\setminus\{0\})$. Therefore $u\in L^{\frac{2^\star}{2}(\beta+1)}(B_{\delta_0}(0)\cap\Omega)$, which proves claim (6.29).

Step 2: We now show that

$$\text{if }\gamma\leq 0,\text{ then }u\in L^p(\Omega\cap B_\delta(0))\text{ for all }p\geq 1,\tag{6.36}$$

$$\text{if }\gamma>0,\text{ then }u\in L^p(\Omega\cap B_\delta(0))\text{ for all }p\in\left(1,\frac{n}{n-2}\frac{n}{\alpha_-(\gamma)}\right).\tag{6.37}$$

The case $\gamma\leq 0$ is standard, so we only consider the case where $\gamma>0$. Fix $p\geq 2$ and set $\beta:=p-1$. we have

$$\frac{4\beta}{(\beta+1)^2}>\frac{4}{n^2}\gamma\ \Leftrightarrow\ \frac{n}{\alpha_+(\gamma)}<p<\frac{n}{\alpha_-(\gamma)}.$$

Since $\alpha_+(\gamma)>n/2$ and $p\geq 2$, then

$$\frac{4\beta}{(\beta+1)^2}>\frac{4}{n^2}\gamma\ \Leftrightarrow\ p<\frac{n}{\alpha_-(\gamma)}.$$

Therefore, it follows from Step 1 that if $u\in L^p(\Omega\cap B_{\delta_0})$, with $p<n/\alpha_-(\gamma)$, then $u\in L^{\frac{n}{n-2}p}(\Omega\cap B_{\delta_0})$. Since $u\in L^2(\Omega\cap B_{\delta_0})$, (6.37) follows.

Step 3: We claim that for any $\lambda>0$, then

$$|x|^{\frac{n-2}{2}}|u(x)|=O(|x|^{\frac{n-2}{n}\left(\frac{n}{2}-\max\{\alpha_-(\gamma),0\}-\lambda\right)}\qquad\text{as }x\to 0.\tag{6.38}$$

Take $p\in\left(2^\star,\frac{n^2}{(n-2)\alpha_-(\gamma)}\right)$ if $\gamma>0$, and $p>2^\star$ if $\gamma\leq 0$. This is possible since $2^\star=2n/(n-2)$ and $\alpha_-(\gamma)<n/2$. We fix a sequence $(\varepsilon_i)_i\in(0,+\infty)$ such that $\lim_{i\to+\infty}\varepsilon_i=0$ and we fix a chart φ as in (5.5) to (5.10). For any $i\in\mathbb{N}$, we define

$$u_i(x):=\varepsilon_i^{\frac{n}{p}}u(\varphi(\varepsilon_i x))\text{ for all }x\in\tilde{B}_{\delta/\varepsilon_i}.$$

Equation (6.28) then rewrites

$$-\Delta_{g_i}u_i-\frac{\epsilon_i^2(\gamma+O(\epsilon_i^\tau|x|^\tau))}{|\varphi(\epsilon_i x)|^2}u_i=f_i(x,u_i)\ ;\ u_i=0\text{ on }\partial\mathbb{R}^n_+\cap\tilde{B}_{\delta/\varepsilon_i}\tag{6.39}$$

where $g_i(x):=\varphi^\star\mathrm{Eucl}(\epsilon_i x)$ and

$$|f_i(x,u_i)|\leq C\epsilon_i^2|u_i|+C\varepsilon_i^{(2^\star(s)-2)\left(\frac{n-2}{2}-\frac{n}{p}\right)}|x|^{-s}|u_i|^{2^\star(s)-1}$$

in $\tilde{B}_{\delta/\varepsilon_i}$. We fix $R>0$ and we define $\omega_R:=\left(\tilde{B}_R\setminus\tilde{B}_{R^{-1}}\right)\cap\mathbb{R}^n_+$. With our choice of p above and using (6.37), we get that

$$\|u_i\|_{L^p(\omega_R)}\leq C,\tag{6.40}$$

and

$$|f_i(x,u_i)| \le C_R|u_i| + C_R|u_i|^{2^\star(s)-1} \quad \text{for all } x\in\omega_R. \tag{6.41}$$

Fix $q \ge p > 2^\star$. It follows from elliptic regularity that

$$\|u_i\|_{L^q(\omega_R)} \le C \;\Rightarrow\; \left\{\begin{array}{ll} \|u_i\|_{L^{q'}(\omega_{R/2})} \le C' & \text{if } q < \frac{n}{2}(2^\star(s)-1) \\ \|u_i\|_{L^{r}(\omega_{R/2})} \le C' & \text{for all } r\ge 1 \text{ if } q = \frac{n}{2}(2^\star(s)-1) \\ \|u_i\|_{L^{\infty}(\omega_{R/2})} \le C' & \text{if } q > \frac{n}{2}(2^\star(s)-1) \end{array}\right.$$

where $\frac{1}{q'} = \frac{2^\star(s)-1}{q} - \frac{2}{n}$ and the constants C, C' are uniform with respect to i. It then follows from the standard bootstrap iterative argument and the initial bound (6.40) that $\|u_i\|_{L^\infty(\omega_{R/4})} \le C'$. Taking $R>0$ large enough and going back to the definition of u_i, we get that for all $i\in\mathbb{N}$,

$$|x|^{\frac{n}{p}}|u(x)| \le C \text{ for all } x \in \Omega\cap B_{2\varepsilon_i}(0)\setminus B_{\varepsilon_i/2}(0)$$

Since this holds for any sequence $(\varepsilon_i)_i$, we get that $|x|^{\frac{n}{p}}|u(x)|\le C$ around 0 for any $2^\star < p < \frac{n^2}{(n-2)\alpha_-(\gamma)}$ when $\gamma>0$. Letting p go to $\frac{n^2}{(n-2)\alpha_-(\gamma)}$ yields (6.38) when $\gamma>0$. For $\gamma\le 0$, we let $p\to+\infty$. This ends Step 3.

To finish the proof of Theorem 6.1, we rewrite equation (6.28) as

$$-\Delta u - \frac{a(x)}{|x|^2}u = 0$$

where

$$\begin{aligned} a(x) &= \gamma + O(|x|^\tau) + O(|x|^2) + O\left(|x|^{2-s}|u|^{2^\star(s)-2}\right) \\ &= \gamma + O(|x|^\tau) + O(|x|^2) + O\left(|x|^{\frac{n-2}{2}}|u(x)|\right)^{2^\star(s)-2} \end{aligned}$$

for all $x\in\Omega$. Since $\alpha_-(\gamma) < \frac{n}{2}$, it then follows from (6.38) that there exists $\tau'>0$ such that $a(x) = \gamma + O(|x|^{\tau'})$ as $x\to 0$. Therefore we are back to the linear case in Lemma 6.6 and we are done. □

Here are a few consequences of Theorem 6.1.

Corollary 6.7. *Suppose $\gamma<\gamma_H(\Omega)$ and consider the first eigenvalue of the operator L_γ, that is*

$$\lambda_1(\Omega,\gamma) := \inf_{u\in D^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \frac{\gamma}{|x|^2}u^2\right)dx}{\int_\Omega u^2\,dx} > 0,$$

and let $u_0\in D^{1,2}(\Omega)\setminus\{0\}$ be a minimizer. Then, there exists $A\neq 0$ such that

$$u_0(x) \sim_{x\to 0} A\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}.$$

Proof of Corollary 6.7: The existence of a u_0 that doesn't change sign is standard. The Euler-Lagrange equation is $-\Delta u - \frac{\gamma}{|x|^2}u = ku$ for some $k\in\mathbb{R}$. We then apply Theorem 6.1. □

Corollary 6.8. *Suppose $u\in D^{1,2}(\mathbb{R}^n_+)$, $u\ge 0$, $u\not\equiv 0$ is a weak solution of*

$$-\Delta u - \frac{\gamma}{|x|^2}u = \frac{u^{2^\star-1}}{|x|^s} \text{ in } \mathbb{R}^n_+.$$

Then, there exist $K_1, K_2 > 0$ such that

$$u(x) \sim_{x\to 0} K_1 \frac{x_1}{|x|^{\alpha_-(\gamma)}} \text{ and } u(x) \sim_{|x|\to+\infty} K_2 \frac{x_1}{|x|^{\alpha_+(\gamma)}}. \tag{6.42}$$

Proof of Corollary 6.8: Theorem 6.1 yields the behavior when $x \to 0$. The Kelvin transform $\hat{u}(x) := |x|^{2-n} u(x/|x|^2)$ is a solution to the same equation in $D^{1,2}(\mathbb{R}^n_+)$, and its behavior at 0 is given by Theorem 6.1. Going back to u yields the behavior at ∞. □

7. A classification of singular solutions of $L_\gamma u = a(x)u$

In this section we describe the profile of any positive solution –variational or not– of linear equations involving L_γ. Here is the main result of this section.

Theorem 7.1. *Let $u \in C^2(B_\delta(0) \cap (\overline{\Omega} \setminus \{0\}))$ be such that*

$$\left\{\begin{array}{rl} -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = 0 & \text{in } \Omega \cap B_\delta(0) \\ u > 0 & \text{in } \Omega \cap B_\delta(0) \\ u = 0 & \text{on } (\partial\Omega \cap B_\delta(0)) \setminus \{0\}. \end{array}\right. \tag{7.1}$$

Then, there exists $K > 0$ such that

$$\text{either } u(x) \sim_{x\to 0} K \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} \quad \text{or} \quad u(x) \sim_{x\to 0} K \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}.$$

In the first case, the solution $u \in D^{1,2}(\Omega)_{loc,0}$ is a variational solution to (7.1).

It is worth noting that Pinchover [43] proved that the quotient of any two positive solutions to (7.1) has a limit at 0.
The proof will require the following two lemmas. The first gives a Harnack-type inequality.

Proposition 7.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, and let $a \in L^\infty(\Omega)$ be such that $\|a\|_\infty \leq M$ for some $M > 0$. Assume U is an open subset of $\mathbb{R}^n$ and consider $u \in C^2(U \cap \overline{\Omega})$ to be a solution of*

$$\left\{\begin{array}{rl} -\Delta_g u + au = 0 & \text{in } U \cap \Omega \\ u \geq 0 & \text{in } U \cap \Omega \\ u = 0 & \text{on } U \cap \partial\Omega. \end{array}\right.$$

Here g is a smooth metric on U. If $U' \subset\subset U$ is such that $U' \cap \Omega$ is connected, then there exists $C > 0$ depending only on Ω, U', M and g such that

$$\frac{u(x)}{d(x,\partial\Omega)} \leq C \frac{u(y)}{d(y,\partial\Omega)} \text{ for all } x, y \in U' \cap \Omega. \tag{7.2}$$

Proof of Proposition 7.2: We first prove a local result. The global result will be the consequence of a covering of U'. Fix $x_0 \in \partial\Omega$. For $\delta > 0$ small enough, there exists a smooth open domain W such that

$$B_\delta(x_0) \cap \Omega \subset W \subset B_{2\delta}(x_0) \cap \Omega \text{ and } B_\delta(x_0) \cap \partial W = B_\delta(x_0) \cap \partial\Omega. \tag{7.3}$$

Let G be the Green's function of $-\Delta_g + a$ with Dirichlet boundary condition on W, then its representation formula reads as

$$u(x) = \int_{\partial W} u(\sigma)\left(-\partial_{\nu,\sigma} G(x,\sigma)\right) d\sigma = \int_{\partial W \setminus \partial\Omega} u(\sigma)\left(-\partial_{\nu,\sigma} G(x,\sigma)\right) d\sigma \tag{7.4}$$

for all $x \in W$, where $\partial_{\nu,\sigma}G(x,\sigma)$ is the normal derivative of $y \mapsto G(x,y)$ at $\sigma \in \partial W$. Estimates of the Green's function (See Robert [46] and Ghoussoub-Robert [24]) yield the existence of $C > 0$ such that for all $x \in W$ and $\sigma \in \partial W$,

$$\frac{1}{C}\frac{d(x,\partial W)}{|x-\sigma|^n} \leq -\partial_{\nu,\sigma}G(x,\sigma) \leq C\frac{d(x,\partial W)}{|x-\sigma|^n}$$

It follows from (7.3) that there exists $C(\delta) > 0$ such that for all $x \in B_{\delta/2}(x_0) \cap \Omega \subset W$ and $\sigma \in \partial W \setminus \partial\Omega$,

$$\frac{1}{C(\delta)}d(x,\partial W) \leq -\partial_{\nu,\sigma}G(x,\sigma) \leq C(\delta)d(x,\partial W)$$

Since u vanishes on $\partial\Omega$, it then follows from (7.4) that for all $x \in B_{\delta/2}(x_0) \cap \Omega$,

$$\frac{1}{C(\delta)}d(x,\partial W)\int_{\partial W} u(\sigma)\, d\sigma \leq u(x) \leq C(\delta)d(x,\partial W)\int_{\partial W} u(\sigma)\, d\sigma.$$

It is easy to check, that under the assumption (7.3), we have that $d(x,\partial\Omega) = d(x,\partial W)$. Therefore, we have for all $x \in B_{\delta/2}(x_0) \cap \Omega$,

$$\frac{1}{C(\delta)}\int_{\partial W} u(\sigma)\, d\sigma \leq \frac{u(x)}{d(x,\partial\Omega)} \leq C(\delta)\int_{\partial W} u(\sigma)\, d\sigma$$

These lower and upper bounds being independent of x, we get inequality (7.2) for any $x, y \in B_{\delta/2}(x_0) \cap \Omega$.

The general case is a consequence of a covering of $U' \cap \Omega$ by finitely many balls. Note that for balls intersecting $\partial\Omega$, we apply the preceding result, while for balls not intersecting $\partial\Omega$, we apply the classical Harnack inequality. This completes the proof of Proposition 7.2. □

Proof of Theorem 7.1: Let u be a solution of (7.1) as in the statement of Theorem 7.1. We claim that

$$u(x) = O(d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}) \text{ for } x \to 0,\ x \in \Omega. \tag{7.5}$$

Indeed, otherwise we can assume that

$$\limsup_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}} = +\infty. \tag{7.6}$$

In particular, there exists $(x_k)_k \in \Omega$ such that for all $k \in \mathbb{N}$,

$$\lim_{k\to+\infty} x_k = 0 \text{ and } \frac{u(x_k)}{d(x_k,\partial\Omega)|x_k|^{-\alpha_+(\gamma)}} \geq k, \tag{7.7}$$

We claim that there exists $C > 0$ such that

$$\tfrac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}} \geq Ck \text{ for all } x \in \Omega \cap \partial B_{r_k}(0), \text{ with } r_k := |x_k| \to 0. \tag{7.8}$$

We prove the claim by using the Harnack inequality (7.2): First take the chart φ at 0 as in (5.5), and define

$$u_k(x) := u \circ \varphi(r_k x) \text{ for } x \in \mathbb{R}^n_+ \cap B_3(0) \setminus \{0\}.$$

Equation (7.1) rewrites

$$-\Delta_{g_k} u_k + a_k u_k = 0 \text{ in } \mathbb{R}^n_+ \cap B_3(0) \setminus \{0\}, \tag{7.9}$$

with $a_k(x) := -r_k^2 \frac{\gamma+O(r_k^\tau|x|^\tau)}{|\varphi(r_k x)|^2}$. In particular, there exists $M>0$ such that $|a_k(x)| \leq M$ for all $x \in \mathbb{R}^n_+ \cap B_3(0) \setminus \overline{B}_{1/3}(0)$. Since $u_k \geq 0$, the Harnack inequality (7.2) yields the existence of $C>0$ such that

$$\frac{u_k(y)}{y_1} \geq C\frac{u_k(x)}{x_1} \quad \text{for all } x,y \in \mathbb{R}^n_+ \cap B_2(0) \setminus \overline{B}_{1/2}(0). \tag{7.10}$$

Let $\tilde{x}_k \in \mathbb{R}^n_+$ be such that $x_k = \varphi(r_k \tilde{x}_k)$. In particular, $|\tilde{x}_k| = 1 + o(1)$ as $k \to +\infty$. It then follows from (7.7), (7.9) and (7.10) that

$$\frac{u \circ \varphi(r_k y)}{d(\varphi(r_k y), \partial\Omega)} \geq C \cdot k \quad \text{for all } y \in \mathbb{R}^n_+ \cap B_2(0) \setminus \overline{B}_{1/2}(0).$$

In particular, (7.8) holds.

We let now W be a smooth domain such that (6.11) holds for $r>0$ small enough. Take the super-solution $u^{(d)}_{\alpha_+(\gamma),-}$ defined in Proposition 6.4. We have that

$$u(x) \geq \frac{C \cdot k}{2} u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x \in W \cap \partial B_{r_k}(0).$$

Since $u^{(d)}_{\alpha_+(\gamma),-}$ vanishes on ∂W, we have

$$u(x) \geq \frac{C \cdot k}{2} u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x \in \partial(W \cap B_{r_k}(0)).$$

Moreover, we have that

$$-\Delta u^{(d)}_{\alpha_+(\gamma),-} - \frac{\gamma+O(|x|^\tau)}{|x|^2} u^{(d)}_{\alpha_+(\gamma),-} < 0 = -\Delta u - \frac{\gamma+O(|x|^\tau)}{|x|^2} u \quad \text{on } W.$$

Up to taking r even smaller, it follows from the coercivity of the operator and the maximum principle that

$$u(x) \geq \frac{C \cdot k}{2} u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x \in W \cap B_{r_k}(0). \tag{7.11}$$

For any $x \in W$, we let $k_0 \in \mathbb{N}$ such that $r_k < |x|$ for all $k \geq k_0$. It then follows from (7.11) that $u(x) \geq \frac{C\cdot k}{2} u^{(d)}_{\alpha_+(\gamma),-}(x)$ for all $k \geq k_0$. Letting $k \to +\infty$ yields that $u^{(d)}_{\alpha_+(\gamma),-}(x)$ goes to zero for all $x \in W$. This is a contradiction with (6.14). Hence (7.6) does not hold, and therefore (7.5) holds.

A straightforward consequence of (7.5) and Lemma 6.5 is that there exists $l \in \mathbb{R}$ such that

$$\lim_{x \to 0} \frac{u(x)}{d(x, \partial\Omega)|x|^{-\alpha_+(\gamma)}} = l. \tag{7.12}$$

We now show the following lemma:

Lemma 7.3. *If* $\lim_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}} = 0$*, then* $u \in D^{1,2}(\Omega)_{loc,0}$ *and there exists* $K>0$ *such that* $u(x) \sim_{x\to 0} K \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}$.

Proof of Lemma 7.3: We shall use Theorem 6.1. Take W as in (6.11) and let $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta(x) = 0$ for $x \in B_{\delta/4}(0)$ and $\eta(x) = 1$ for $x \in \mathbb{R}^n \setminus B_{\delta/3}(0)$. Define

$$f(x) := \left(-\Delta - \frac{\gamma+O(|x|^\tau)}{|x|^2}\right)(\eta u) \text{ for } x \in W.$$

The function $f \in C^\infty(\overline{W})$ and vanishes around 0. Let $v \in D^{1,2}(\Omega)$ be such that

$$\begin{cases} -\Delta v - \frac{\gamma+O(|x|^\tau)}{|x|^2} v = f & \text{in } W \\ \qquad\qquad\qquad v = 0 & \text{on } \partial W. \end{cases}$$

Note again that for $r > 0$ small enough, $-\Delta - (\gamma + O(|x|^\tau))|x|^{-2}$ is coercive on W, and therefore, the existence of v is ensured for small r. Define

$$\tilde{u} := u - \eta u + v.$$

The properties of W and the definition of η and v yield

$$\begin{cases} -\Delta \tilde{u} - \frac{\gamma+O(|x|^\tau)}{|x|^2} \tilde{u} = 0 & \text{in } W \\ \qquad\qquad\qquad \tilde{u} = 0 & \text{in } \partial W \setminus \{0\}. \end{cases}$$

Moreover, since $-\Delta v - (\gamma + O(|x|^\tau))|x|^{-2} v = 0$ around 0 and $v \in D^{1,2}(W)$, it follows from Theorem 6.1 that there exists $C > 0$ such that $|v(x)| \leq C d(x, W)|x|^{-\alpha_-(\gamma)}$ for all $x \in W$. Therefore, we have that

$$\lim_{x \to 0} \frac{\tilde{u}(x)}{d(x, \partial\Omega)|x|^{-\alpha_+(\gamma)}} = 0. \tag{7.13}$$

It then follows from Lemma 6.5 that

$$\lim_{x \to 0} |x|^{\alpha_+(\gamma)} |\nabla \tilde{u}(x)| = 0. \tag{7.14}$$

Let $\psi \in C_c^\infty(W)$ and $w \in D^{1,2}(W)$ be such that

$$\begin{cases} -\Delta w - \frac{\gamma+O(|x|^\tau)}{|x|^2} w = \psi & \text{in } W \\ \qquad\qquad\qquad w = 0 & \text{on } \partial W. \end{cases}$$

Since ψ vanishes around 0, it follows from Theorem 6.1 and Lemma 6.5 that

$$w(x) = O(d(x, \partial W)|x|^{-\alpha_-(\gamma)}) \quad \text{and} \quad |\nabla w(x)| = O(|x|^{-\alpha_-(\gamma)}) \quad \text{as } x \to 0. \tag{7.15}$$

Fix $\epsilon > 0$ small and integrate by parts using that both $\tilde{u}$ and w vanish on ∂W, to get

$$\begin{aligned} 0 &= \int_{W \setminus B_\epsilon(0)} \left(-\Delta \tilde{u} - \frac{\gamma + O(|x|^\tau)}{|x|^2} \tilde{u} \right) w \, dx \\ &= \int_{W \setminus B_\epsilon(0)} \left(-\Delta w - \frac{\gamma + O(|x|^\tau)}{|x|^2} w \right) \tilde{u} \, dx + \int_{\partial(W \setminus B_\epsilon(0))} (-w \partial_\nu \tilde{u} + \tilde{u} \partial_\nu w) \, d\sigma \\ &= \int_{W \setminus B_\epsilon(0)} \psi \tilde{u} \, dx - \int_{\Omega \cap \partial B_\epsilon(0)} (-w \partial_\nu \tilde{u} + \tilde{u} \partial_\nu w) \, d\sigma. \end{aligned}$$

Using the limits and estimates (7.13), (7.14) and (7.15), and that ψ vanishes around 0, we get

$$\begin{aligned} 0 &= \int_{W \setminus B_\epsilon(0)} \psi \tilde{u} \, dx + o\left(\epsilon^{n-1} (\epsilon^{1-\alpha_-(\gamma)} \epsilon^{-\alpha_+(\gamma)} + \epsilon^{1-\alpha_+(\gamma)} \epsilon^{-\alpha_-(\gamma)}) \right) \\ &= \int_{W \setminus B_\epsilon(0)} \psi \tilde{u} \, dx + o(1), \quad \text{as } \epsilon \to 0. \end{aligned}$$

Therefore, we have $\int_W \psi \tilde{u} \, dx = 0$ for all $\psi \in C_c^\infty(W)$. Since $\tilde{u} \in L^p$ is smooth outside 0, we then get that $\tilde{u} \equiv 0$, and therefore $u = \eta u + v$. In particular, $u \in D^{1,2}(\Omega)_{loc,0}$ is a distributional positive solution to $-\Delta u - \frac{\gamma+O(|x|^\tau)}{|x|^2} u = 0$ on

W. It then follows from Theorem 6.1 that there exists $K > 0$ such that $u(x) \sim_{x\to 0} K\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}$. This proves Lemma 7.3. $\Box$

Combining Lemma 7.3 with (7.12) completes the proof of Theorem 7.1. $\Box$

As a consequence of Theorem 7.1, we improve Lemma 6.2 as follows.

Proposition 7.4. *Let $u \in C^2(\overline{\mathbb{R}^n_+} \setminus \{0\})$ be a nonnegative function such that*

$$-\Delta u - \frac{\gamma}{|x|^2}u = 0 \text{ in } \mathbb{R}^n_+ \ ; \ u = 0 \text{ on } \partial\mathbb{R}^n_+. \tag{7.16}$$

Then there exist $\lambda_-, \lambda_+ \geq 0$ such that

$$u(x) = \lambda_- x_1|x|^{-\alpha_-(\gamma)} + \lambda_+ x_1|x|^{-\alpha_+(\gamma)} \text{ for all } x \in \mathbb{R}^n_+.$$

Proof of Proposition 7.4: Without loss of generality, we assume that $u \not\equiv 0$, so that $u > 0$. We consider the Kelvin transform of u defined by $\hat{u}(x) := |x|^{2-n}u(x/|x|^2)$ for all $x \in \mathbb{R}^n_+$. Both u and $\hat{u}$ are then nonnegative solutions of (7.16). It follows from Theorem 7.1 that, after performing back the Kelvin transform, there exist $\alpha_1, \alpha_2 \in \{\alpha_+(\gamma), \alpha_-(\gamma)\}$ such that

$$\lim_{x\to 0} \frac{u(x)}{x_1|x|^{-\alpha_1}} = l_1 > 0 \text{ and } \lim_{|x|\to\infty} \frac{u(x)}{x_1|x|^{-\alpha_2}} = l_2 > 0.$$

If $\alpha_1 \leq \alpha_2$, then $u(x) \leq Cx_1|x|^{-\alpha_1}$ for all $x \in \mathbb{R}^n_+$. The result then follows from Lemma 6.2. If $\alpha_1 > \alpha_2$, then $\alpha_1 = \alpha_+(\gamma)$ and $\alpha_2 = \alpha_-(\gamma)$. We define

$$\tilde{u}(x) := u(x) - l_1x_1|x|^{-\alpha_+(\gamma)} \text{ for all } x \in \mathbb{R}^n_+.$$

to obtain that

$$-\Delta\tilde{u} - \frac{\gamma}{|x|^2}\tilde{u} = 0 \text{ in } \mathbb{R}^n_+ \ ; \ \tilde{u} = 0 \text{ on } \partial\mathbb{R}^n_+,$$

and $\tilde{u}(x) = o(x_1|x|^{-\alpha_+(\gamma)})$ as $x \to 0$. Arguing as in the proof of Lemma 7.3, we get that $\tilde{u} \in D^{1,2}(\mathbb{R}^n_+)_{loc,0}$ and $\tilde{u}(x) = O(x_1|x|^{-\alpha_-(\gamma)})$ as $x \to 0$. Moreover, we have that $\tilde{u}(x) = (l_2 + o(1))x_1|x|^{-\alpha_-(\gamma)}$ as $|x| \to +\infty$, therefore $\tilde{u}(x) > 0$ for $|x| >> 1$. Since $\tilde{u} \in D^{1,2}(\mathbb{R}^n_+)_{loc,0}$, the comparison principle then yields $\tilde{u} > 0$ everywhere. We also have that $\tilde{u}(x) \leq Cx_1|x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$. It then follows from Lemma 6.2 that there exists $\lambda_- \geq 0$ such that $\tilde{u}(x) = \lambda_- x_1|x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$. We then get the conclusion of Proposition 7.4. $\Box$

8. The Hardy singular b-mass of a domain in the case $\gamma > \frac{n^2-1}{4}$

We shall proceed in the following theorem to define the mass of a smooth bounded domain Ω of $\mathbb{R}^n$ such as $0 \in \partial\Omega$. It will involve the expansion of positive singular solutions of the Dirichlet boundary problem $L_\gamma u = 0$.

Theorem 8.1. *Let Ω be a smooth bounded domain Ω of $\mathbb{R}^n$ such as $0 \in \partial\Omega$, and assume that $\frac{n^2-1}{4} < \gamma < \gamma_H(\Omega)$. Then, up to multiplication by a positive constant, there exists a unique function $H \in C^2(\overline{\Omega} \setminus \{0\})$ such that*

$$-\Delta H - \frac{\gamma}{|x|^2}H = 0 \text{ in } \Omega \ , \ H > 0 \text{ in } \Omega \ , \ H = 0 \text{ on } \partial\Omega \setminus \{0\}. \tag{8.1}$$

Moreover, there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$H(x) = c_1\frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}} + c_2\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \text{ as } x \to 0. \tag{8.2}$$

The quantity $m_\gamma(\Omega) := \frac{c_2}{c_1} \in \mathbb{R}$, which is independent of the choice of H satisfying (8.1)*, will be called the Hardy b-mass of Ω associated to L_γ.*

Proof of Theorem 8.1. First, we start by constructing a singular solution H_0 for (8.1). For that, consider $u_{\alpha_+(\gamma)}$ as in (5.12) and let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 1$ for $x \in B_{\delta/2}(0)$ and $\eta(x) = 0$ for $x \in \mathbb{R}^n \setminus B_\delta(0)$. Set

$$f := -\Delta(\eta u_{\alpha_+(\gamma)}) - \frac{\gamma}{|x|^2}(\eta u_{\alpha_+(\gamma)}) \text{ in } \overline{\Omega} \setminus \{0\}.$$

It follows from (5.17) and (5.3) that f is smooth outside 0 and that

$$f(x) = O\left(d(x, \partial\Omega)|x|^{-\alpha_+(\gamma)-1}\right) = O\left(|x|^{-\alpha_+(\gamma)}\right) \text{ in } \Omega \cap B_{\delta/2}(0).$$

Since $\gamma > \frac{n^2-1}{4}$, we have that $\alpha_+(\gamma) < \frac{n+1}{2}$, and therefore $f \in L^{\frac{2n}{n+2}}(\Omega) = \left(L^{2^\star}(\Omega)\right)' \subset \left(D^{1,2}(\Omega)\right)'$. It then follows from the coercivity assumption $\gamma < \gamma_H(\Omega)$ that there exists $v \in D^{1,2}(\Omega)$ such that

$$-\Delta v - \frac{\gamma}{|x|^2} v = f \text{ in } \left(D^{1,2}(\Omega)\right)'.$$

Let $v_1, v_2 \in D^{1,2}(\Omega)$ be such that

$$-\Delta v_1 - \frac{\gamma}{|x|^2} v_1 = f_+ \text{ and } -\Delta v_2 - \frac{\gamma}{|x|^2} v_2 = f_- \text{ in } \left(D^{1,2}(\Omega)\right)'. \tag{8.3}$$

In particular, $v = v_1 - v_2$ and $v_1, v_2 \in C^1(\overline{\Omega} \setminus \{0\})$, and they vanish on $\partial\Omega \setminus \{0\}$. Assume that $f_+ \not\equiv 0$. Since $f_+ \geq 0$, the comparison principle yields $v_1 > 0$ on $\Omega \setminus \{0\}$ and $\partial_\nu v_1 < 0$ on $\partial\Omega \setminus \{0\}$. Therefore, for any $\delta > 0$ small enough, there exists $C(\delta) > 0$ such that

$$v_1(x) \geq C(\delta) d(x, \partial\Omega) \text{ for all } x \in \partial B_\delta(0) \cap \Omega.$$

Let $u_{\alpha_-(\gamma),-}$ be the sub-solution defined in (5.2). It follows from the asymptotic (5.3) that there exists $C'(\delta) > 0$ such that

$$v_1 \geq C'(\delta) u_{\alpha_-(\gamma),-} \text{ in } \partial B_\delta(0) \cap \Omega.$$

Since this inequality also holds on $\partial(B_\delta(0) \cap \Omega)$ and that

$$(-\Delta - \tfrac{\gamma}{|x|^2})(v_1 - C'(\delta) u_{\alpha_-(\gamma),-}) \geq 0 \quad \text{in } B_\delta(0) \cap \Omega,$$

coercivity and the maximum principle yield $v_1 \geq C'(\delta) u_{\alpha_-(\gamma),-}$ in $B_\delta(0) \cap \Omega$. It then follows from (5.3) that there exists $c > 0$ such that

$$v_1(x) \geq c \cdot d(x, \partial\Omega)|x|^{-\alpha_-(\gamma)} \text{ in } B_\delta(0) \cap \Omega.$$

Therefore, we have that

$$f_+(x) \leq C d(x, \partial\Omega)|x|^{-\alpha_+(\gamma)-1} \leq \frac{C}{c}|x|^{\alpha_-(\gamma)-\alpha_+(\gamma)-1} v_1(x) \leq \frac{C}{c}|x|^{\alpha_-(\gamma)-\alpha_+(\gamma)+1} \frac{v_1(x)}{|x|^2}$$

in $B_\delta(0) \cap \Omega$. Therefore, (8.3) yields

$$-\Delta v_1 + \frac{\gamma + O(|x|^{\alpha_-(\gamma)-\alpha_+(\gamma)+1})}{|x|^2} v_1 = 0 \text{ in } B_\delta(0) \cap \Omega.$$

Since $\gamma > \frac{n^2-1}{4}$, we have that $\alpha_-(\gamma) - \alpha_+(\gamma) + 1 > 0$. Since $v_1 \in D^{1,2}(\Omega)$, $v_1 \geq 0$ and $v_1 \not\equiv 0$, it follows from Theorem 6.1 that there exists $K_1 > 0$ such that

$$v_1(x) = K_1 \tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \quad \text{as } x \to 0. \tag{8.4}$$

If $f_+ \equiv 0$, then $v_1 \equiv 0$ and (8.4) holds with $K_1 = 0$. Arguing similarly for f_-, we then get that there exists $K_1, K_2 \geq 0$ such that for any $i = 1, 2$, we have that

$$v_i(x) = K_i \tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \quad \text{when } x \to 0.$$

Since $v = v_1 - v_2$, we then get that there exists $K \in \mathbb{R}$ such that

$$v(x) = -K \tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \quad \text{as } x \to 0. \tag{8.5}$$

Set

$$H_0(x) := \eta(x) u_{\alpha_+(\gamma)}(x) - v(x) \text{ for all } x \in \overline{\Omega} \setminus \{0\}. \tag{8.6}$$

It follows from the definition of v and the regularity outside 0 that

$$-\Delta H_0 - \frac{\gamma}{|x|^2} H_0 = 0 \text{ in } \Omega\,;\; H_0(x) = 0 \text{ in } \partial\Omega \setminus \{0\}.$$

Moreover, the asymptotics (5.3) and (8.5) yield $H_0(x) > 0$ on $\Omega \cap B_{\delta'}(0)$ for some $\delta' > 0$ small enough. It follows from the comparison principle that $H_0 > 0$ in Ω. We now perform an expansion of H_0. First note that from the definition (5.12) of $u_{\alpha_+(\gamma)}$, the asymptotic (8.5) of v and the fact that $\alpha_+(\gamma) - \alpha_-(\gamma) < 1$, we have

$$\begin{aligned} H_0(x) &= \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}(1 + O(|x|)) + K \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \\ &= \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}} + K \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \end{aligned}$$

as $x \to 0$. In particular, since in addition $H_0 > 0$ in Ω, there exists $c > 1$ such that

$$\tfrac{1}{c} \tfrac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}} \leq H_0(x) \leq c \tfrac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}} \qquad \text{for all } x \in \Omega. \tag{8.7}$$

Finally, we establish the uniqueness. For that, we let $H \in C^2(\overline{\Omega} \setminus \{0\})$ be as in (8.1) and set

$$\lambda_0 := \max\{\lambda \geq 0/\ H \geq \lambda H_0\}.$$

The number λ_0 is clearly defined, and so we set $\tilde{H} := H - \lambda_0 H_0 \geq 0$. Assume that $\tilde{H} \not\equiv 0$. Since $-\Delta \tilde{H} - \gamma |x|^{-2} \tilde{H} = 0$, it follows from Theorem 7.1 that there exists $\alpha \in \{\alpha_+(\gamma), \alpha_-(\gamma)\}$ and $K > 0$ such that

$$H(x) \sim_{x \to 0} K \frac{d(x,\partial\Omega)}{|x|^{\alpha}}. \tag{8.8}$$

If $\alpha = \alpha_-(\gamma)$, then $\tilde{H} \in D^{1,2}(\Omega)$ is a variational solution to $-\Delta \tilde{H} - \frac{\gamma}{|x|^2} \tilde{H} = 0$ in Ω. Then coercivity then yields that $\tilde{H} \equiv 0$, contradicting the initial hypothesis.

Therefore $\alpha = \alpha_+(\gamma)$. Since $\tilde{H} > 0$ vanishes on $\partial\Omega \setminus \{0\}$, then for any $\delta > 0$, there exists $c(\delta) > 0$ such that

$$\tilde{H}(x) \geq c(\delta) d(x, \partial\Omega) \text{ for } x \in \Omega \setminus B_\delta(0). \tag{8.9}$$

Therefore, (8.8), (8.9) and (8.7) yield the existence of $c > 0$ such that $\tilde{H} \geq c H_0$, and then $H \geq (\lambda_0 + c) H_0$, contradicting the definition of λ_0. It follows that $\tilde{H} \equiv 0$, which means that $H = \lambda_0 H_0$ for some $\lambda_0 > 0$. This proves uniqueness and completes the proof of Theorem 8.1. $\square$

Now we establish the monotonicity of the mass with respect to set inclusion.

Proposition 8.2. *The Hardy b-mass is strictly increasing in the following sense: Assume Ω_1, Ω_2 are two smooth bounded domains such that $0 \in \partial\Omega_1 \cap \partial\Omega_2$, and $\frac{n^2-1}{4} < \gamma < \min\{\gamma_H(\Omega_1), \gamma_H(\Omega_2)\}$, then*

$$\Omega_1 \subsetneq \Omega_2 \ \Rightarrow \ m_\gamma(\Omega_1) < m_\gamma(\Omega_2). \tag{8.10}$$

Moreover, if $\Omega \subsetneq \mathbb{R}^n_+$ and $\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$, then $m_\gamma(\Omega) < 0$.

Proof of Proposition 8.2: It follows from the definition of the mass that for $i = 1, 2$, there exists $H_i \in C^2(\overline{\Omega_i} \setminus \{0\})$ such that

$$-\Delta H_i - \frac{\gamma}{|x|^2} H_i = 0 \text{ in } \Omega_i \ , \ H_i > 0 \text{ in } \Omega_i \ , \ H_i = 0 \text{ on } \partial\Omega_i, \tag{8.11}$$

with

$$H_i(x) = \frac{d(x, \partial\Omega_i)}{|x|^{\alpha_+(\gamma)}} + m_\gamma(\Omega_i) \frac{d(x, \partial\Omega_i)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x, \partial\Omega_i)}{|x|^{\alpha_-(\gamma)}}\right) \tag{8.12}$$

as $x \to 0$, $x \in \Omega_i$. Set $h := H_2 - H_1$ on Ω_1. Since $\Omega_1 \subsetneq \Omega_2$, we have that

$$\begin{cases} -\Delta h - \frac{\gamma}{|x|^2} h = 0 & \text{in } \Omega_1 \\ \qquad\qquad h \geq 0 & \text{on } \partial\Omega_1 \\ \qquad\qquad h \not\equiv 0 & \text{in } \partial\Omega_1. \end{cases} \tag{8.13}$$

First, we claim that $h \in H^{1,2}(\Omega_1)$. Indeed, it follows from the construction of the singular function (see (8.6)), that there exists $w \in H^{1,2}(\Omega_1)$ such that

$$h(x) = \frac{d(x, \partial\Omega_2) - d(x, \partial\Omega_1)}{|x|^{\alpha_+(\gamma)}} + w(x) \text{ for all } x \in \Omega_1. \tag{8.14}$$

Since $\Omega_1 \subset \Omega_2$ and 0 is on the boundary of both domains, then the tangent spaces at 0 of Ω_1 and Ω_2 are equal, and one gets that

$$d(x, \partial\Omega_1) - d(x, \partial\Omega_2) = O(|x|^2) \quad \text{as } x \to 0.$$

Since $\alpha_+(\gamma) - \alpha_-(\gamma) < 1$, we then get that

$$\tilde{h}(x) := \frac{d(x, \partial\Omega_2) - d(x, \partial\Omega_1)}{|x|^{\alpha_+(\gamma)}} = O(|x|^{1-\alpha_-(\gamma)}) \text{ as } x \to 0.$$

Similarly, $|\nabla \tilde{h}(x)| = O(|x|^{-\alpha_-(\gamma)})$ as $x \to 0$. Therefore, we deduce that $\tilde{h} \in H^{1,2}(\Omega_1)$. It then follows from (8.14) that $h \in H^{1,2}(\Omega_1)$.
To prove the monotonicity, note first that since $\gamma < \gamma_H(\Omega_1)$ and $h \in H^{1,2}(\Omega_1)$, it follows from (8.13) and the comparison principle that $h \geq 0$ in Ω_1 (indeed, this is obtained by multiplying (8.13) by $h_- \in D^2_1(\Omega)$ and integrating: therefore, coercivity yields $h_- \equiv 0$). Since $h \not\equiv 0$, it follows from Hopf's maximum principle that for any $\delta > 0$ small, there exists $C(\delta) > 0$ such that

$$h(x) \geq C(\delta) d(x, \partial\Omega_1) \text{ for all } x \in \partial B_\delta(0) \cap \Omega_1.$$

We define the sub-solution $u_{\alpha_-(\gamma),-}$ as in Proposition 5.1. It then follows from the inequality above and the asymptotics in (5.3) that there exists $\epsilon_0 > 0$ such that

$$h(x) \geq 2\epsilon_0 u_{\alpha_-(\gamma),-}(x) \text{ for all } x \in \partial B_\delta(0) \cap \Omega_1.$$

This inequality also holds on $B_\delta(0) \cap \partial\Omega_1$ since $u_{\alpha_-(\gamma),-}$ vanishes on $\partial\Omega_1$. It then follows from the maximum principle that $h(x) \geq 2\epsilon_0 u_{\alpha_-(\gamma),-}(x)$ for all $x \in$

$B_\delta(0)\cap\Omega_1$. With the definition of h and the asymptotic (5.3), we then have that for $\delta'>0$ small enough

$$H_2(x)-H_1(x)\geq \epsilon_0\frac{d(x,\partial\Omega_1)}{|x|^{\alpha_-(\gamma)}} \text{ for all } x\in B_{\delta'}(0)\cap\Omega_1. \tag{8.15}$$

We let $\vec{\nu}$ be the inner unit normal vector of $\partial\Omega_1$ at 0. This is also the inner unit normal vector of $\partial\Omega_2$ at 0. Therefore, for any $t>0$ small enough, we have that $d(t\vec{\nu},\partial\Omega_i)=t$ for $i=1,2$. It then follows from the expressions (8.12) and (8.15) that

$$(m_\gamma(\Omega_2)-m_\gamma(\Omega_1))\frac{t}{t^{\alpha_-(\gamma)}}+o\left(\frac{t}{t^{\alpha_-(\gamma)}}\right)\geq \epsilon_0\frac{t}{t^{\alpha_-(\gamma)}} \quad \text{as } t\downarrow 0.$$

We then get that $m_\gamma(\Omega_2)-m_\gamma(\Omega_1)\geq\epsilon_0$, and therefore $m_\gamma(\Omega_2)>m_\gamma(\Omega_1)$. This proves (8.10) and ends the first part of Proposition 8.2.

The proof of the second part is similar. Indeed, we take $\Omega_2:=\mathbb{R}^n_+$ and we define $H_2(x):=\frac{x_1}{|x|^{\alpha_+(\gamma)}}$. Arguing as above, we get that $0>m_\gamma(\Omega)$, which completes the proof of Proposition 8.2. □

In Section 10, we will prove that one can define the mass $m_\gamma(\mathbb{R}^n_+)$ of $\mathbb{R}^n_+$, and that $m_\gamma(\mathbb{R}^n_+)=0$.

9. Test functions and the existence of extremals

Let Ω be a domain of $\mathbb{R}^n$ such that $0\in\partial\Omega$. For $\gamma\in\mathbb{R}$ and $s\in[0,2)$, recall that

$$\mu_{\gamma,s}(\Omega):=\inf_{u\in D^{1,2}(\Omega)\setminus\{0\}} J^\Omega_{\gamma,s}(u), \tag{9.1}$$

where

$$J^\Omega_{\gamma,s}(u):=\frac{\int_\Omega\left(|\nabla u|^2-\frac{\gamma}{|x|^2}u^2\right)dx}{\left(\int_\Omega\frac{|u|^{2^\star}}{|x|^s}\,dx\right)^{\frac{2}{2^\star}}}.$$

Note that critical points $u\in D^{1,2}(\Omega)$ of $J^\Omega_{\gamma,s}$ are weak solutions to the pde

$$-\Delta u-\frac{\gamma}{|x|^2}=\lambda\frac{|u|^{2^\star-2}u}{|x|^s}, \tag{9.2}$$

for some $\lambda\in\mathbb{R}$, which can be rescaled to be equal to 1 if $\lambda>0$ and to be -1 if $\lambda<0$. In this section, we investigate the existence of minimizers for $J^\Omega_{\gamma,s}$. We start with the following easy case, where we don't have extremals.

Proposition 9.1. *Let $\Omega\subset\mathbb{R}^n$ be a smooth domain such that $0\in\partial\Omega$ (No boundedness is assumed). When $s=0$ and $\gamma\leq 0$, we have that $\mu_{\gamma,0}(\Omega)=\frac{1}{K(n,2)^2}$ (where $K(n,2)^{-2}=\mu_{0,0}(\mathbb{R}^n)$ is the best constant in the Sobolev inequality* (1.14)*) and there is no extremal.*

Proof of Proposition 9.1: Note that $2^\star(s)=2^\star(0)=2^\star$. Since $\gamma\leq 0$, we have for any $u\in C^\infty_c(\Omega)\setminus\{0\}$,

$$\frac{\int_\Omega\left(|\nabla u|^2-\gamma\frac{u^2}{|x|^2}\right)dx}{\left(\int_\Omega|u|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}}\geq\frac{\int_\Omega|\nabla u|^2\,dx}{\left(\int_\Omega|u|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}}\geq\frac{1}{K(n,2)^2}, \tag{9.3}$$

and therefore $\mu_{\gamma,0}(\Omega)\geq\frac{1}{K(n,2)^2}$. Fix now $x_0\in\Omega$ and let $\eta\in C^\infty_c(\Omega)$ be such that $\eta(x)=1$ around x_0. Set $u_\varepsilon(x):=\eta(x)\left(\frac{\varepsilon}{\varepsilon^2+|x-x_0|^2}\right)^{\frac{n-2}{2}}$ for all $x\in\Omega$ and $\varepsilon>0$.

Since $x_0 \neq 0$, it is easy to check that $\lim_{\varepsilon\to 0}\int_\Omega \frac{u_\varepsilon^2}{|x|^2}\,dx = 0$. It is also classical (see for example Aubin [2]) that

$$\lim_{\varepsilon\to 0}\frac{\int_\Omega |\nabla u_\varepsilon|^2\,dx}{\left(\int_\Omega |u_\varepsilon|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}} = \frac{1}{K(n,2)^2}.$$

It follows that $\mu_{\gamma,0}(\Omega) \leq \frac{1}{K(n,2)^2}$. This proves that $\mu_{\gamma,0}(\Omega) = \frac{1}{K(n,2)^2}$.

Assume now that there exists an extremal u_0 for $\mu_{\gamma,0}(\Omega)$ in $D^{1,2}(\Omega)\setminus\{0\}$. The inequalities in (9.3) and the fact that

$$\frac{\int_\Omega |\nabla u_0|^2\,dx}{\left(\int_\Omega |u_0|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}} = \frac{1}{K(n,2)^2},$$

means that $u_0 \in D^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$ is an extremal for the classical Sobolev inequality on $\mathbb{R}^n$. But these extremals are known (see Aubin [2] or Talenti [52]) and their support is the whole of $\mathbb{R}^n$, which is a contradiction since u_0 has support in $\Omega \neq \mathbb{R}^n$. It follows that there is no extremal for $\mu_{\gamma,0}(\Omega)$. This proves Proposition 9.1. □

The remainder of the section is devoted to the proof of the following.

Theorem 9.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n\geq 3)$ such that $0\in\partial\Omega$ and let $0\leq s<2$ and $\gamma<\frac{n^2}{4}$. Assume that either $s>0$, or that $\{s=0,\ n\geq 4$ and $\gamma>0\}$. There are then extremals for $\mu_{\gamma,s}(\Omega)$ under one of the following two conditions:*

1. *$\gamma \leq \frac{n^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
2. *$\gamma > \frac{n^2-1}{4}$ and the mass $m_\gamma(\Omega)$ of Ω is positive.*

Moreover, if $\gamma < \gamma_H(\Omega)$ (resp., $\gamma\geq\gamma_H(\Omega)$), then such extremals are positive solutions for (9.2) with $\lambda>0$ (resp., $\lambda\leq 0$).

The remaining case $n=3$, $s=0$ and $\gamma>0$ will be dealt with in section 11.

According to Theorem 4.4, in order to establish existence of extremals, it suffices to show that $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+)$. The rest of the section consists of showing that the above mentioned geometric conditions lead to such gap.

In the sequel, $h_\Omega(0)$ will denote the mean curvature of $\partial\Omega$ at 0. The orientation is chosen such that the mean curvature of the canonical sphere (as the boundary of the ball) is positive. Since $\{s>0\}$, or that $\{s=0,\ n\geq 4$ and $\gamma>0\}$, it follows from Theorem 12.1 in Section 12 of the appendix (see also Bartsch-Peng-Zhang [3] and Chern-Lin [10]) that there are extremals for $\mu_{\gamma,s}(\mathbb{R}^n_+)$. The following proposition combined with Theorem 4.4 clearly yield the claims in Theorem 9.2.

Proposition 9.3. *We fix $\gamma<\frac{n^2}{4}$. Assume that there are extremals for $\mu_{\gamma,s}(\mathbb{R}^n_+)$. There exist then two families $(u_\varepsilon^1)_{\varepsilon>0}$ and $(u_\varepsilon^2)_{\varepsilon>0}$ in $D^{1,2}(\Omega)$, and two positive constants $c^1_{\gamma,s}$ and $c^2_{\gamma,s}$ such that:*

(1) *For $\gamma<\frac{n^2-1}{4}$, we have that*

$$J(u^1_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 + c^1_{\gamma,s}\cdot h_\Omega(0)\cdot\varepsilon + o(\varepsilon)\right) \text{ when } \varepsilon\to 0. \tag{9.4}$$

(2) *For $\gamma=\frac{n^2-1}{4}$, we have that*

$$J(u^1_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 + c^1_{\gamma,s}\cdot h_\Omega(0)\cdot\varepsilon\ln\frac{1}{\varepsilon} + o\left(\varepsilon\ln\frac{1}{\varepsilon}\right)\right) \text{ when } \varepsilon\to 0. \tag{9.5}$$

(3) *For $\gamma > \frac{n^2-1}{4}$, we have as $\epsilon \to 0$, that*

$$J(u_\epsilon^2) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 - c_{\gamma,s}^2 \cdot m_\gamma(\Omega) \cdot \varepsilon^{\alpha_+(\gamma)-\alpha_-(\gamma)} + o(\varepsilon^{\alpha_+(\gamma)-\alpha_-(\gamma)})\right). \tag{9.6}$$

Remark: When $\gamma < \frac{n^2-1}{4}$, this result is due to Chern-Lin [10]. Actually, they stated the result for $\gamma < \frac{(n-2)^2}{4}$, but their proof works for $\gamma < \frac{n^2-1}{4}$. However, when $\gamma \geq \frac{n^2-1}{4}$, we need the exact asymptotic profile of U that was described by Corollary 6.8.

Proof of Proposition 9.3: By assumption, there exists $U \in D^{1,2}(\mathbb{R}^n_+) \setminus \{0\}$, $U \geq 0$, that is a minimizer for $\mu_{\gamma,s}(\mathbb{R}^n_+)$. In other words,

$$J_{\gamma,s}^{\mathbb{R}^n_+}(U) = \frac{\int_{\mathbb{R}^n_+}\left(|\nabla U|^2 - \frac{\gamma}{|x|^2}U^2\right)dx}{\left(\int_{\mathbb{R}^n_+}\frac{|U|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}} = \mu_{\gamma,s}(\mathbb{R}^n_+).$$

Therefore (see Corollary 6.8), there exists $\lambda > 0$ such that

$$\left\{\begin{array}{rl} -\Delta U - \frac{\gamma}{|x|^2}U = \lambda\frac{U^{2^\star(s)-1}}{|x|^s} & \text{in } \mathbb{R}^n_+\\ U > 0 & \text{in } \mathbb{R}^n_+\\ U = 0 & \text{in } \partial\mathbb{R}^n_+ \end{array}\right\} \tag{9.7}$$

and there exist $K_1, K_2 > 0$ such that

$$U(x) \sim_{x\to 0} K_1\frac{x_1}{|x|^{\alpha_-}} \text{ and } U(x) \sim_{|x|\to+\infty} K_2\frac{x_1}{|x|^{\alpha_+}}, \tag{9.8}$$

where here and in the sequel, we write for convenience

$$\alpha_+ := \alpha_+(\gamma) \text{ and } \alpha_- := \alpha_-(\gamma).$$

In particular, it follows from Lemma 6.5 (after reducing all limits to happen at 0 via the Kelvin transform) that there exists $C > 0$ such that

$$U(x) \leq Cx_1|x|^{-\alpha_+} \text{ and } |\nabla U(x)| \leq C|x|^{-\alpha_+} \text{ for all } x \in \mathbb{R}^n_+. \tag{9.9}$$

We shall now construct a suitable test-function for each range of γ. First note that

$$\gamma < \frac{n^2-1}{4} \quad\Leftrightarrow\quad \alpha_+ - \alpha_- > 1$$
$$\gamma = \frac{n^2-1}{4} \quad\Leftrightarrow\quad \alpha_+ - \alpha_- = 1.$$

Concerning terminology, here and in the sequel, we define as in (5.4)

$$\tilde{B}_r := (-r,r)\times B_r^{(n-1)}(0) \subset \mathbb{R}\times\mathbb{R}^{n-1},$$

for all $r > 0$ and

$$V_+ := V \cap \mathbb{R}^n_+$$

for all $V \subset \mathbb{R}^n$. Since Ω is smooth, up to a rotation, there exists $\delta > 0$ and $\varphi_0 : B_\delta^{(n-1)}(0) \to \mathbb{R}$ such that $\varphi_0(0) = |\nabla\varphi_0(0)| = 0$ and

$$\left\{\begin{array}{cccc} \varphi: & \tilde{B}_{3\delta} & \to & \mathbb{R}^n\\ & (x_1,x') & \mapsto & (x_1+\varphi_0(x'),x'), \end{array}\right. \tag{9.10}$$

that realizes a diffeomorphism onto its image and such that

$$\varphi(\tilde{B}_{3\delta}\cap\mathbb{R}^n_+) = \varphi(\tilde{B}_{3\delta})\cap\Omega \text{ and } \varphi(\tilde{B}_{3\delta}\cap\partial\mathbb{R}^n_+) = \varphi(\tilde{B}_{3\delta})\cap\partial\Omega.$$

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 1$ for all $x \in \tilde{B}_\delta$, $\eta(x) = 0$ for all $x \notin \tilde{B}_{2\delta}$.

Case 1: $\gamma \leq \frac{n^2-1}{4}$**.** As in Chern-Lin [10], for any $\epsilon > 0$, we define

$$u_\epsilon(x) := \left(\eta \epsilon^{-\frac{n-2}{2}} U(\epsilon^{-1}x)\right) \circ \varphi^{-1}(x) \text{ for } x \in \varphi(\tilde{B}_{2\delta}) \cap \Omega \text{ and } 0 \text{ elsewhere.}$$

This subsection is devoting to give a Taylor expansion of $J^\Omega_{\gamma,s}(u_\varepsilon)$ as $\epsilon \to 0$. In the sequel, we adopt the following notation: given $(a_\epsilon)_{\epsilon>0} \in \mathbb{R}$, $\Theta_\gamma(a_\epsilon)$ denotes a quantity such that, as $\epsilon \to 0$.

$$\Theta_\gamma(a_\epsilon) := \begin{cases} o(a_\epsilon) & \text{if } \gamma < \frac{n^2-1}{4} \\ O(a_\epsilon) & \text{if } \gamma = \frac{n^2-1}{4} \end{cases}$$

Estimate of $\int_\Omega |\nabla u_\varepsilon|^2 \, dx$**:**

It follows from (9.9) that

$$|\nabla u_\varepsilon(x)| \leq C \varepsilon^{\alpha_+ - \frac{n}{2}} |x|^{-\alpha_+} \text{ for all } x \in \Omega \text{ and } \varepsilon > 0. \tag{9.11}$$

Therefore,

$$\int_{\varphi\left(\left(\tilde{B}_{3\delta} \setminus \tilde{B}_\delta\right) \cap \mathbb{R}^n_+\right)} |\nabla u_\varepsilon|^2 \, dx = \Theta_\gamma(\epsilon) \text{ as } \varepsilon \to 0.$$

It follows that

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx = \int_{\tilde{B}_{\delta,+}} |\nabla (u_\varepsilon \circ \varphi)|^2_{\varphi^\star \mathrm{Eucl}} |\mathrm{Jac}(\varphi)| \, dx + \Theta_\gamma(\epsilon) \quad \text{as } \varepsilon \to 0,$$

where $\tilde{B}_{\delta,+} := \tilde{B}_\delta \cap \mathbb{R}^n_+$. The definition (9.10) of φ yields $\mathrm{Jac}(\varphi) = 1$. Moreover, for any $\theta \in (0,1)$, we have as $x \to 0$,

$$\varphi^\star \mathrm{Eucl} := \begin{pmatrix} 1 & \partial_j \varphi_0 \\ \partial_i \varphi_0 & \delta_{ij} + \partial_i \varphi_0 \partial_j \varphi_0 \end{pmatrix} = Id + H + O(|x|^{1+\theta})$$

where

$$H := \begin{pmatrix} 0 & \partial_j \varphi_0 \\ \partial_i \varphi_0 & 0 \end{pmatrix}.$$

It follows that

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^2 \, dx &= \int_{\tilde{B}_{\delta,+}} |\nabla (u_\varepsilon \circ \varphi)|^2_{\mathrm{Eucl}} \, dx - \int_{\tilde{B}_{\delta,+}} H^{ij} \partial_i (u_\varepsilon \circ \varphi) \partial_j (u_\varepsilon \circ \varphi) \, dx \\ &\quad + O\left(\int_{\tilde{B}_{\delta,+}} |x|^{1+\theta} |\nabla (u_\varepsilon \circ \varphi)|^2 \, dx\right) + \Theta_\gamma(\epsilon) \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{9.12}$$

We have that

$$\begin{aligned}
&\int_{\tilde{B}_{\delta,+}} H^{ij}\partial_i(u_\varepsilon\circ\varphi)\partial_j(u_\varepsilon\circ\varphi)\,dx\\
&= 2\sum_{i\geq 2}\int_{\tilde{B}_{\delta,+}} H^{1i}\partial_1(u_\varepsilon\circ\varphi)\partial_i(u_\varepsilon\circ\varphi)\,dx\\
&= 2\sum_{i\geq 2}\int_{\tilde{B}_{\delta,+}} \partial_i\varphi_0(x')\partial_1(u_\varepsilon\circ\varphi)\partial_i(u_\varepsilon\circ\varphi)\,dx\\
&= 2\sum_{i,j\geq 2}\int_{\tilde{B}_{\delta,+}} \partial_{ij}\varphi_0(0)(x')^j\partial_1(u_\varepsilon\circ\varphi)\partial_i(u_\varepsilon\circ\varphi)\,dx\\
&+O\left(\int_{\tilde{B}_{\delta,+}}|x|^2|\nabla(u_\varepsilon\circ\varphi)|^2\,dx\right)\quad\text{as }\varepsilon\to 0.
\end{aligned}\tag{9.13}$$

We let II be the second fundamental form at 0 of the oriented boundary $\partial\Omega$. By definition, for any $X,Y\in T_0\partial\Omega$, we have that

$$II(X,Y):=(d\vec{\nu}_0(X),Y)_{\text{Eucl}}$$

where $\vec{\nu}:\partial\Omega\to\mathbb{R}^n$ is the outer unit normal vector of $\partial\Omega$. In particular, we have that $\vec{\nu}(0)=(-1,0,\cdot,0)$. For any $i,j\geq 2$, we have that

$$II_{ij}:=II(\partial_i\varphi(0),\partial_j\varphi(0))=(\partial_i(\vec{\nu}\circ\varphi)(0),\partial_j\varphi(0))=-(\vec{\nu}(0),\partial_{ij}\varphi(0))=\partial_{ij}\varphi_0(0).$$

Plugging (9.13) in (9.12), and using a change of variables, we get that

$$\begin{aligned}
\int_\Omega|\nabla u_\varepsilon|^2\,dx &= \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}|\nabla U|^2\,dx-2II_{ij}\sum_{i,j\geq 2}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}(x')^j\partial_1U\partial_iU\,dx\\
&+O\left(\int_{\tilde{B}_{\delta,+}}|x|^{1+\theta}|\nabla(u_\varepsilon\circ\varphi)|^2\,dx\right)+\Theta_\gamma(\epsilon)\quad\text{as }\varepsilon\to 0.
\end{aligned}\tag{9.14}$$

We now choose θ:
(i) If $\gamma<\frac{n^2-1}{4}$, then choose $0<\theta<\alpha_+-\alpha_--1$;
(ii) If $\gamma=\frac{n^2-1}{4}$, we take any $\theta\in(0,1)$.
In both cases, we get by using (9.11), that

$$\int_{\tilde{B}_{\delta,+}}|x|^{1+\theta}|\nabla(u_\varepsilon\circ\varphi)|^2\,dx=\Theta_\gamma(\epsilon)\quad\text{as }\varepsilon\to 0.\tag{9.15}$$

Moreover, using (9.9), we have that

$$\textstyle\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}|\nabla U|^2\,dx=\int_{\mathbb{R}^n_+}|\nabla U|^2\,dx+\Theta_\gamma(\varepsilon)\text{ as }\varepsilon\to 0.\tag{9.16}$$

Plugging together (9.14), (9.15), (9.16) yields

$$\begin{aligned}
\int_\Omega|\nabla u_\varepsilon|^2\,dx &= \int_{\mathbb{R}^n_+}|\nabla U|^2\,dx\\
&-2II_{ij}\sum_{i,j\geq 2}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}(x')^j\partial_1U\partial_iU\,dx+\Theta_\gamma(\epsilon).
\end{aligned}\tag{9.17}$$

Estimate of $\int_\Omega\frac{|u_\varepsilon|^{2^\star(s)}}{|x|^s}\,dx$:

Fix $\sigma\in[0,2]$. We will apply the estimates below to $\sigma=s\in[0,2)$ or to $\sigma:=2$. The first estimate in (9.9) yields

$$|u_\varepsilon(x)|\leq C\varepsilon^{\alpha_+-\frac{n}{2}}d(x,\partial\Omega)|x|^{-\alpha_+}\leq C\varepsilon^{\alpha_+-\frac{n}{2}}|x|^{1-\alpha_+} \tag{9.18}$$

for all $\varepsilon>0$ and all $x\in\Omega$. Since Jac $\varphi=1$, this estimate then yields

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\,dx &= \int_{\varphi(\tilde{B}_{\delta,+})}\frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\,dx+\Theta_\gamma(\varepsilon)\\ &= \int_{\tilde{B}_{\delta,+}}\tfrac{|u_\varepsilon\circ\varphi|^{2^\star(\sigma)}}{|\varphi(x)|^\sigma}\,dx+\Theta_\gamma(\varepsilon)\quad\text{ as }\varepsilon\to0.\end{aligned} \tag{9.19}$$

If $\gamma<\frac{n^2-1}{4}$ or if $\gamma=\frac{n^2-1}{4}$ and $\sigma<2$, we choose $\theta\in(0,(\alpha_+-\alpha_-)\frac{2^\star(\sigma)}{2}-1)\cap(0,1)$. If $\gamma=\frac{n^2-1}{4}$ and $\sigma=2$, we choose any $\theta\in(0,1)$. Using the expression of $\varphi(x_1,x')$, a Taylor expansion yields

$$|\varphi(x)|^{-\sigma}=|x|^{-\sigma}\left(1-\frac{\sigma}{2}\frac{x_1}{|x|^2}\sum_{i,j\geq2}\partial_{ij}\varphi_0(0)(x')^i(x')^j+O(|x|^{1+\theta})\right)\quad\text{ as }\varepsilon\to0. \tag{9.20}$$

The choice of θ yields

$$\int_{\tilde{B}_{\delta,+}}\tfrac{|u_\varepsilon\circ\varphi|^{2^\star(\sigma)}}{|\varphi(x)|^\sigma}|x|^{1+\theta}\,dx=\Theta_\gamma(\varepsilon)\quad\text{ as }\varepsilon\to0. \tag{9.21}$$

Plugging together (9.19), (9.20), (9.21), and using a change of variable, we get as $\varepsilon\to0$ that

$$\begin{aligned}\int_\Omega\frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\,dx &= \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\,dx\\ &\quad -\frac{\sigma}{2}\sum_{i,j\geq2}\varepsilon II_{ij}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\frac{x_1}{|x|^2}(x')^i(x')^j\,dx+\Theta_\gamma(\varepsilon).\end{aligned}$$

Moreover, (9.9) yields

$$\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\,dx=\int_{\mathbb{R}^n_+}\frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\,dx+\Theta_\gamma(\varepsilon)\quad\text{ as }\varepsilon\to0.$$

Therefore, we get that

$$\begin{aligned}\int_\Omega\frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\,dx &= \int_{\mathbb{R}^n_+}\frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\,dx\\ &\quad -\frac{\sigma}{2}\sum_{i,j\geq2}\varepsilon II_{ij}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\frac{x_1}{|x|^2}(x')^i(x')^j\,dx+\Theta_\gamma(\varepsilon).\end{aligned} \tag{9.22}$$

We now compute the terms in U by using its symmetry property established in Chern-Lin [10] (see also Theorem 13.1 in the Appendix). Indeed, it follows from (9.7) that there exists $\tilde{U}:(0,+\infty)\times\mathbb{R}$ such that $U(x_1,x')=\tilde{U}(x_1,|x'|)$ for all $(x_1,x')\in\mathbb{R}^n_+$. Therefore, for any $i,j\geq2$, we get that

$$\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\frac{x_1}{|x|^2}(x')^i(x')^j\,dx=\frac{\delta_{ij}}{n-1}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\frac{x_1}{|x|^2}|x'|^2\,dx$$

and that

$$\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} (x')^j \partial_1 U \partial_i U\, dx = \frac{\delta_{ij}}{n-1} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U (x', \nabla U)\, dx$$

where $x = (x_1, x') \in \mathbb{R}^n_+$. Therefore, the identities (9.17) and (9.22) rewrite as

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx = \int_{\mathbb{R}^n_+} |\nabla U|^2\, dx - \frac{2h_\Omega(0)}{n-1} \varepsilon \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U (x', \nabla U)\, dx + \Theta_\gamma(\epsilon) \tag{9.23}$$

and

$$\begin{aligned} \int_\Omega \frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\, dx &= \int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\, dx \\ &\quad - \frac{\sigma h_\Omega(0)}{2(n-1)} \varepsilon \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|U|^{2^\star(\sigma)}}{|x|^\sigma} \frac{x_1}{|x|^2} |x'|^2\, dx + \Theta_\gamma(\varepsilon) \end{aligned} \tag{9.24}$$

as $\varepsilon \to 0$, where $h_\Omega(0) = \sum_i II_{ii}$ is the mean curvature at 0.

An intermediate identity. We now claim that as $\varepsilon \to 0$,

$$\begin{aligned} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U (x', \nabla U)\, dx &= \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2 x_1}{2|x|^2} \left(\lambda \frac{s}{2^\star(s)} \frac{U^{2^\star(s)}}{|x|^s} + \gamma \frac{U^2}{|x|^2} \right) dx \\ &\quad - \int_{\partial\mathbb{R}^n_+ \cap \tilde{B}_{\varepsilon^{-1}\delta}} \frac{|x'|^2 (\partial_1 U)^2}{4}\, dx + \Theta_\gamma(1) \end{aligned} \tag{9.25}$$

where $\lambda > 0$ is as in (9.7). This was shown by Chern-Lin [10], and we include it for the sake of completeness. Here and in the sequel, ν_i denotes the i^{th} coordinate of the direct outward normal vector on the boundary of the relevant domain (for instance, on $\partial\mathbb{R}^n_+$, we have that $\nu_i = -\delta_{1i}$). We write

$$\begin{aligned} &\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U (x', \nabla U)\, dx = \sum_{j \geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U (x')^j \partial_j U\, dx \\ &= \sum_{j\geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U \partial_j \left(\frac{|x'|^2}{2} \right) \partial_j U\, dx \\ &= \sum_{j\geq 2} \int_{\partial(\tilde{B}_{\varepsilon^{-1}\delta,+})} \partial_1 U \frac{|x'|^2}{2} \partial_j U \nu_j\, d\sigma - \sum_{j\geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2}{2} \partial_j \left(\partial_1 U \partial_j U \right)\, dx \\ &= \sum_{j\geq 2} \int_{\partial\mathbb{R}^n_+ \cap \tilde{B}_{\varepsilon^{-1}\delta}} \partial_1 U \frac{|x'|^2}{2} \partial_j U \nu_j\, d\sigma + O\left(\int_{\mathbb{R}^n_+ \cap \partial\tilde{B}_{\varepsilon^{-1}\delta}} |x'|^2 |\nabla U|^2(x)\, d\sigma \right) \\ &\quad - \sum_{j\geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2}{2} \left(\partial_{1j} U \partial_j U + \partial_1 U \partial_{jj} U \right)\, dx. \end{aligned} \tag{9.26}$$

Since $U(0,x')=0$ for all $x'\in\mathbb{R}^{n-1}$, using the upper-bound (9.9) and writing $\nabla'=(\partial_2,\dots,\partial_n)$, we get that

$$\begin{aligned}
&\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\partial_1 U(x',\nabla U)\,dx\\
&=-\sum_{j\geq 2}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\left(\partial_{1j}U\partial_j U+\partial_1 U\partial_{jj}U\right)\,dx+\Theta_\gamma(1)\\
&=-\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{4}\partial_1\left(|\nabla' U|^2\right)\,dx\\
&+\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U\left(-\Delta U+\partial_{11}U\right)\,dx+\Theta_\gamma(1)\\
&=-\int_{\partial(\tilde{B}_{\varepsilon^{-1}\delta,+})}\frac{|x'|^2|\nabla' U|^2}{4}\nu_1\,dx+\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U(-\Delta U)\,dx\\
&+\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\partial_1\left(\frac{|x'|^2(\partial_1 U)^2}{4}\right)\,dx+\Theta_\gamma(1).
\end{aligned}\tag{9.27}$$

Using again that U vanishes on $\partial\mathbb{R}^n_+$ and the bound (9.9), we get that

$$\begin{aligned}
\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\partial_1 U(x',\nabla U)\,dx &= \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U(-\Delta U)\,dx+\int_{\partial\mathbb{R}^n_+\cap\tilde{B}_{\varepsilon^{-1}\delta}}\frac{|x'|^2(\partial_1 U)^2}{4}\nu_1\,dx\\
&+O\left(\int_{\partial(\tilde{B}_{\varepsilon^{-1}\delta})\cap\mathbb{R}^n_+}|x'|^2|\nabla U|^2\,dx\right)+\Theta_\gamma(1)\\
&=\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U(-\Delta U)\,dx\\
&-\int_{\partial\mathbb{R}^n_+\cap\tilde{B}_{\varepsilon^{-1}\delta}}\frac{|x'|^2(\partial_1 U)^2}{4}\,dx+\Theta_\gamma(1)
\end{aligned}\tag{9.28}$$

as $\varepsilon\to 0$. Now use equation (9.7) to get that

$$\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U(-\Delta U)\,dx=\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U\left(\lambda\frac{U^{2^\star(s)-1}}{|x|^s}+\gamma\frac{U}{|x|^2}\right)\,dx.\tag{9.29}$$

Integrating by parts, using that U vanishes on $\partial\mathbb{R}^n_+$ and the upper-bound (9.9), for $\sigma\in[0,2]$, we get that

$$\begin{aligned}
&\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}|x'|^2\partial_1 U\frac{U^{2^\star(\sigma)-1}}{|x|^\sigma}\,dx=\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}|x'|^2|x|^{-\sigma}\partial_1\left(\frac{U^{2^\star(\sigma)}}{2^\star(\sigma)}\right)\,dx\\
&=\int_{\partial(\tilde{B}_{\varepsilon^{-1}\delta,+})}|x'|^2|x|^{-\sigma}\frac{U^{2^\star(\sigma)}}{2^\star(\sigma)}\nu_1\,dx-\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\partial_1(|x'|^2|x|^{-\sigma})\left(\frac{U^{2^\star(\sigma)}}{2^\star(\sigma)}\right)\,dx\\
&=O\left(\int_{\mathbb{R}^n_+\cap\partial\tilde{B}_{\varepsilon^{-1}\delta,+}}|x|^{2-\sigma}U^{2^\star(\sigma)}\,d\sigma\right)+\frac{\sigma}{2^\star(s)}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2x_1}{|x|^{\sigma+2}}U^{2^\star(\sigma)}\,dx\\
&=\frac{\sigma}{2^\star(s)}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2x_1}{|x|^{\sigma+2}}U^{2^\star(\sigma)}\,dx+\Theta_\gamma(1)\quad\text{as }\varepsilon\to 0.
\end{aligned}\tag{9.30}$$

Putting together (9.28) to (9.30) yields (9.25).

Estimate of $J^{\Omega}_{\gamma,s}(u_\varepsilon)$: Since $U\in D^{1,2}(\mathbb{R}^n)$, it follows from (9.7) that
$$\int_{\mathbb{R}^n_+}\left(|\nabla U|^2-\frac{\gamma}{|x|^2}U^2\right)dx=\lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx.$$
This equality, combined with (9.23) and (9.24) gives
$$\begin{aligned}J^{\Omega}_{\gamma,s}(u_\varepsilon) &= \frac{\int_\Omega\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx}{\left(\int_\Omega\frac{|u_\varepsilon|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}}\\
&= \frac{\int_{\mathbb{R}^n_+}\left(|\nabla U|^2-\frac{\gamma}{|x|^2}U^2\right)dx}{\left(\int_{\mathbb{R}^n_+}\frac{|U|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}}\left(1+\epsilon\frac{h_\Omega(0)}{(n-1)\lambda\int_{\mathbb{R}^n_+}\frac{|U|^{2^\star(s)}}{|x|^s}\,dx}C_\varepsilon+\Theta_\gamma(\varepsilon)\right)\end{aligned}\tag{9.31}$$
where for all $\varepsilon>0$,
$$\begin{aligned}C_\varepsilon &:= -2\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\partial_1U(x',\nabla U)\,dx+\gamma\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2x_1}{|x|^2}\frac{U^2}{|x|^2}\,dx\\
&\quad+\lambda\frac{s}{2^\star(s)}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2x_1}{|x|^2}\frac{U^{2^\star(s)}}{|x|^s}\,dx.\end{aligned}$$
The identity (9.25) then yields
$$C_\varepsilon=\int_{\partial\mathbb{R}^n_+\cap\tilde{B}_{\varepsilon^{-1}\delta}}\frac{|x'|^2(\partial_1U)^2}{2}\,dx+\Theta_\gamma(1)$$
as $\varepsilon\to0$. Therefore, (9.31) yields that as $\varepsilon\to0$,
$$J^{\Omega}_{\gamma,s}(u_\varepsilon)=\mu_{\gamma,s}(\mathbb{R}^n_+)\left(1+\epsilon\frac{h_\Omega(0)\int_{\partial\mathbb{R}^n_+\cap\tilde{B}_{\varepsilon^{-1}\delta}}|x'|^2(\partial_1U)^2\,dx'}{2(n-1)\lambda\int_{\mathbb{R}^n_+}\frac{|U|^{2^\star(s)}}{|x|^s}\,dx}+\Theta_\gamma(\varepsilon)\right).\tag{9.32}$$

We now distinguish two cases:

Case 1': $\gamma<\frac{n^2-1}{4}$. The bound (9.9) yields $|x'|^2|\partial_1U|^2=O(|x'|^{2-2\alpha_+})$ when $|x'|\to+\infty$. Since $\partial\mathbb{R}^n_+=\mathbb{R}^{n-1}$, we then get that $x'\mapsto|x'|^2|\partial_1U(x')|^2$ is in $L^1(\partial\mathbb{R}^n_+)$, and therefore, (9.32) yields
$$J^{\Omega}_{\gamma,s}(u_\varepsilon)=\mu_{\gamma,s}(\mathbb{R}^n_+)\left(1+C_0\cdot h_\Omega(0)\cdot\epsilon+o(\varepsilon)\right)\text{ as }\varepsilon\to0,\tag{9.33}$$
with
$$C_0:=\frac{\int_{\partial\mathbb{R}^n_+}|x'|^2(\partial_1U)^2\,dx'}{2(n-1)\lambda\int_{\mathbb{R}^n_+}\frac{|U|^{2^\star(s)}}{|x|^s}\,dx}>0.$$

Case 1": $\gamma=\frac{n^2-1}{4}$. It follows from (9.8), Lemma 6.5 and a Kelvin transform that
$$\lim_{|x'|\to+\infty}|x'|^{\alpha_+}|\partial_1U(0,x')|=K_2>0.$$
Since $2\alpha_+-2=n-1$, we get that
$$\int_{\partial\mathbb{R}^n_+\cap\tilde{B}_{\varepsilon^{-1}\delta}}|x'|^2(\partial_1U)^2\,dx'=\omega_{n-1}K_2^2\ln\frac{1}{\varepsilon}+o\left(\ln\frac{1}{\varepsilon}\right)$$

as $\varepsilon \to 0$. Therefore, (9.32) yields

$$J^{\Omega}_{\gamma,s}(u_\varepsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 + C_0' h_\Omega(0)\varepsilon \ln \tfrac{1}{\varepsilon} + o\left(\ln \tfrac{1}{\varepsilon}\right)\right) \quad \text{as } \varepsilon \to 0, \tag{9.34}$$

where

$$C_0' := \frac{\omega_{n-1} K_2^2}{2(n-1)\lambda \int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(s)}}{|x|^s}\, dx} > 0.$$

Cases 1 and 2 prove Proposition 9.3 when $\gamma \le \frac{n^2-1}{4}$.

Case 2: $\gamma > \frac{n^2-1}{4}$. In this case, the test-functions are more subtle. First, use Theorem 8.1 to obtain $H \in C^2(\overline{\Omega} \setminus \{0\})$ such that (8.1) holds and

$$H(x) = \tfrac{d(x,\partial\Omega)}{|x|^{\alpha_+}} + m_\gamma(\Omega)\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-}} + o\left(\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-}}\right) \quad \text{when } x \to 0. \tag{9.35}$$

As above, we fix $\eta \in C^\infty_c(\mathbb{R}^n)$ such that $\eta(x) = 1$ for all $x \in \tilde{B}_\delta$, $\eta(x) = 0$ for all $x \notin \tilde{B}_{2\delta}$. We then define β such that

$$H(x) = \left(\eta \frac{x_1}{|x|^{\alpha_+}}\right) \circ \varphi^{-1}(x) + \beta(x) \quad \text{for all } x \in \Omega.$$

Here φ is as in (5.5) to (5.10). Note that $\beta \in D^{1,2}(\Omega)$ and

$$\beta(x) = m_\gamma(\Omega)\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-}} + o\left(\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-}}\right) \quad \text{as } x \to 0. \tag{9.36}$$

Indeed, since $\alpha_+ - \alpha_- < 1$, an essential point underlying all this subsection is that

$$|x| = o\left(|x|^{\alpha_+ - \alpha_-}\right) \quad \text{as } x \to 0.$$

We choose U as in (9.7). Ny multiplying by a constant if necessary, we assume that $K_2 = 1$, that is

$$U(x) \sim_{x \to 0} K_1 \frac{x_1}{|x|^{\alpha_-}} \text{ and } U(x) \sim_{|x| \to +\infty} \frac{x_1}{|x|^{\alpha_+}}. \tag{9.37}$$

Now define

$$u_\epsilon(x) := \left(\eta \epsilon^{-\frac{n-2}{2}} U(\epsilon^{-1}\cdot)\right) \circ \varphi^{-1}(x) + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \beta(x) \text{ for } x \in \Omega \text{ and } \epsilon > 0. \tag{9.38}$$

We start by showing that for any $k \ge 0$

$$\lim_{\varepsilon \to 0} \frac{u_\varepsilon}{\varepsilon^{\frac{\alpha_+ - \alpha_-}{2}}} = H \text{ in } C^k_{loc}(\overline{\Omega} \setminus \{0\}). \tag{9.39}$$

Indeed, the convergence in $C^0_{loc}(\overline{\Omega} \setminus \{0\})$ is a consequence of the definition of u_ε, the choice $K_2 = 1$ and the asymptotic behavior (9.37). For convergence in C^k, we need in addition that $\nabla^i(U - x_1|x|^{-\alpha_+}) = o\left(|x|^{1-\alpha_+ - i}\right)$ as $x \to +\infty$ for all $i \ge 0$. This estimate follows from (9.37) and Lemma 6.5.

In the sequel, we adopt the following notation: θ^ε_c will denote any quantity such that there exists $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{c \to 0} \lim_{\varepsilon \to 0} \theta^\varepsilon_c = 0.$$

We first claim that for any $c > 0$, we have that

$$\begin{aligned} &\int_{\Omega \setminus \varphi(B_c(0)_+)} \left(|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2} u_\varepsilon^2\right) dx \\ &= \varepsilon^{\alpha_+ - \alpha_-}\left((\alpha_+ - 1)c^{n-2\alpha_+}\frac{\omega_{n-1}}{2n} + m_\gamma(\Omega)\frac{(n-2)\omega_{n-1}}{2n}\right) + \theta^\varepsilon_c \varepsilon^{\alpha_+ - \alpha_-}. \end{aligned} \tag{9.40}$$

Indeed, it follows from (9.39) that

$$\lim_{\varepsilon\to 0}\frac{\int_{\Omega\setminus\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)\,dx}{\varepsilon^{\alpha_+-\alpha_-}}=\int_{\Omega\setminus\varphi(B_c(0)_+)}\left(|\nabla H|^2-\frac{\gamma}{|x|^2}H^2\right)\,dx. \tag{9.41}$$

Since H vanishes on $\partial\Omega\setminus\{0\}$ and satisfies $-\Delta H-\frac{\gamma}{|x|^2}H=0$, integrating by parts yields

$$\begin{aligned}\int_{\Omega\setminus\varphi(B_c(0)_+)}\left(|\nabla H|^2-\frac{\gamma}{|x|^2}H^2\right)\,dx &= -\int_{\varphi(\mathbb{R}^n_+\cap\partial B_c(0))}H\partial_\nu H\,d\sigma\\ &= -\int_{\mathbb{R}^n_+\cap\partial B_c(0)}H\circ\varphi\,\partial_{\varphi_\star\nu}(H\circ\varphi)\,d(\varphi^\star\sigma),\end{aligned}\tag{9.42}$$

where in the two last equalities, $\nu(x)$ is the outer normal vector of $B_c(0)$ at $x\in\partial B_c(0)$.

We now estimate $H\circ\varphi\,\partial_{\varphi_\star\nu}H\circ\varphi$. It follows from (9.35) that

$$H\circ\varphi(x)=\frac{x_1}{|x|^{\alpha_+}}+m_\gamma(\Omega)\frac{x_1}{|x|^{\alpha_-}}+o\left(\frac{x_1}{|x|^{\alpha_-}}\right)\quad\text{as } x\to 0. \tag{9.43}$$

It follows from elliptic theory and (9.36) that for any $i=1,...,n$, we have that

$$\partial_i(\beta\circ\varphi)=\partial_i\left(m_\gamma(\Omega)\tfrac{x_1}{|x|^{\alpha_-}}\right)+o\left(|x|^{-\alpha_-}\right)\quad\text{as } x\to 0. \tag{9.44}$$

Therefore,

$$\begin{aligned}\partial_i(H\circ\varphi) &= \delta_{i1}|x|^{-\alpha_+}-\alpha_+x_1x_i|x|^{-\alpha_+-2}\\ &\quad +m_\gamma(\Omega)\left(\delta_{i1}|x|^{-\alpha_-}-\alpha_-x_1x_i|x|^{-\alpha_--2}\right)+o\left(|x|^{-\alpha_-}\right)\end{aligned}\tag{9.45}$$

as $x\to 0$. Moreover, $\varphi_\star\nu(x)=\frac{x}{|x|}+O(|x|)$ as $x\to 0$. Therefore, the estimate (9.45) yields

$$\partial_{\varphi_\star\nu}(H\circ\varphi)=-(\alpha_+-1)\frac{x_1}{|x|^{\alpha_++1}}-(\alpha_--1)m_\gamma(\Omega)\frac{x_1}{|x|^{\alpha_-+1}}+o\left(|x|^{-\alpha_-}\right) \tag{9.46}$$

as $x\to 0$. By using that $\alpha_++\alpha_-=n$ and $\alpha_+-\alpha_-<1$, (9.43) and (9.46) yield

$$-H\circ\varphi\partial_{\varphi_\star\nu}(H\circ\varphi)=\tfrac{(\alpha_+-1)x_1^2}{|x|^{2\alpha_++1}}+(n-2)m_\gamma(\Omega)\tfrac{x_1^2}{|x|^{n+1}}+o\left(|x|^{1-n}\right)\quad\text{as } x\to 0.$$

Integrating this expression on $B_c(0)_+=\mathbb{R}^n_+\cap\partial B_c(0)$ and plugging into (9.42) yield

$$\int_{\Omega\setminus\varphi(B_c(0)_+)}\left(|\nabla H|^2-\frac{\gamma}{|x|^2}H^2\right)\,dx = \frac{(\alpha_+-1)c^{n-2\alpha_+}\omega_{n-1}}{2n}+(n-2)m_\gamma(\Omega)\frac{\omega_{n-1}}{2n}+\theta_c$$

where $\lim_{c\to 0}\theta_c=0$. Here, we have used that

$$\int_{\mathbb{S}^{n-1}_+}x_1^2\,d\sigma=\frac{1}{2}\int_{\mathbb{S}^{n-1}}x_1^2\,d\sigma=\frac{1}{2n}\int_{\mathbb{S}^{n-1}}|x|^2\,d\sigma=\frac{\omega_{n-1}}{2n},\ \omega_{n-1}:=\int_{\mathbb{S}^{n-1}}d\sigma.$$

This equality and (9.41) prove (9.40).

We now claim that

$$\begin{aligned}\int_\Omega\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)\,dx &= \lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx\\ &\quad +m_\gamma(\Omega)\tfrac{(n-2)\omega_{n-1}}{2n}\varepsilon^{\alpha_+-\alpha_-}+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\quad\text{as }\varepsilon\to 0.\end{aligned}\tag{9.47}$$

Indeed, define $U_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}} U(\varepsilon^{-1}x)$ for all $x \in \mathbb{R}^n_+$. The definition (9.38) of u_ε rewrites as:

$$u_\varepsilon \circ \varphi(x) = U_\epsilon(x) + \varepsilon^{\frac{\alpha_+ - \alpha_-}{2}} \beta \circ \varphi(x) \quad \text{for all } x \in \mathbb{R}^n_+ \cap \tilde{B}_\delta.$$

Fix $c \in (0, \delta)$ that we will eventually let go to 0. Since $d\varphi_0$ is an isometry, we get that

$$\begin{aligned}
(9.48) \qquad & \int_{\varphi(B_c(0)_+)} \left(|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2} u_\varepsilon^2 \right) dx \\
&= \int_{B_c(0)_+} \left(|\nabla (u_\varepsilon \circ \varphi)|^2_{\varphi^\star \mathrm{Eucl}} - \frac{\gamma}{|\varphi(x)|^2} (u_\varepsilon \circ \varphi)^2 \right) |\mathrm{Jac}(\varphi)|\, dx \\
&= \int_{B_c(0)_+} \left(|\nabla U_\varepsilon|^2_{\varphi^\star \mathrm{Eucl}} - \frac{\gamma}{|\varphi(x)|^2} U_\varepsilon^2 \right) |\mathrm{Jac}(\varphi)|\, dx \\
&\quad + 2\varepsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_c(0)_+} \left((\nabla U_\varepsilon, \nabla(\beta \circ \varphi))_{\varphi^\star \mathrm{Eucl}} - \frac{\gamma}{|\varphi(x)|^2} U_\varepsilon (u_\varepsilon \circ \varphi) \right) |\mathrm{Jac}(\varphi)|\, dx \\
&\quad + \varepsilon^{\alpha_+ - \alpha_-} \int_{B_c(0)_+} \left(|\nabla(\beta \circ \varphi)|^2_{\varphi^\star \mathrm{Eucl}} - \frac{\gamma}{|\varphi(x)|^2} (\beta \circ \varphi)^2 \right) |\mathrm{Jac}(\varphi)|\, dx
\end{aligned}$$

Since $\varphi^\star \mathrm{Eucl} = \mathrm{Eucl} + O(|x|)$, $|\varphi(x)| = |x| + O(|x|^2)$ and $\beta \in D^{1,2}(\Omega)$, we get that

$$\begin{aligned}
(9.49) & \int_{\varphi(B_c(0)_+)} \left(|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2} u_\varepsilon^2 \right) dx = \int_{B_c(0)_+} \left(|\nabla U_\varepsilon|^2_{\mathrm{Eucl}} - \frac{\gamma}{|x|^2} U_\varepsilon^2 \right) |\, dx \\
&\quad + O\left(\int_{B_c(0)_+} |x| \left(|\nabla U_\varepsilon|^2_{\mathrm{Eucl}} + \frac{U_\varepsilon^2}{|x|^2} \right) |\, dx \right) \\
&\quad + 2\varepsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_c(0)_+} \left((\nabla U_\varepsilon, \nabla(\beta \circ \varphi))_{\mathrm{Eucl}} - \frac{\gamma}{|x|^2} U_\varepsilon (\beta \circ \varphi) \right) dx \\
&\quad + O\left(\varepsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_c(0)_+} |x| \left(|\nabla U_\varepsilon| \cdot |\nabla(\beta \circ \varphi)| + \frac{U_\varepsilon |\beta \circ \varphi|}{|x|^2} \right) dx \right) + \varepsilon^{\alpha_+ - \alpha_-} \theta_c^\varepsilon
\end{aligned}$$

as $\varepsilon \to 0$. The pointwise estimates (9.37) and (9.44) yield

$$\begin{aligned}
& \int_{\varphi(B_c(0)_+)} \left(|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2} u_\varepsilon^2 \right) dx = \int_{B_c(0)_+} \left(|\nabla U_\varepsilon|^2_{\mathrm{Eucl}} - \frac{\gamma}{|x|^2} U_\varepsilon^2 \right) dx \\
& + 2\varepsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_c(0)_+} \left((\nabla U_\varepsilon, \nabla(\beta \circ \varphi))_{\mathrm{Eucl}} - \frac{\gamma}{|x|^2} U_\varepsilon (\beta \circ \varphi) \right) dx \\
& + \varepsilon^{\alpha_+ - \alpha_-} \theta_c^\varepsilon
\end{aligned}$$

as $\varepsilon \to 0$. Integrating by parts yields

$$\begin{aligned}
& \int_{\varphi(B_c(0)_+)} \left(|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2} u_\varepsilon^2 \right) dx \\
&= \int_{B_c(0)_+} \left(-\Delta U_\varepsilon - \frac{\gamma}{|x|^2} U_\varepsilon \right) U_\varepsilon\, dx + \int_{\partial(B_c(0)_+)} U_\varepsilon \partial_\nu U_\varepsilon\, d\sigma \\
&+ 2\varepsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left(\int_{B_c(0)_+} \left(-\Delta U_\varepsilon - \frac{\gamma}{|x|^2} U_\varepsilon \right) \beta \circ \varphi\, dx + \int_{\partial(B_c(0)_+)} \beta \circ \varphi \partial_\nu U_\varepsilon\, d\sigma \right) \\
&+ \varepsilon^{\alpha_+ - \alpha_-} \theta_c^\varepsilon
\end{aligned}$$

as $\varepsilon\to 0$. Since both U and $\beta\circ\varphi$ vanish on $\partial\mathbb{R}^n_+\setminus\{0\}$, we get that

$$\int_{\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx \tag{9.50}$$
$$=\int_{B_c(0)_+}\left(-\Delta U_\varepsilon-\frac{\gamma}{|x|^2}U_\varepsilon\right)U_\varepsilon\,dx+\int_{\mathbb{R}^n_+\cap\partial B_c(0)}U_\varepsilon\partial_\nu U_\varepsilon\,d\sigma$$
$$+2\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(\int_{B_c(0)_+}\left(-\Delta U_\varepsilon-\frac{\gamma}{|x|^2}U_\varepsilon\right)\beta\circ\varphi\,dx+\int_{\mathbb{R}^n_+\cap\partial B_c(0)}\beta\circ\varphi\partial_\nu U_\varepsilon\,d\sigma\right)$$
$$+\varepsilon^{\alpha_+-\alpha_-}\theta_c^\varepsilon$$

as $\varepsilon\to 0$. The asymptotic estimate (9.37) of U and Lemma 6.5 yield (after a Kelvin transform)

$$\partial_\nu U_\varepsilon=-(\alpha_+-1)\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}x_1|x|^{-\alpha_+-1}+o\left(\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}|x|^{-\alpha_+}\right)$$

as $\varepsilon\to 0$ uniformly on compact subsets of $\overline{\mathbb{R}^n_+}\setminus\{0\}$. We then get that

$$\beta\circ\varphi\partial_\nu U_\varepsilon=\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(-m_\gamma(\Omega)(\alpha_+-1)x_1^2|x|^{-n-1}+o\left(|x|^{1-n}\right)\right)$$

and

$$U_\varepsilon\partial_\nu U_\varepsilon=\varepsilon^{\alpha_+-\alpha_-}\left(-(\alpha_+-1)x_1^2|x|^{-2\alpha_+-1}+o\left(|x|^{1-2\alpha_+}\right)\right)$$

as $\varepsilon\to 0$ uniformly on compact subsets of $\overline{\mathbb{R}^n_+}\setminus\{0\}$. Plugging these identities in (9.51) and using equation (9.7) yield, as $\varepsilon\to 0$,

$$\begin{aligned}\int_{\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx &= \int_{B_c(0)_+}\lambda\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx-(\alpha_+-1)\frac{\omega_{n-1}}{2n}c^{n-2\alpha_+}\varepsilon^{\alpha_+-\alpha_-}\\ &\quad+2\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\lambda\frac{U_\varepsilon^{2^\star(s)-1}}{|x|^s}\beta\circ\varphi\,dx\\ &\quad-(\alpha_+-1)\frac{\omega_{n-1}}{n}m_\gamma(\Omega)\varepsilon^{\alpha_+-\alpha_-}+\varepsilon^{\alpha_+-\alpha_-}\theta_c^\varepsilon\end{aligned}\tag{9.51}$$

Note that as $\varepsilon\to 0$,

$$\begin{aligned}\int_{B_c(0)_+}\lambda\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx &= \int_{\mathbb{R}^n_+}\lambda\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx+O\left(\int_{\mathbb{R}^n_+\setminus B_c(0)_+}\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx\right)\\ &=\int_{\mathbb{R}^n_+}\lambda\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\end{aligned}\tag{9.52}$$

The expansion (9.36) and the change of variable $x:=\varepsilon y$ yield as $\varepsilon\to 0$,

$$\int_{B_c(0)_+}\lambda\frac{U_\varepsilon^{2^\star(s)-1}}{|x|^s}\beta\circ\varphi\,dx=\lambda m_\gamma(\Omega)\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)-1}}{|y|^s}\frac{y_1}{|y|^{\alpha_-}}\,dy+\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\theta_\varepsilon^c \tag{9.53}$$

Integrating by parts, and using the asymptotics (9.37) for U yield

$$\begin{aligned}\lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)-1}}{|y|^s}\frac{y_1}{|y|^{\alpha_-}}\,dy &= \lim_{R\to+\infty}\int_{B_R(0)_+}\lambda\frac{U^{2^\star(s)-1}}{|y|^s}\frac{y_1}{|y|^{\alpha_-}}\,dy\\ &= \lim_{R\to+\infty}\int_{B_R(0)_+}\left(-\Delta U-\frac{\gamma}{|y|^2}U\right)\frac{y_1}{|y|^{\alpha_-}}\,dy\\ &= \lim_{R\to+\infty}\int_{B_R(0)_+}U\left(-\Delta-\frac{\gamma}{|y|^2}\right)\left(\frac{y_1}{|y|^{\alpha_-}}\right)dy\\ &\quad -\int_{\partial B_R(0)_+}\partial_\nu U\frac{y_1}{|y|^{\alpha_-}}\,d\sigma\\ &= (\alpha_+-1)\,\frac{\omega_{n-1}}{2n}. \end{aligned}\tag{9.54}$$

Putting together (9.52), (9.53) and (9.54) yield

$$\begin{aligned}\int_\Omega\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx &= \lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx\\ &\quad +m_\gamma(\Omega)\frac{(n-2)\omega_{n-1}}{2n}\varepsilon^{\alpha_+-\alpha_-}+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\end{aligned}$$

as $\varepsilon\to0$. This finally yields (9.47).

We finally claim that

$$\begin{aligned}\int_\Omega\frac{u_\varepsilon^{2^\star(s)}}{|x|^s}\,dx &= \int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx+\frac{2^\star(s)}{\lambda}m_\gamma(\Omega)\frac{(\alpha_+-1)\omega_{n-1}}{2n}\varepsilon^{\alpha_+-\alpha_-}\\ &\quad +o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\qquad\text{as }\varepsilon\to0.\end{aligned}\tag{9.55}$$

Indeed, fix $c>0$. Due to estimates (9.36) and (9.37), we have that

$$\begin{aligned}\int_\Omega\frac{u_\varepsilon^{2^\star(s)}}{|x|^s}\,dx &= \int_{\varphi(B_c(0)_+)}\frac{u_\varepsilon^{2^\star(s)}}{|x|^s}\,dx+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\\ &= \int_{B_c(0)_+}\frac{|U_\varepsilon+\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\beta\circ\varphi|^{2^\star(s)}}{|\varphi(x)|^s}|\mathrm{Jac}(\varphi)|\,dx+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\\ &= \int_{B_c(0)_+}\frac{|U_\varepsilon+\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\beta\circ\varphi|^{2^\star(s)}}{|x|^s}|(1+O(|x|))\,dx+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\end{aligned}$$

as $\varepsilon\to0$. One can easily check that there exists $C>0$ such that for all $X,Y\in\mathbb{R}$,

$$||X+Y|^{2^\star(s)}-|X|^{2^\star(s)}-2^\star(s)|X|^{2^\star(s)-2}XY|\le C\left(|X|^{2^\star(s)-2}|Y|^2+|Y|^{2^\star(s)}\right)\tag{9.56}$$

Therefore, using the asymptotics (9.36) and (9.37) of U and β, we get that

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^{2^\star(s)}}{|x|^s}\,dx &= \int_{B_c(0)_+} \frac{U_\varepsilon^{2^\star(s)}}{|x|^s}|(1+O(|x|))\,dx\\ &\quad +2^\star(s)\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\frac{U_\varepsilon^{2^\star(s)-1}}{|x|^s}\beta\circ\varphi(1+O(|x|))\,dx\\ &\quad +\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\theta_\varepsilon^c\\ &= \int_{B_c(0)_+}\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx + 2^\star(s)\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\frac{U_\varepsilon^{2^\star(s)-1}}{|x|^s}\beta\circ\varphi\,dx\\ &\quad +\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\theta_\varepsilon^c \quad \text{as } \varepsilon\to 0.\end{aligned}$$

Then (9.55) follows from this latest identity, combined with (9.52), (9.53), and (9.54).

We can finally use (9.47) and (9.55), and the fact that

$$\int_{\mathbb{R}^n_+}(|\nabla U|^2 - \frac{\gamma}{|x|^2}U^2)\,dx = \lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx,$$

to get

$$J_{\gamma,s}^\Omega(u_\varepsilon) = J_{\gamma,s}^{\mathbb{R}^n_+}(U)\left(1 - \frac{\left(\alpha_+ - \frac{n}{2}\right)\omega_{n-1}}{n\lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx}m_\gamma(\Omega)\varepsilon^{\alpha_+-\alpha_-} + o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\right) \quad \text{as } \varepsilon\to 0,$$

which proves (9.6). This ends the proof of Proposition 9.3, and therefore, as already mentioned, of Theorem 9.2.

10. Examples of domains with positive mass

We now assume that $\gamma \in (\frac{n^2-1}{4}, \frac{n^2}{4})$. We have seen in Proposition 8.2 that the mass is negative when $\Omega \subset \mathbb{R}^n_+$. In particular, $m_\gamma(\Omega) < 0$ if Ω is convex and $\gamma < \gamma_H(\Omega)$. In this section, we give examples of domains Ω with positive mass.

For any $x_0 \in \mathbb{R}^n \setminus \{0\}$, we define the inversion

$$i_{x_0}(x) := x_0 + |x_0|^2\frac{x - x_0}{|x-x_0|^2}$$

for all $x \in \mathbb{R}^n \setminus \{x_0\}$. The inversion i_{x_0} is the identity map on $\partial B_{|x_0|}(x_0)$ (the ball of center x_0 and of radius $|x_0|$), and in particular $i(0) = 0$.

Definition 10.1. *We shall say that a domain $\Omega \subset \mathbb{R}^n$ ($0 \in \partial\Omega$) is smooth at infinity if there exists $x_0 \notin \overline{\Omega}$ such that $i_{x_0}(\Omega)$ is a smooth bounded domain of $\mathbb{R}^n$ having both 0 and x_0 being on its boundary $\partial(i_{x_0}(\Omega))$.*

One can easily check that $\mathbb{R}^n_+$ is a smooth domain at infinity (take $x_0 := (-1, 0, \dots, 0)$). We now state and prove three propositions that are crucial for the constructions that follow. The first one indicates that the notion of mass defined in Theorem 8.1 extends to unbounded domains that are smooth at infinity.

Proposition 10.2. *Let Ω be a domain that is smooth at infinity such that $0 \in \partial\Omega$. We assume that $\gamma_H(\Omega) > \frac{n^2-1}{4}$ and fix $\gamma \in \left(\frac{n^2-1}{4}, \gamma_H(\Omega)\right)$. Then, up to a multiplicative constant, there exists a unique function $H \in C^2(\overline{\Omega} \setminus \{0\})$ such that*

$$
\left\{
\begin{array}{rl}
-\Delta H - \frac{\gamma}{|x|^2} H = 0 & \text{in } \Omega \\
H > 0 & \text{in } \Omega \\
H = 0 & \text{on } \partial\Omega \setminus \{0\} \\
H(x) \leq C|x|^{1-\alpha_+(\gamma)} & \text{for all } x \in \Omega.
\end{array}
\right.
\tag{10.1}
$$

Moreover, there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$
H(x) = c_1 \frac{d(x, \partial\Omega)}{|x|^{\alpha_+(\gamma)}} + c_2 \frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \quad \text{as } x \to 0.
$$

We define the mass $m_\gamma(\Omega) := \frac{c_2}{c_1}$, which is independent of the choice of H in (10.1).

With this notion of mass, we will be in a position to prove the following continuity result.

Proposition 10.3. *Let $\Omega \subset \mathbb{R}^n$ be smooth at infinity such that $0 \in \partial\Omega$. We assume that $\gamma_H(\Omega) > \frac{n^2-1}{4}$, and fix $\gamma \in \left(\frac{n^2-1}{4}, \gamma_H(\Omega)\right)$. For any $R > 0$, let D_R be a smooth domain of $\mathbb{R}^n$ such that*

- *$B_R(x_0) \subset D_R \subset B_{2R}(x_0)$,*
- *$\Omega \cap D_R$ is a smooth domain of $\mathbb{R}^n$.*

Let $\Phi \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be such that

- *$\Phi_t := \Phi(t, \cdot)$ is a smooth diffeomorphism of $\mathbb{R}^n$,*
- *$\Phi_t(x) = x$ for all $|x| > 1/2$ and all $t \in \mathbb{R}$,*
- *$\Phi_t(0) = 0$ for all $t \in \mathbb{R}$,*
- *$\Phi_0 = Id_{\mathbb{R}^n}$.*

Set $\Omega_{t,R} := \Phi_t(\Omega) \cap D_R$. Then

$$
\liminf_{t \to 0,\, R \to +\infty} \gamma_H(\Omega_{t,R}) \geq \gamma_H(\Omega).
$$

Therefore, for $t \to 0$, $R \to +\infty$, we have that $\gamma_H(\Omega_{t,R}) > \frac{n^2-1}{4}$ and $m_\gamma(\Omega_{t,R})$ is well defined. In addition,

$$
\lim_{t \to 0,\, R \to +\infty} m_\gamma(\Omega_{t,R}) = m_\gamma(\Omega).
$$

As a consequence of the above, we shall be able to construct smooth bounded domains with positive or negative mass with any behavior at 0.

Proposition 10.4. *Let ω be a smooth open set of $\mathbb{R}^n$. Then, there exist Ω_+, Ω_- two smooth bounded domains of $\mathbb{R}^n$ with Hardy constants $> \frac{n^2-1}{4}$, and there exists $r_0 > 0$ such that*

$$
\Omega_+ \cap B_{r_0}(0) = \Omega_- \cap B_{r_0}(0) = \omega \cap B_{r_0}(0),
$$

and for any $\gamma \in (\frac{n^2-1}{4}, \min\{\gamma_H(\Omega_+), \gamma_H(\Omega_-)\})$, we have that

$$
m_\gamma(\Omega_+) > 0 > m_\gamma(\Omega_-).
$$

The remainder of this section will be devoted to the proof of these three propositions. As a preliminary remark, we claim that if Ω is a domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$ and Ω is smooth at infinity, then

$$\liminf_{t\to 0, R\to\infty} \gamma_H(\Omega_{t,R}) \geq \gamma_H(\Omega), \tag{10.2}$$

where $\Omega_{t,R}$ are defined as in Proposition 10.3. Indeed, by definition, $\gamma_H(\Omega_{t,R}) \geq \gamma_H(\Omega_t) = \gamma_H(\Phi_t(\Omega))$. Inequality (10.2) then follows from the limit (3.7) of Lemma 3.2. We shall proceed in 7 steps.

Step 1: Reformulation via the inversion. For convenience, up to a rotation and a dilation, we can assume that $x_0 := (-1, 0, \dots, 0) \in \mathbb{R}^n$ and we define the inversion

$$i(x) := x_0 + \frac{x - x_0}{|x - x_0|^2} \quad \text{for all } x \in \mathbb{R}^n \setminus \{x_0\}.$$

For any $u \in C^2(U)$, where U is a domain of $\mathbb{R}^n$ we define its Kelvin transform $\hat{u} : \hat{U} \to \mathbb{R}$ by

$$\hat{u}(x) := |x - x_0|^{2-n} u(i(x)) \text{ for all } x \in \hat{U} := i^{-1}(U \setminus \{x_0\}).$$

The Kelvin transform leaves the Laplacian invariant in the following sense:

$$-\Delta \hat{u}(x) = |x - x_0|^{-(n+2)} (-\Delta u)(i(x)) \text{ for all } x \in \hat{U}. \tag{10.3}$$

Define $\tilde{\Omega} := i(\Omega)$, $\tilde{\Phi}(t, x) := i \circ \Phi(t, i(x))$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, and $\tilde{D}_r := \mathbb{R}^n \setminus i(D_{r^{-1}})$ (i.e., the complement in $\mathbb{R}^n$). Here, note that $R \to +\infty$ in Proposition 10.3 is equivalent to $r \to 0$ in here. We then have that

$$0, x_0 \in \partial\tilde{\Omega} \text{ and } \tilde{\Omega} \text{ is a smooth bounded domain of } \mathbb{R}^n.$$

Note that $\tilde{\Phi} \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is such that

- For any $t \in (-2, 2)$, $\tilde{\Phi}_t := \tilde{\Phi}(t, \cdot)$ is a $C^\infty-$diffeomorphism onto its open image $\tilde{\Phi}_t(\mathbb{R}^n)$.
- $\tilde{\Phi}_0 = \mathrm{Id}$,
- $\tilde{\Phi}_t(0) = 0$ for all $t \in (-2, 2)$,
- $\tilde{\Phi}_t(x) = x$ for all $t \in (-2, 2)$ and all $x \in B_{2\delta}(x_0)$ with $\delta < 1/4$.

We define

$$\tilde{\Omega}_t := \tilde{\Phi}_t(\tilde{\Omega}).$$

The sets $\tilde{D}_r$ satisfy the following properties:

- $B_{r/2}(x_0) \subset \tilde{D}_r \subset B_r(x_0)$,
- $\tilde{\Omega}_{t,r} := \tilde{\Omega}_t \setminus \tilde{D}_r$ is a smooth domain of $\mathbb{R}^n$.

In particular, we have that

$$\tilde{\Omega}_{t,r} = i(\Omega_{t,r^{-1}}).$$

Let $u \in C^2(\overline{\Omega_{t,r}} \setminus \{0\})$ be such that

$$-\Delta u - \frac{\gamma}{|x|^2} u = 0 \text{ in } \Omega_{t,r}\ ,\ u > 0 \text{ in } \Omega_{t,r}\ ,\ u = 0 \text{ on } \partial\Omega_{t,r}. \tag{10.4}$$

The existence of u follows from Theorem 8.1. Consider the Kelvin transform of u, that is

$$\tilde{u}(x) := |x - x_0|^{2-n} u(i(x)) \text{ for all } x \in \tilde{\Omega}_{t,r}.$$

It then follows from (10.3) that

$$-\Delta \tilde{u} - V \tilde{u} = 0 \text{ in } \tilde{\Omega}_{t,r},$$

where

$$V(x) := \frac{\gamma}{|x|^2|x-x_0|^2} \text{ for } x \in \mathbb{R}^n \setminus \{0, x_0\}. \tag{10.5}$$

It is easy to check that

$$V(x) = \frac{\gamma + O(|x|)}{|x|^2} \text{ as } x \to 0 \text{ and } V(x) = \frac{\gamma + O(|x-x_0|)}{|x-x_0|^2} \text{ as } x \to x_0$$

The coercivity of $-\Delta - \gamma|x|^{-2}$ on Ω (since $\gamma < \gamma_H(\Omega)$) yields the coercivity of $-\Delta - V$ on $\tilde{\Omega}$, that is there exists $c_0 > 0$ such that

$$\int_{\tilde{\Omega}} \left(|\nabla u|^2 - V(x)u^2\right) dx \geq c_0 \int_{\tilde{\Omega}} |\nabla u|^2 \, dx \text{ for all } u \in D^{1,2}(\tilde{\Omega}).$$

From now on, we should be able to transfer the analysis to the bounded domain $\tilde{\Omega}$.

Step 2: Perturbation of the domain via the two singular points 0 and x_0. We shall need the following.

Proposition 10.5. *For any $t \in (-1,1)$, there exists $u_t \in C^2(\overline{\tilde{\Omega}_t} \setminus \{0, x_0\})$ such that*

$$\left\{\begin{array}{lll} -\Delta u_t - V u_t & = 0 & \text{in } \tilde{\Omega}_t \\ u_t & > 0 & \text{in } \tilde{\Omega}_t \\ u_t & = 0 & \text{on } \partial\tilde{\Omega}_t \setminus \{0, x_0\} \\ u_t(x) & \leq C|x|^{1-\alpha_+(\gamma)} + C|x-x_0|^{1-\alpha_-(\gamma)} & \text{for } x \in \tilde{\Omega}_t. \end{array}\right. \tag{10.6}$$

Moreover, we have that

$$u_t(x) = \frac{d(x, \partial\tilde{\Omega}_t)}{|x|^{\alpha_+(\gamma)}}(1 + O(|x|^{\alpha_+(\gamma)-\alpha_-(\gamma)})) \tag{10.7}$$

as $x \to 0$, uniformly wrt $t \in (-1,1)$.

Proof of Proposition 10.5. We construct approximate singular solutions as in Section 5. For all $t \in (-2,2)$, there exists a chart φ_t that satisfies (5.5) to (5.10) for $\tilde{\Omega}_t$. Without restriction, we assume that $\lim_{t\to 0} \varphi_t = \varphi_0$ in $C^k(\tilde{B}_{2\delta}, \mathbb{R}^n)$. We define a cut-off function η_δ such that $\eta_\delta(x) = 1$ for $x \in \tilde{B}_\delta$ and $\eta_\delta(x) = 0$ for $x \notin \tilde{B}_{2\delta}$. As in (5.12), we define $u_{\alpha_+(\gamma),t} \in C^2(\overline{\tilde{\Omega}_t} \setminus \{0\})$ with compact support in $\varphi_t(\tilde{B}_{2\delta})$ such that

$$u_{\alpha_+,t} \circ \varphi_t(x_1, x') := \eta_\delta(x_1, x') x_1 |x|^{-\alpha_+}(1 + \Theta_t(x)) \text{ for all } (x_1, x') \in \tilde{B}_{2\delta} \setminus \{0\}, \tag{10.8}$$

where $\Theta_t(x_1, x') := e^{-\frac{1}{2}x_1 H_t(x')} - 1$ for all $x = (x_1, x') \in \tilde{B}_{2\delta}$ and all $t \in (-2,2)$. Here, $H_t(x')$ is the mean curvature of $\partial\tilde{\Omega}_t$ at the point $\varphi_t(0, x')$. Note that $\lim_{t\to 0} \Theta_t = \Theta_0$ in $C^k(U)$. Arguing as is Section 5, we get that

$$\left\{\begin{array}{lll} (-\Delta - V)u_{\alpha_+,t} & = O(d(x, \partial\tilde{\Omega}_t)|x|^{-\alpha_+(\gamma)-1}) & \text{in } \tilde{\Omega}_t \cap \tilde{B}_\delta \\ u_{\alpha_+,t} & > 0 & \text{in } \tilde{\Omega}_t \cap \tilde{B}_\delta \\ u_{\alpha_+,t} & = 0 & \text{on } \partial\tilde{\Omega}_t \setminus \{0\}, \end{array}\right.$$

and

$$u_{\alpha_+,t}(x) = \frac{d(x, \partial\tilde{\Omega}_t)}{|x|^{\alpha_+(\gamma)}}(1 + O(|x|) \text{ as } x \to 0.$$

The construction in Section 5 also yields

$$\lim_{t\to 0} u_{\alpha_+,t} \circ \Phi_t = u_{\alpha_+,0} \text{ in } C^2_{loc}(\overline{\tilde{\Omega}} \setminus \{0\}). \tag{10.9}$$

Note also that all these estimates are uniform in $t \in (-1,1)$. In particular, defining

$$f_t := -\Delta u_{\alpha_+,t} - V u_{\alpha_+,t}, \tag{10.10}$$

then there exists $C>0$ such that

$$|f_t(x)| \leq C d(x, \partial\tilde{\Omega}_t)|x|^{-\alpha_+(\gamma)-1} \leq C|x|^{-\alpha_+(\gamma)} \tag{10.11}$$

for all $t \in (-1,1)$ and all $x \in \tilde{\Omega}_t \cap \tilde{B}_\delta$. Therefore, since $\gamma > \frac{n^2-1}{4}$, it follows from (10.9) and this pointwise control that $f_t \in L^{\frac{2n}{n+2}}(\tilde{\Omega}_t)$ for all $t \in (-1,1)$ and that

$$\lim_{t\to 0} \|f_t \circ \Phi_t - f_0\|_{L^{\frac{2n}{n+2}}(\tilde{\Omega})} = 0. \tag{10.12}$$

For any $t \in (-1,1)$, we let $v_t \in D^{1,2}(\tilde{\Omega}_t)$ be such that

$$-\Delta v_t - V v_t = f_t \text{ weakly in } D^{1,2}(\tilde{\Omega}_t). \tag{10.13}$$

The existence follows from the coercivity of $-\Delta - V$ on $\tilde{\Omega}_t$, which follows itself from the coercivity on $\tilde{\Omega} = \tilde{\Omega}_0$. We then get from (10.12) and the uniform coercivity on $\tilde{\Omega}_t$ that

$$\lim_{t\to 0} v_t \circ \Phi_t = v_0 \text{ in } D^{1,2}(\tilde{\Omega}) \text{ and } C^1_{loc}(\overline{\tilde{\Omega}} \setminus \{0, x_0\}).$$

It follows from the construction of the mass in Section 8 (see the proof of Theorem 8.1) that around 0, $|v_t(x)|$ is bounded by $|x|^{1-\alpha_-(\gamma)}$. Around x_0, $-\Delta v_t - V v_t = 0$ and the regularity Theorem 6.1 yields a control by $|x-x_0|^{1-\alpha_-(\gamma)}$. These controls are uniform with respect to $t \in (-1,1)$. Therefore, there exists $C>0$ such that

$$|v_t(x)| \leq C d(x, \partial\tilde{\Omega}_t)\left(|x|^{-\alpha_-(\gamma)} + |x-x_0|^{-\alpha_-(\gamma)}\right)$$

for all $t \in (-1,1)$ and all $x \in \tilde{\Omega}_t$. Now define

$$u_t(x) := u_{\alpha_+,t}(x) - v_t(x)$$

for all $t \in (-1,1)$ and $x \in \tilde{\Omega}_t$. This function satisfies all the requirements of Proposition 10.5. $\square$

Step 3: Chopping off a neighborhood of x_0: We now study $\tilde{\Omega}_{t,r} = \tilde{\Omega}_t \setminus \tilde{D}_r$. For $r \in (0, \delta/2)$, note that $\tilde{\Omega}_{t,r} \cap B_\delta(0) = \tilde{\Omega} \cap B_\delta(0)$. We shall now define a mass associated to the potential V, and prove its continuity.

Step 3.1: The function $f_t : \tilde{\Omega}_t \to \mathbb{R}$ defined in (10.10) has compact support in $B_{2\delta}(0)$, therefore, it is well-defined also on $\tilde{\Omega}_{t,r}$. We define $v_{t,r} \in D^{1,2}(\tilde{\Omega}_{t,r})$ such that

$$-\Delta v_{t,r} - V v_{t,r} = f_t \text{ weakly in } D^{1,2}(\tilde{\Omega}_{t,r}). \tag{10.14}$$

Since the operator $-\Delta - V$ is uniformly coercive on $\tilde{\Omega}_t$, it is also uniformly coercive on $\tilde{\Omega}_{t,r}$ with respect to (t,r), so that the definition of $v_{t,r}$ via (10.14) makes sense. The uniform coercivity and (10.10)-(10.11) yield the existence of $C>0$ such that $\|v_{t,r}\|_{D^{1,2}(\tilde{\Omega}_{t,r})} \leq C$ for all t,r. Since $x_0 \notin \tilde{\Omega}_{t,r}$, (10.10)-(10.11) and regularity theory yield $v_{t,r} \in C^1(\overline{\tilde{\Omega}_{t,r}} \setminus \{0\})$ and for all $\rho>0$, there exists $C(\rho)>0$ independent of t and r such that

$$\|v_{t,r}\|_{C^1(\tilde{\Omega}_{t,r} \setminus (B_\rho(0) \cup B_\rho(x_0)))} \leq C(\rho). \tag{10.15}$$

Step 3.2: We claim that there exists $C>0$ such that

$$|v_{t,r}(x)| \leq C d(x, \partial\Omega_t)\left(|x|^{-\alpha_-(\gamma)} + |x-x_0|^{-\alpha_-(\gamma)}\right) \tag{10.16}$$

for all $t \in (-1,1)$ and all $x \in \tilde{\Omega}_{t,r}$. Indeed, around 0, $\tilde{\Omega}_{t,r}$ coincides with $\tilde{\Omega}_t$, and the proof of the control goes as in the construction of the mass in Section 8 (see the proof of Proposition 8.1). The argument is different around x_0. We let $r_0 > 0$ be such that $\tilde{\Omega}_t \cap B_{2r_0}(x_0) = \tilde{\Omega} \cap B_{2r_0}(x_0)$. Therefore, for $r \in (0, r_0)$, we have that

$$\tilde{\Omega}_{t,r} \cap B_{2r_0}(x_0) = (\tilde{\Omega} \setminus \tilde{D}_r) \cap B_{2r_0}(x_0).$$

Arguing as in the proof of Proposition 5.1, there exists $\tilde{u}_{\alpha_-} \in C^\infty(\overline{\tilde{\Omega}} \setminus \{0\})$ and $\tau' > 0$ such that

$$\begin{cases} \tilde{u}_{\alpha_-} & > 0 \quad \text{in } \tilde{\Omega} \cap B_{2r_0}(x_0) \\ \tilde{u}_{\alpha_-} & = 0 \quad \text{in } (\partial\tilde{\Omega}) \cap B_{2r_0}(x_0) \\ -\Delta\tilde{u}_{\alpha_-} - V\tilde{u}_{\alpha_-} & > 0 \quad \text{in } \tilde{\Omega} \cap B_{2r_0}(x_0). \end{cases}$$

Moreover, we have that

$$\tilde{u}_{\alpha_-}(x) = \frac{d(x, \partial\tilde{\Omega})}{|x - x_0|^{\alpha_-}}(1 + O(|x - x_0|)) \quad \text{as } x \to x_0,\, x \in \tilde{\Omega}. \tag{10.17}$$

Therefore, since $v_{t,r}$ vanishes on $B_{2r_0}(x_0) \cap \partial(\tilde{\Omega} \setminus \tilde{D}_r)$, it follows from (10.15) and the properties of $\tilde{u}_{\alpha_-}$ that there exists $C > 0$ such that

$$v_{t,r} \leq C\tilde{u}_{\alpha_-} \text{ on } \partial\left((\tilde{\Omega} \cap \tilde{D}_r) \cap B_{2r_0}(x_0)\right).$$

Since in addition $(-\Delta - V)v_{t,r} = 0 < (-\Delta - V)(C\tilde{u}_{\alpha_-})$, it follows from the comparison principle that $v_{t,r} \leq C\tilde{u}_{\alpha_-}$ in $(\tilde{\Omega} \setminus \tilde{D}_r) \cap B_{2r_0}(x_0)$. Arguing similarly with $-v_{t,r}$ and using the asymptotic (10.17), we get (10.16).

Step 3.3: We claim that

$$\lim_{t,r \to 0} v_{t,r} \circ \Phi_t = v_0 \text{ in } D^{1,2}(\tilde{\Omega})_{loc,\{x_0\}^c} \cap C^1_{loc}(\overline{\tilde{\Omega}} \setminus \{0, x_0\}), \tag{10.18}$$

where v_0 was defined in (10.13) and convergence in $D^{1,2}(\tilde{\Omega})_{loc,\{x_0\}^c}$ means that $\lim_{t,r\to 0} \eta v_{t,r} \circ \Phi_t = \eta v_0$ in $D^{1,2}(\tilde{\Omega})$ for all $\eta \in C^\infty(\mathbb{R}^n)$ vanishing around x_0. Indeed, $v_{t,r} \circ \Phi_t \in D^{1,2}(\tilde{\Omega} \setminus \tilde{D}_r) \subset D^{1,2}(\tilde{\Omega})$. Uniform coercivity yields weak convergence in $D^{1,2}(\tilde{\Omega})$ to $\tilde{v} \in D^{1,2}(\tilde{\Omega})$. Passing to the limit, one gets $(-\Delta - V)\tilde{v} = f_0$, so that $\tilde{v} = v_0$. Uniqueness then yields convergence in $C^1_{loc}(\overline{\tilde{\Omega}} \setminus \{0, x_0\})$. With a change of variable, equation (10.14) yields an elliptic equation for $v_{t,r} \circ \Phi_t$. Multiplying this equation by $\eta^2 \cdot (v_{t,r} \circ \Phi_t - v_0)$ for $\eta \in C^\infty(\mathbb{R}^n)$ vanishing around x_0, one gets convergence of $\eta v_{t,r} \circ \Phi_t$ to ηv_0 in $D^{1,2}(\tilde{\Omega})$. This proves the claim.

It follows from the construction of the mass (see Theorem 8.1) and the regularity Theorem 6.1 that there exists $K_0 \in \mathbb{R}$ and for all (t, r) small, there exists $K_{t,r} \in \mathbb{R}$ such that

$$v_{t,r}(x) = K_{t,r} \frac{d(x, \partial\tilde{\Omega}_t)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x, \partial\tilde{\Omega}_t)}{|x|^{\alpha_-(\gamma)}}\right) \tag{10.19}$$

and

$$v_0(x) = K_0 \frac{d(x, \partial\tilde{\Omega})}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x, \partial\tilde{\Omega})}{|x|^{\alpha_-(\gamma)}}\right) \tag{10.20}$$

as $x \in \tilde{\Omega}$ goes to 0. Note that around 0, $\tilde{\Omega}_{t,r}$ coincides with $\tilde{\Omega}_t$.

Step 3.4: We claim that

$$\lim_{t,r\to 0} K_{t,r} = K_0. \tag{10.21}$$

We only give a sketch. Noting $\tilde{v}_{t,r} := v_{t,r}\circ\Phi_t$, the proof relies on (10.18) and the fact that

$$-\Delta_{\Phi_t^\star \mathrm{Eucl}}\tilde{v}_{t,r} - V\circ\Phi_t\tilde{v}_{t,r} = f_t\circ\Phi_t \text{ in } \tilde{\Omega}\cap B_\delta(0).$$

The comparison principle and the definitions (10.19) and (10.20) then yield (10.21).

Step 4: Proof of Proposition 10.2. We define $\tilde{H}_0(x) := u_{\alpha_+(\gamma),0}(x) - v_0(x)$ for all $x\in\overline{\tilde{\Omega}}\setminus\{0,x_0\}$, and consider its Kelvin transform

$$H_0(x) := |x-x_0|^{2-n}\tilde{H}_0(i(x)) = |x-x_0|^{2-n}\left(u_{\alpha_+(\gamma),0} - v_0\right)(i(x)) \tag{10.22}$$

for all $x\in\Omega$. It follows from (10.3), the definitions of $u_{\alpha_+(\gamma),0}$ and v_0 that H_0 satisfies the following properties:

$$\begin{cases} -\Delta H_0 - \frac{\gamma}{|x|^2}H_0 & = 0 \quad \text{in } \Omega\\ H_0 & > 0 \quad \text{in } \Omega\\ H_0 & = 0 \quad \text{in } \partial\Omega\setminus\{0\}. \end{cases} \tag{10.23}$$

Concerning the pointwise behavior, we have that

$$H_0(x) = \frac{d(x,\partial\Omega)}{|x|^{\alpha_+}} - K_0\frac{d(x,\partial\Omega)}{|x|^{\alpha_-}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-}}\right) \tag{10.24}$$

as $x\to 0$, $x\in\Omega$, and

$$H_0(x)\leq C|x|^{1-\alpha_+} \text{ for all } x\in\Omega,\ |x|>1. \tag{10.25}$$

This proves the existence part in Proposition 10.2. We now deal with the uniqueness. We let $H\in C^2(\overline{\Omega}\setminus\{0\})$ be as in Proposition 10.2, and consider its Kelvin transform $\tilde{H}(x) := |x-x_0|^{2-n}H(i(x))$ for all $x\in\overline{\tilde{\Omega}}\setminus\{0,x_0\}$. The transformation law (10.3) yields

$$\begin{cases} -\Delta\tilde{H} - V\tilde{H} & = 0 \quad \text{in } \tilde{\Omega}\\ \tilde{H} & > 0 \quad \text{in } \tilde{\Omega}\\ \tilde{H} & = 0 \quad \text{in } \partial\tilde{\Omega}\setminus\{0,x_0\}. \end{cases} \tag{10.26}$$

Moreover, we have that $\tilde{H}(x)\leq C|x|^{1-\alpha_+(\gamma)} + C|x-x_0|^{1-\alpha_-(\gamma)}$ for all $x\in\tilde{\Omega}$. It then follows from Theorem 7.1 that there exist $C_1, C_2>0$ such that

$$\tilde{H}(x)\sim_{x\to 0} C_1\frac{d(x,\partial\tilde{\Omega})}{|x|^{\alpha}} \text{ and } \tilde{H}(x)\sim_{x\to x_0} C_2\frac{d(x,\partial\tilde{\Omega})}{|x-x_0|^{\alpha_-(\gamma)}}, \tag{10.27}$$

where $\alpha\in\{\alpha_-(\gamma),\alpha_+(\gamma)\}$. We claim that $\alpha=\alpha_+(\gamma)$. Indeed, otherwise, we would have $\tilde{H}\in D^{1,2}(\tilde{\Omega})$ (see Theorem 7.1) and then (10.26) and coercivity would yield $\tilde{H}\equiv 0$, which is a contradiction. Therefore $\alpha=\alpha_+(\gamma)$. By the same reasoning, the estimates (10.27) hold for $\tilde{H}_0$ (with different constants C_1, C_2). Arguing as in the proof of Theorem 8.1, we get that there exists $\lambda>0$ such that $\tilde{H}=\lambda\tilde{H}_0$, and therefore $H=\lambda H_0$. This proves uniqueness and completes the proof of Proposition 10.2.

As a consequence of (10.24), the mass $m_\gamma(\Omega)$ is well-defined and we have that

$$m_\gamma(\Omega) = -K_0. \tag{10.28}$$

Step 5: convergence of the mass: We claim that

$$\lim_{t\to 0, R\to\infty} m_\gamma(\Omega_{t,R}) = m_\gamma(\Omega). \tag{10.29}$$

We define $\tilde{H}_{t,r} := u_{\alpha_+,t} - v_{t,r}$ so that

$$-\Delta \tilde{H}_{t,r} - V\tilde{H}_{t,r} = 0 \text{ in } \tilde{\Omega}_{t,r}.$$

It follows from (10.7) and (10.19) that $\tilde{H}_{t,r} > 0$ around 0. From the maximum principle, we deduce that $\tilde{H}_{t,r} > 0$ on $\tilde{\Omega}_{t,r}$ and that it vanishes on $\partial\tilde{\Omega}_{t,r} \setminus \{0, x_0\}$.

It follows from (10.7) and (10.19) that

$$\tilde{H}_{t,r}(x) = \frac{d(x, \partial\tilde{\Omega}_{t,r})}{|x|^{\alpha_+}} - K_{t,r}\frac{d(x, \partial\tilde{\Omega}_{t,r})}{|x|^{\alpha_-}} + o\left(\frac{d(x, \partial\tilde{\Omega}_{t,r})}{|x|^{\alpha_-}}\right)$$

as $x \to 0$, $x \in \tilde{\Omega}_{t,r}$. Coming back to $\Omega_{t,R}$ with $R = r^{-1}$ via the inversion i with $H_{t,R}(x) := |x - x_0|^{2-n}\tilde{H}_{t,r}(i(x))$ for all $x \in \Omega_{t,R}$, we get that

$$\left\{\begin{array}{rll} -\Delta H_{t,R} - \frac{\gamma}{|x|^2}H_{t,R} & = 0 & \text{in } \Omega_{t,R} \\ H_{t,R} & > 0 & \text{in } \Omega_{t,R} \\ H_{t,R} & = 0 & \text{in } \partial\Omega_{t,R} \setminus \{0\} \end{array}\right.$$

and

$$H_{t,R}(x) = \frac{d(x, \partial\Omega_{t,R})}{|x|^{\alpha_+}} - K_{t,r}\frac{d(x, \partial\Omega_{t,R})}{|x|^{\alpha_-}} + o\left(\frac{d(x, \partial\Omega_{t,R})}{|x|^{\alpha_-}}\right)$$

as $x \to 0$, $x \in \Omega_{t,R}$. Therefore, it follows from the definition of the mass (see Theorem 8.1) that $m_\gamma(\Omega_{t,R}) = -K_{t,r}$ for all t, r, $R = r^{-1}$. Claim (10.29) then follows from (10.21) and (10.28).

This ends the proofs of Propositions 10.2 and 10.3.

Step 6: In order to prove Proposition 10.4, we need to exhibit prototypes of unbounded domains with either positive or negative mass.

Proposition 10.6. *Let Ω be a domain such that $0 \in \partial\Omega$ and Ω is smooth at infinity. Assume that $\gamma_H(\Omega) > \frac{n^2-1}{4}$ and fix $\gamma \in \left(\frac{n^2-1}{4}, \gamma_H(\Omega)\right)$. Then $m_\gamma(\Omega) > 0$ if $\mathbb{R}^n_+ \subsetneq \Omega$, and $m_\gamma(\Omega) < 0$ if $\Omega \subsetneq \mathbb{R}^n_+$.*

Proof of Proposition 10.6 : With H_0 defined as in (10.22), we set

$$\mathcal{U}(x) := H_0(x) - x_1|x|^{-\alpha_+} \text{ for all } x \in \Omega.$$

We first assume that $\mathbb{R}^n_+ \subsetneq \Omega$. We then have that

$$\left\{\begin{array}{ll} -\Delta\mathcal{U} - \frac{\gamma}{|x|^2}\mathcal{U} = 0 & \text{in } \mathbb{R}^n_+ \\ \mathcal{U} \gneqq 0 & \text{in } \partial\mathbb{R}^n_+ \setminus \{0\}. \end{array}\right. \tag{10.30}$$

We claim that

$$\int_{\mathbb{R}^n_+} |\nabla\mathcal{U}|^2\, dx < +\infty. \tag{10.31}$$

Indeed, at infinity, this is the consequence of the fact that $|\nabla\mathcal{U}|(x) \leq C|x|^{-\alpha_+}$ for all $x \in \mathbb{R}^n_+$ large, this latest bound being a consequence of (10.25) combined with elliptic regularity theory. At zero, the argument is different. Indeed, one first notes that $d(x, \partial\Omega') = x_1 + O(|x|^2)$ for $x \in \mathbb{R}^n_+$ close to 0, and therefore, $\mathcal{U}(x) = O(|x|^{1-\alpha_-})$ for $x \to 0$. The control on the gradient $|\nabla\mathcal{U}|(x) \leq C|x|^{-\alpha_-}$

at 0 follows from the construction of $\tilde{H}_0$. This yields integrability at 0 and proves (10.31).

We claim that $\mathcal{U} > 0$ in $\mathbb{R}^n_+$. Indeed, it follows from (10.30) and (10.31) that $\mathcal{U}_- \in D^{1,2}(\mathbb{R}^n_+)$. Multiplying equation (10.23) by $\mathcal{U}_-$, integrating by parts on $(B_R(0) \setminus B_\epsilon(0)) \cap \mathbb{R}^n_+$, and letting $\epsilon \to 0$ and $R \to +\infty$ by using (10.31), one gets $\mathcal{U}_- \equiv 0$, and then $\mathcal{U} \geq 0$. The result follows from Hopf's maximum principle.

We now claim that

$$m_\gamma(\Omega) > 0. \tag{10.32}$$

Indeed, since $\mathcal{U} > 0$ in $\mathbb{R}^n_+$, there exists $c_0 > 0$ such that $\mathcal{U}(x) \geq c_0 x_1 |x|^{-\alpha_-}$ for all $x \in \partial(B_1(0)_+)$. It then follows from (10.31), (10.30) and the comparison principle that $\mathcal{U}(x) \geq c_0 x_1 |x|^{-\alpha_-}$ for all $x \in B_1(0)_+$. The expansion (10.24) then yields $-K_0 \geq c_0 > 0$. This combined with (10.28) proves the claim.

When $\Omega \subset \mathbb{R}^n_+$, the argument is similar except that one works on Ω (and not $\mathbb{R}^n_+$) and that $\mathcal{U} \lneqq 0$ in $\partial\Omega \setminus \{0\}$. This ends the proof of Proposition 10.6.

Step 7: Proof of Proposition 10.4: Let ω be a smooth domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. Up to a rotation, there exists $\varphi \in C^\infty(\mathbb{R}^{n-1})$ such that $\varphi(0) = 0$, $\nabla\varphi(0) = 0$ and there exists $\delta_0 > 0$ such that

$$\omega \cap B_{\delta_0}(0) = \{x_1 > \varphi(x') / \, (x_1, x') \in B_{\delta_0}(0)\}.$$

Let $\eta \in C^\infty_c(B_{\delta_0}(0))$ be such that $\eta(x) = 1$ for all $x \in B_{\delta_0/2}(0)$, and define

$$\Phi_t(x) := \left(x_1 + \eta(x)\frac{\varphi(tx')}{t}, x'\right) \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^n,$$

and $\Phi_0 := Id_{\mathbb{R}^n}$. It is easy to see that Φ_t satisfies the hypotheses of Proposition 10.3. Moreover, for $0 < t < 1$, we have that

$$\frac{\omega}{t} \cap \Phi_t(B_{\delta_0/2}(0)) = \Phi_t(\mathbb{R}^n_+ \cap B_{\delta_0/2}(0)).$$

We let Ω be a smooth domain at infinity such that

$$\Omega \cap B_1(0) = \mathbb{R}^n_+ \cap B_1(0) \text{ and } \gamma_H(\Omega) > \frac{n^2-1}{4}. \tag{10.33}$$

(for example, $\mathbb{R}^n_+$), and let $\Omega_{t,R}$ be as in Proposition 10.3. It is easy to see that

$$\omega \cap t\Phi_t(B_{\delta_0/2}(0)) = t\Omega_{t,R} \cap t\Phi_t(B_{\delta_0/2}(0)).$$

Therefore, for $t > 0$ small enough, we have that

$$\omega \cap B_{t\delta_0/3}(0) = t\Omega_{t,R} \cap B_{t\delta_0/3}(0).$$

Moreover, $\gamma_H(t\Omega_{t,R}) = \gamma_H(\Omega_{t,R}) > (n^2-1)/4$ as $t \to 0$ and $R \to +\infty$ (see (10.2)). Concerning the mass, we have that

$$t^{\alpha_+(\gamma)-\alpha_-(\gamma)} m_\gamma(t\Omega_{t,R}) = m_\gamma(\Omega_{t,R}) \to m_\gamma(\Omega) \text{ as } t \to 0, R \to +\infty.$$

We now choose Ω appropriately.
To get a negative mass, we choose Ω smooth at infinity such that $\Omega \cap B_1(0) = \mathbb{R}^n_+ \cap B_1(0)$ and $\Omega \subsetneq \mathbb{R}^n_+$. Then $\gamma_H(\Omega) = n^2/4$, (10.33) holds and Proposition 10.6 yields $m_\gamma(\Omega) < 0$. With this choice of Ω, we take $\Omega_- := \Omega_{t,R}$ for t small and R large.

To get a positive mass, we choose $\mathbb{R}^n_+ \subsetneq \Omega$ such that (10.33) holds (this is possible for any value of $\gamma_H(\Omega)$ arbitrarily close to $\frac{n^2}{4}$, see point (5) of Proposition 3.1). Then

Proposition 10.6 yields $m_\gamma(\Omega) > 0$. With this choice of Ω, we take $\Omega_+ := \Omega_{t,R}$ for t small and R large.

This proves Proposition 10.4.

11. The remaining cases corresponding to $s = 0$ and $n = 3$

The remaining situation not covered by Proposition 9.1 and Theorem 9.2 is $s = 0$, $n = 3$ and $\gamma \in (0, \frac{n^2}{4})$. Note first, that if $\gamma \geq \gamma_H(\Omega)$, we have from Proposition 4.1 and Theorem 4.4 that $\mu_{\gamma,0}(\Omega) \leq 0 < \mu_{\gamma,0}(\mathbb{R}^n_+)$ and the existence of extremals is guaranteed. Another situation is when $\mu_{\gamma,0}(\mathbb{R}^n_+)$ does have an extremal U. In this case, Proposition 9.3 provides sufficient conditions for $\mu_{\gamma,0}(\Omega) < \mu_{\gamma,0}(\mathbb{R}^n_+)$, and hence there are extremals by again using Theorem 4.4. The rest of this section addresses the remaining case, that is when $\gamma \in (0, \gamma_H(\Omega))$ and when $\mu_{\gamma,0}(\mathbb{R}^n_+)$ has no extremal, and therefore $\mu_{\gamma,0}(\mathbb{R}^3_+) = K(3,2)^{-2}$ according to Theorem 12.1.

We first define the "interior" mass in the spirit of Schoen-Yau [49].

Proposition 11.1. *Let $\Omega \subset \mathbb{R}^3$ be an open smooth bounded domain such that $0 \in \partial\Omega$. Fix $x_0 \in \Omega$. If $\gamma \in (0, \gamma_H(\Omega))$, then the equation*

$$\begin{cases} -\Delta G - \frac{\gamma}{|x|^2} G = 0 & \text{in } \Omega \setminus \{x_0\} \\ G > 0 & \text{in } \Omega \setminus \{x_0\} \\ G = 0 & \text{on } \partial\Omega \setminus \{0\} \end{cases}$$

has a solution $G \in C^2(\overline{\Omega} \setminus \{0, x_0\}) \cap D^2_1(\Omega \setminus \{x_0\})_{loc,0}$, that is unique up to multiplication by a constant. Moreover, for any $x_0 \in \Omega$, there exists a unique $R_\gamma(x_0) \in \mathbb{R}$ independent of the choice of G and $c_G > 0$ such that

$$G(x) = c_G \left(\frac{1}{|x - x_0|} + R_\gamma(x_0) \right) + o(1) \text{ as } x \to x_0.$$

Proof of Proposition 11.1. Since $\gamma < \gamma_H(\Omega)$, the operator $-\Delta - \gamma|x|^{-2}$ is coercive and we can consider G to be its Green's function at x_0 on Ω with Dirichlet boundary condition. In particular, for any $\varphi \in C^\infty_c(\Omega)$, we have that

$$\varphi(x) = \int_\Omega G_x(y) \left(-\Delta\varphi(y) - \gamma \frac{\varphi(y)}{|y|^2} \right) dy \quad \text{for } x \in \Omega,$$

where $G_x := G(x, \cdot)$. Fix $x_0 \in \Omega$ and let $\eta \in C^\infty_c(\Omega)$ be such that $\eta(x) = 1$ around x_0. Define the distribution $\beta_{x_0} : \Omega \to \mathbb{R}$ as

$$G_{x_0}(x) = \frac{1}{\omega_2} \left(\frac{\eta(x)}{|x - x_0|} + \beta_{x_0}(x) \right) \quad \text{for all } x \in \Omega,$$

where $\omega_2 := 4\pi$ is the volume of the canonical $2-$sphere. Set

$$f(x) := -\left(-\Delta - \frac{\gamma}{|x|^2} \right) \left(\frac{\eta(x)}{|x - x_0|} \right) \quad \text{for all } x \neq x_0.$$

In particular,

$$\left(-\Delta - \frac{\gamma}{|x|^2} \right) \beta_{x_0} = f \text{ in the distributional sense.}$$

On can easily see that there exists $C > 0$ such that

$$|f(x)| \leq C|x - x_0|^{-1} \quad \text{for all } x \in \Omega.$$

Therefore $f \in L^2(\Omega)$ and, by uniqueness of the Green's function (since the operator is coercive), we have that $\beta_{x_0} \in D^{1,2}(\Omega)$. It follows from standard elliptic theory that $\beta_{x_0} \in C^\infty(\overline{\Omega}\setminus\{0, x_0\}) \cap C^{0,\theta}(\overline{\Omega}\setminus B_\delta(0))$ for all $\theta \in (0,1)$ and $\delta > 0$. In addition, for any $\theta \in (0,1)$ and $\delta > 0$, there exists $C_\theta > 0$ such that

$$|\nabla \beta_{x_0}(x)| \leq C_\theta |x|^{\theta-1} \quad \text{for all } x \in \Omega \setminus B_\delta(0). \tag{11.1}$$

Since f vanishes around 0, it follows from Theorem 6.1 and Lemma 6.5 that

$$\beta_{x_0}(x) = O(|x|^{1-\alpha_-(\gamma)}) \quad \text{and} \quad |\nabla \beta_{x_0}(x)| = O(|x|^{-\alpha_-(\gamma)}) \quad \text{when } x \to 0. \tag{11.2}$$

We can therefore define the *mass of* Ω at x_0 associated to the operator L_γ by

$$R_\gamma(\Omega, x_0) := \beta_{x_0}(x_0).$$

One can easily check that this quantity is independent of the choice of η.

The uniqueness is proved as in Theorem 8.1. The behavior on the boundary is given by Theorem 6.1 and the interior behavior around x_0 is classical. This ends the proof of Proposition 11.1. □

Lemma 11.2. *Let $\Omega \subset \mathbb{R}^3$ be an open smooth bounded domain such that $0 \in \partial\Omega$ and $x_0 \in \Omega$. Assume that $\gamma \in (0, \gamma_H(\Omega))$ and that $\mu_{\gamma,0}(\mathbb{R}^3_+) = K(3,2)^{-2}$. Then, there exists a family $(u_\epsilon)_\epsilon$ in $D^{1,2}(\Omega)$ such that*

$$J^\Omega_{\gamma,0}(u_\varepsilon) = \tfrac{1}{K(n,2)^2}\left(1 - \tfrac{\omega_2 R_\gamma(x_0)}{3\int_{\mathbb{R}^3} U^{2^\star}\,dx}\varepsilon + o(\varepsilon)\right) \text{ as } \varepsilon \to 0, \tag{11.3}$$

where $U(x) := (1+|x|^2)^{-1/2}$ for all $x \in \mathbb{R}^3$ and $2^\star = 2^\star(0) = \frac{2n}{n-2}$.

Proof of Lemma 11.2: We proceed as in Schoen [48] (see Druet [15, 16] and Jaber [34]). The computations are similar to the case $\gamma > \frac{n^2-1}{4}$ performed in Section 9. For $\varepsilon > 0$, define the functions

$$u_\varepsilon(x) := \eta(x)\left(\tfrac{\varepsilon}{\varepsilon^2 + |x-x_0|^2}\right)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}\beta_{x_0}(x) \quad \text{for all } x \in \Omega.$$

One can easily check that $u_\varepsilon \in D^{1,2}(\Omega)$. We now estimate $J^\Omega_{\gamma,0}(u_\varepsilon)$.

In the sequel, $\Theta_c(\varepsilon)$ will denote any quantity such that

$$\lim_{c\to 0}\lim_{\varepsilon\to 0}\frac{\Theta_c(\varepsilon)}{\varepsilon} = 0.$$

We first claim that

$$\int_{\Omega\setminus B_c(x_0)}\left(|\nabla u_\varepsilon|^2 - \gamma\frac{u_\varepsilon^2}{|x|^2}\right)dx = \omega_2 c^{-1}\varepsilon + \omega_2 R_\gamma(x_0)\varepsilon + \Theta_c(\varepsilon). \tag{11.4}$$

Indeed, it is clear that $\varepsilon^{-\frac{1}{2}}u_\varepsilon \to G'_{x_0} := \omega_2 G_{x_0}$ in $C^2_{loc}(\overline{\Omega}\setminus\{0, x_0\})$. Therefore, Lebesgue's dominated convergence theorem yields

$$\lim_{\varepsilon\to 0}\frac{\int_{\Omega\setminus B_c(x_0)}\left(|\nabla u_\varepsilon|^2 - \gamma\frac{u_\varepsilon^2}{|x|^2}\right)dx}{\varepsilon} = \int_{\Omega\setminus B_c(x_0)}\left(|\nabla G'_{x_0}|^2 - \gamma\frac{(G'_{x_0})^2}{|x|^2}\right)dx. \tag{11.5}$$

Integrating by parts and using (11.2), as $\delta > 0$ goes to 0, we have that

$$\begin{aligned}
&\int_{\Omega\setminus B_c(x_0)} \left(|\nabla G'_{x_0}|^2 - \gamma\frac{(G'_{x_0})^2}{|x|^2}\right) dx \\
&= \int_{\Omega\setminus(B_c(x_0)\cup B_\delta(0))} \left(|\nabla G'_{x_0}|^2 - \gamma\frac{(G'_{x_0})^2}{|x|^2}\right) dx + o(1) \\
&= \int_{\Omega\setminus(B_c(x_0)\cup B_\delta(0))} \left(-\Delta G'_{x_0} - \gamma\frac{G'_{x_0}}{|x|^2}\right) G'_{x_0}\, dx - \int_{\partial B_c(x_0)} G'_{x_0}\partial_\nu G'_{x_0}\, d\sigma \\
&- \int_{\partial B_\delta(0)} G'_{x_0}\partial_\nu G'_{x_0}\, d\sigma + o(1) \\
&= -\int_{\partial B_c(x_0)} G'_{x_0}\partial_\nu G'_{x_0}\, d\sigma + O(\delta^{n-1}\delta^{1-\alpha_-(\gamma)}\delta^{-\alpha_-(\gamma)}) + o(1) \quad \text{as } \delta \to 0.
\end{aligned}$$

Since $\alpha_-(\gamma) < n/2$, we then have that

$$\int_{\Omega\setminus B_c(x_0)} \left(|\nabla G'_{x_0}|^2 - \gamma\frac{(G'_{x_0})^2}{|x|^2}\right) dx = -\int_{\partial B_c(x_0)} G'_{x_0}\partial_\nu G'_{x_0}\, d\sigma.$$

With the definition of $R_\gamma(x_0)$ and (11.1), we have that $G'_{x_0} = c^{-1} + R_\gamma(x_0) + O(c^\theta)$ and $\partial_\nu G'_{x_0}(x) = -c^{-2} + O(c^{\theta-1})$ on $\partial B_c(x_0)$ as $c \to 0$. Therefore

$$-\textstyle\int_{\partial B_c(x_0)} G'_{x_0}\partial_\nu G'_{x_0}\, d\sigma = \omega_2 c^{-1} + \omega_2 R_\gamma(x_0) + O(c^\theta) \quad \text{as } c \to 0.$$

Combined with (11.5), this proves (11.4).
Now define for each $\varepsilon > 0$, the function

$$U_\varepsilon(x) := \left(\tfrac{\varepsilon}{\varepsilon^2+|x-x_0|^2}\right)^{\frac{1}{2}} \quad \text{for all } x \in \mathbb{R}^3,$$

and set $U(x) := (1+|x|^2)^{-1/2}$ for all $x \in \mathbb{R}^3$. It is clear that $\Delta U = 3U^{2^\star-1}$. We claim that

$$\int_{B_c(x_0)} \left(|\nabla u_\varepsilon|^2 - \gamma\frac{u_\varepsilon^2}{|x|^2}\right) dx = 3\int_{\mathbb{R}^3} U^{2^\star}\, dx - \omega_2 c^{-1}\varepsilon + \Theta_c(\varepsilon). \tag{11.6}$$

Indeed, note first $|u_\varepsilon(x)| \leq C\sqrt{\varepsilon}|x-x_0|^{-1}$ for all $\varepsilon > 0$ and all $x \in \Omega$ close to x_0. Therefore, for $c > 0$ small enough, we have that

$$\int_{B_c(x_0)} \frac{u_\varepsilon^2}{|x|^2}\, dx = \Theta_c(\varepsilon). \tag{11.7}$$

Using that $\beta_{x_0}\in D^{1,2}(\Omega)$ and integrating by parts, we get that

$$\begin{aligned}\int_{B_c(x_0)}|\nabla u_\varepsilon|^2\,dx &= \int_{B_c(x_0)}|\nabla(U_\varepsilon+\sqrt{\varepsilon}\beta_{x_0})|^2\,dx\\ &= \int_{B_c(x_0)}|\nabla U_\varepsilon|^2\,dx+2\sqrt{\varepsilon}\int_{B_c(x_0)}\nabla U_\varepsilon\nabla\beta_{x_0}\,dx\\ &\quad +\varepsilon\int_{B_c(x_0)}|\nabla\beta_{x_0}|^2\,dx\\ &= \int_{B_c(x_0)}U_\varepsilon(-\Delta U_\varepsilon)\,dx+\int_{\partial B_c(x_0)}U_\varepsilon\partial U_\varepsilon\,d\sigma\\ &\quad +2\sqrt{\varepsilon}\int_{B_c(x_0)}\beta_{x_0}(-\Delta U_\varepsilon)\,dx\\ &\quad +2\sqrt{\varepsilon}\int_{\partial B_c(x_0)}\beta_{x_0}\partial_\nu U_\varepsilon\,d\sigma+\Theta_c(\varepsilon)\end{aligned}\tag{11.8}$$

Since $\varepsilon^{-1/2}U_\varepsilon\to|\cdot-x_0|^{-1}$ in $C^1_{loc}(\mathbb{R}^3\setminus\{0\})$, we get that

$$\int_{\partial B_c(x_0)}U_\varepsilon\partial_\nu U_\varepsilon\,d\sigma=-\omega_2c^{-1}\varepsilon+o(\varepsilon)\quad\text{as }\varepsilon\to0.\tag{11.9}$$

Using in addition that $\beta_{x_0}\in C^{0,\theta}$ around x_0, we get as $\varepsilon\to0$ and for $c>0$ small, that

$$\int_{\partial B_c(x_0)}\beta_{x_0}\partial_\nu U_\varepsilon\,d\sigma=-\sqrt{\varepsilon}\omega_2R_\gamma(x_0)+O(c^\theta\sqrt{\varepsilon})\tag{11.10}$$

Plugging (11.9) and (11.10) into (11.8) yields

$$\begin{aligned}\int_{B_c(x_0)}|\nabla u_\varepsilon|^2\,dx &= 3\int_{B_{c/\varepsilon}(0)}U^{2^\star}\,dx-\omega_2c^{-1}\varepsilon+o(\varepsilon)\\ &\quad +2\sqrt{\varepsilon}\int_{B_c(x_0)}\beta_{x_0}(-\Delta U_\varepsilon)\,dx-2\varepsilon\omega_2R_\gamma(x_0)+\Theta_c(\varepsilon).\end{aligned}\tag{11.11}$$

It is easy to check that $\int_{B_{c/\varepsilon}(0)}U^{2^\star}\,dx=\int_{\mathbb{R}^3}U^{2^\star}\,dx+o(\varepsilon)$ as $\varepsilon\to0$. For $\theta\in(1/2,1)$ we have that

$$\int_{B_c(x_0)}|\Delta U_\varepsilon|\cdot|x-x_0|^\theta\,dx=o(\varepsilon)\quad\text{as }\varepsilon\to0.\tag{11.12}$$

Integrating by parts and using that $\varepsilon^{-1/2}U_\varepsilon(x)\to|x-x_0|^{-1}$ in $C^1_{loc}(\mathbb{R}^3\setminus\{0\})$, we get that as $\varepsilon\to0$,

$$\begin{aligned}\int_{B_c(x_0)}-\Delta U_\varepsilon\,dx &= -\int_{\partial B_c(x_0)}\partial_\nu U_\varepsilon\,d\sigma\\ &= -\sqrt{\varepsilon}\int_{\partial B_c(x_0)}\partial_\nu|x-x_0|^{-1}\,d\sigma+o(\varepsilon)\\ &= \omega_2\sqrt{\varepsilon}+o(\varepsilon)\end{aligned}\tag{11.13}$$

Plugging (11.12) and (11.13) into (11.11) and using that $\beta_{x_0}(x)=R_\gamma(x_0)+O(|x-x_0|^\theta)$, we get (11.6).

Putting together (11.4) and (11.5) yields

$$\int_\Omega\left(|\nabla u_\varepsilon|^2-\gamma\frac{u_\varepsilon^2}{|x|^2}\right)dx=3\int_{\mathbb{R}^3}U^{2^\star}\,dx+\omega_2R_\gamma(x_0)\varepsilon+o(\varepsilon)\quad\text{as }\varepsilon\to0.\tag{11.14}$$

We now claim that

$$\int_\Omega u_\varepsilon^{2^\star}\,dx = \int_{\mathbb{R}^3} U^{2^\star}\,dx + \tfrac{2^\star}{3}\omega_2 R_\gamma(x_0)\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{11.15}$$

Using (9.56), the boundedness of β_{x_0} around x_0, and the above computations, we get that

$$\begin{aligned}
\int_\Omega u_\varepsilon^{2^\star}\,dx &= \int_{B_c(x_0)} u_\varepsilon^{2^\star}\,dx + o(\varepsilon)\\
&= \int_{B_c(x_0)} |U_\varepsilon + \sqrt{\varepsilon}\beta_{x_0}|^{2^\star}\,dx + o(\varepsilon)\\
&= \int_{B_c(x_0)} U_\varepsilon^{2^\star}\,dx + 2^\star\sqrt{\varepsilon}\int_{B_c(x_0)} \beta_{x_0} U_\varepsilon^{2^\star-1}\,dx\\
&\quad + O\left(\int_\Omega \left(\varepsilon U_\varepsilon^{2^\star-2}\beta_{x_0}^2 + |\sqrt{\varepsilon}\beta_{x_0}|^{2^\star}\right)dx\right) + o(\varepsilon)\\
&= \int_{B_{c/\varepsilon}(0)} U^{2^\star}\,dx + \frac{2^\star}{3}\omega_2 R_\gamma(x_0)\varepsilon + o(\varepsilon)\\
&= \int_{\mathbb{R}^3} U^{2^\star}\,dx + \frac{2^\star}{3}\omega_2 R_\gamma(x_0)\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

which proves (11.15).

Putting together (11.14) and (11.15) yields

$$\frac{\int_\Omega \left(|\nabla u_\varepsilon|^2 - \gamma\frac{u_\varepsilon^2}{|x|^2}\right)dx}{\left(\int_\Omega u_\varepsilon^{2^\star}\,dx\right)^{\frac{2}{2^\star}}} = \frac{3\int_{\mathbb{R}^3} U^{2^\star}\,dx}{\left(\int_{\mathbb{R}^3} U^{2^\star}\,dx\right)^{\frac{2}{2^\star}}}\left(1 - \frac{\omega_2 R_\gamma(x_0)}{3\int_{\mathbb{R}^3} U^{2^\star}\,dx}\varepsilon + o(\varepsilon)\right) \tag{11.16}$$

as $\varepsilon \to 0$. Since $\Delta U = 3U^{2^\star-1}$ and U is an extremal for the Sobolev inequality $\mu_{0,0}(\mathbb{R}^3)$, we have that

$$J_{\gamma,0}^\Omega(u_\varepsilon) = \tfrac{1}{K(n,2)^2}\left(1 - \tfrac{\omega_2 R_\gamma(x_0)}{3\int_{\mathbb{R}^3} U^{2^\star}\,dx}\varepsilon + o(\varepsilon)\right) \quad \text{as } \varepsilon \to 0.$$

This proves Lemma 11.2. □

We finally get the following.

Theorem 11.3. *Let Ω be a bounded smooth domain of $\mathbb{R}^3$ such that $0 \in \partial\Omega$.*

1. *If $\gamma \ge \gamma_H(\Omega)$, then there are extremals for $\mu_{\gamma,0}(\Omega)$.*
2. *If $\gamma \le 0$, then there are no extremals for $\mu_{\gamma,0}(\Omega)$.*
3. *If $0 < \gamma < \gamma_H(\Omega)$ and there are extremals for $\mu_{\gamma,0}(\mathbb{R}^n_+)$, then there are extremals for $\mu_{\gamma,0}(\Omega)$ under either one of the following conditions:*
   - *$\gamma \le \frac{n^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
   - *$\gamma > \frac{n^2-1}{4}$ and the mass $m_\gamma(\Omega)$ is positive.*
4. *If $0 < \gamma < \gamma_H(\Omega)$ and there are no extremals for $\mu_{\gamma,0}(\mathbb{R}^n_+)$, then there are extremals for $\mu_{\gamma,0}(\Omega)$ if there exists $x_0 \in \Omega$ such that $R_\gamma(\Omega, x_0) > 0$.*

Proof of Theorem 11.3: The two first points of the theorem follow from Proposition 9.1 and Theorem 4.4. The third point follows from Proposition 9.3. For the fourth point, in this situation, it follows from Theorem 12.1 below that $\mu_{\gamma,0}(\mathbb{R}^n_+) = \frac{1}{K(n,2)^2}$, and then Lemma 11.2 yields $\mu_{\gamma,0}(\Omega) < \mu_{\gamma,0}(\mathbb{R}^n_+)$, which yields the existence of extremals by Theorem 4.4. This proves Theorem 11.3. □

12. Appendix 1: Existence of extremals for $\mu_{\gamma,s}(\mathbb{R}^n_+)$ and other cones

The following result is used frequently throughout this memoir in the case of $\mathbb{R}^n_+$. In this appendix we give proofs for any open connected cone of $\mathbb{R}^n$, $n \geq 3$, centered at 0, that is

$$\left\{ \begin{array}{c} \mathcal{C} \text{ is a domain (that is open and connected)} \\ \forall x \in \mathcal{C},\, \forall r > 0,\ rx \in \mathcal{C}. \end{array} \right. \tag{12.1}$$

Fix $\gamma < \gamma_H(\mathcal{C})$, then by the Hardy-Sobolev inequality, there exists $\mu_{\gamma,s}(\mathcal{C}) > 0$ such that

$$\mu_{\gamma,s}(\mathcal{C}) := \inf_{u \in D^{1,2}(\mathcal{C})\setminus\{0\}} \frac{\int_{\mathcal{C}} \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx}{\left(\int_{\mathcal{C}} \frac{|u|^{2^\star(s)}}{|x|^s}\, dx \right)^{\frac{2}{2^\star(s)}}}. \tag{12.2}$$

We consider the question of whether there is an extremal $u_0 \in D^{1,2}(\mathcal{C}) \setminus \{0\}$ for $\mu_{\gamma,s}(\mathcal{C})$, that is if the latter achieves its infimum in (12.2). The question of the extremals on general cones has been tackled by Egnell [17] in the case $\{\gamma = 0 \text{ and } s > 0\}$. Theorem 12.1 below has been noted in several contexts by Bartsch-Peng-Zhang [3] and Lin-Wang [10]. We sketch an independent proof for the convenience of the reader.

Theorem 12.1. *We let $\mathcal{C}$ be a cone of $\mathbb{R}^n$, $n \geq 3$, as in* (12.1)*, $s \in [0, 2)$ and $\gamma < \gamma_H(\mathcal{C})$. Then,*

1. *If $\{s > 0\}$ or $\{s = 0,\ \gamma > 0$ and $n \geq 4\}$, then extremals for $\mu_{\gamma,s}(\mathcal{C})$ exist.*
2. *If $\{s = 0$ and $\gamma < 0\}$, there are no extremals for $\mu_{\gamma,0}(\mathcal{C})$.*
3. *If $\{s = 0$ and $\gamma = 0\}$, there are extremals for $\mu_{0,0}(\mathcal{C})$ if and only if there exists $z \in \mathbb{R}^n$ such that $(1+|x-z|^2)^{1-n/2} \in D^{1,2}(\mathcal{C})$ (in particular $\overline{\mathcal{C}} = \mathbb{R}^n$).*

Moreover, if there are no extremals for $\mu_{\gamma,0}(\mathcal{C})$, then $\mu_{\gamma,0}(\mathcal{C}) = \mu_{0,0}(\mathcal{C})$, that is

$$\mu_{\gamma,0}(\mathcal{C}) = \frac{1}{K(n,2)^2} := \inf_{u \in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\star}\, dx \right)^{\frac{2}{2^\star}}}. \tag{12.3}$$

Remark: Note that the case when $\{s = 0,\ n = 3 \text{ and } \gamma > 0\}$ remains unsettled. We isolate two corollaries. The first one is essentially what we need in our context

$(\mathcal{C} = \mathbb{R}^n_+)$. The second deals with the case $\mathcal{C} = \mathbb{R}^n$. There is no issue for $n = 3$ in the second corollary.

Corollary 12.2. *We let $\mathcal{C}$ be a cone of $\mathbb{R}^n$, $n \geq 3$, as in* (12.1) *such that $\overline{\mathcal{C}} \neq \mathbb{R}^n$. We let $s \in [0, 2)$ and $\gamma < \gamma_H(\mathcal{C})$. Then,*

1. *If $\{s > 0\}$ or $\{s = 0,\ \gamma > 0$ and $n \geq 4\}$, then there are extremals for $\mu_{\gamma,s}(\mathcal{C})$.*
2. *If $\{s = 0$ and $\gamma \leq 0\}$, there are no extremals for $\mu_{\gamma,0}(\mathcal{C})$.*

Corollary 12.3. *We let $\mathcal{C}$ be a cone of $\mathbb{R}^n$, $n \geq 3$, as in* (12.1)*. We assume that there exists $z \in \mathbb{R}^n$ such that $(1 + |x - z|^2)^{1-n/2} \in D^{1,2}(\mathcal{C})$ (in particular $\overline{\mathcal{C}} = \mathbb{R}^n$). We fix $s \in [0, 2)$ and $\gamma < \gamma_H(\mathcal{C})$. Then,*

1. *If $\{s > 0\}$ or $\{s = 0$ and $\gamma \geq 0\}$, then there are extremals for $\mu_{\gamma,s}(\mathcal{C})$.*
2. *If $\{s = 0$ and $\gamma < 0\}$, there are no extremals for $\mu_{\gamma,0}(\mathcal{C})$.*

Proof of Theorem 12.1: This goes as the classical proof of the existence of extremals for the Sobolev inequalities using Lions's concentration-compactness Lemmas ([40, 41], see also Struwe [51] for an exposition in book form).

We let $(\tilde{u}_k)_k \in D^{1,2}(\mathbb{R}^n_+)$ be a minimizing sequence for $\mu_{\gamma,s}(\mathcal{C})$ such that

$$\int_{\mathcal{C}} \frac{|\tilde{u}_k|^{2^\star(s)}}{|x|^s}\, dx = 1 \text{ and } \lim_{k\to+\infty} \int_{\mathcal{C}} \left(|\nabla \tilde{u}_k|^2 - \frac{\gamma}{|x|^2}\tilde{u}_k^2 \right) dx = \mu_{\gamma,s}(\mathcal{C}).$$

We use a concentration compactness argument in the spirit of Lions [40, 41]. For any k, there exists $r_k > 0$ such that $\int_{B_{r_k}(0)\cap\mathcal{C}} \frac{|\tilde{u}_k|^{2^\star(s)}}{|x|^s}\, dx = 1/2$. We define $u_k(x) := r_k^{\frac{n-2}{2}} \tilde{u}_k(r_k x)$ for all $x \in \mathcal{C}$. Since $\mathcal{C}$ is a cone, we have that $u_k \in D^{1,2}(\mathcal{C})$. We then have that

$$\lim_{k\to+\infty} \int_{\mathcal{C}} \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx = \mu_{\gamma,s}(\mathcal{C}), \tag{12.4}$$

and

$$\int_{\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 1\ ,\ \int_{B_1(0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = \frac{1}{2}. \tag{12.5}$$

We first claim that, up to a subsequence,

$$\lim_{R\to+\infty} \lim_{k\to+\infty} \int_{B_R(0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 1. \tag{12.6}$$

Indeed, for $k \in \mathbb{N}$ and $r \geq 0$, we define

$$Q_k(r) := \int_{B_r(0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx.$$

Since $0 \leq Q_k \leq 1$ and $r \mapsto Q_k(r)$ is nondecreasing for all $k \in \mathbb{N}$, then, up to a subsequence, there exists $Q : [0, +\infty) \to \mathbb{R}$ nondecreasing such that $Q_k(r) \to Q(r)$ as $k \to +\infty$ for a.e. $r > 0$. We define

$$\alpha := \lim_{r\to+\infty} Q(r).$$

It follows from (12.4) and (12.5) that $\frac{1}{2} \leq \alpha \leq 1$. Up to taking another subsequence, there exist $(R_k)_k, (R'_k)_k \in (0, +\infty)$ such that

$$\begin{cases} 2R_k \leq R'_k \leq 3R_k \text{ for all } k \in \mathbb{N}, \\ \lim_{k\to+\infty} R_k = \lim_{k\to+\infty} R'_k = +\infty, \\ \lim_{k\to+\infty} Q_k(R_k) = \lim_{k\to+\infty} Q_k(R'_k) = \alpha. \end{cases}$$

In particular,

$$\lim_{k\to+\infty} \int_{B_{R_k}(0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = \alpha \text{ and } \lim_{k\to+\infty} \int_{(\mathbb{R}^n\setminus B_{R'_k}(0))\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 1-\alpha. \tag{12.7}$$

We claim that

$$\lim_{k\to+\infty} R_k^{-2} \int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} u_k^2\, dx = 0. \tag{12.8}$$

Indeed, for all $x \in B_{R'_k}(0) \setminus B_{R_k}(0)$, we have that $R_k \le |x| \le 3R_k$. Therefore, Hölder's inequality yields

$$\begin{aligned}\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} u_k^2\,dx &\le \left(\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} dx\right)^{1-\frac{2}{2^\star(s)}}\left(\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} |u_k|^{2^\star(s)}\,dx\right)^{\frac{2}{2^\star(s)}}\\ &\le CR_k^2\left(\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}\end{aligned}$$

for all $k \in \mathbb{N}$. The conclusion (12.8) then follows from (12.7).

We now let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi(x) = 1$ for $x \in B_1(0)$ and $\varphi(x) = 0$ for $x \in \mathbb{R}^n \setminus B_2(0)$. For $k \in \mathbb{N}$, we define

$$\varphi_k(x) := \varphi\left(\frac{|x|}{R'_k - R_k} + \frac{R'_k - 2R_k}{R'_k - R_k}\right) \text{ for all } x \in \mathbb{R}^n.$$

One can easily check that $\varphi_k u_k, (1-\varphi_k)u_k \in D^{1,2}(\mathcal{C})$ for all $k \in \mathbb{N}$. Therefore, we have that

$$\begin{aligned}\int_{\mathcal{C}} \frac{|\varphi_k u_k|^{2^\star(s)}}{|x|^s}\,dx &\ge \int_{B_{R_k}(0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\,dx = \alpha + o(1),\\ \int_{\mathcal{C}} \frac{|(1-\varphi_k)u_k|^{2^\star(s)}}{|x|^s}\,dx &\ge \int_{(\mathbb{R}^n\setminus B_{R'_k}(0))\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\,dx = 1 - \alpha + o(1)\end{aligned}$$

as $k \to +\infty$. The Hardy-Sobolev inequality (12.2) and (12.8) yield

$$\begin{aligned}&\mu_{\gamma,s}(\mathcal{C})\left(\int_{\mathcal{C}} \frac{|\varphi_k u_k|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \le \int_{\mathcal{C}}\left(|\nabla(\varphi_k u_k)|^2 - \frac{\gamma}{|x|^2}\varphi_k^2 u_k^2\right)dx\\ &\le \int_{\mathcal{C}} \varphi_k^2\left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2}u_k^2\right)dx + O\left(R_k^{-2}\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} u_k^2\,dx\right)\\ &\le \int_{\mathcal{C}} \varphi_k^2\left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2}u_k^2\right)dx + o(1)\end{aligned}$$

as $k \to +\infty$. Similarly,

$$\mu_{\gamma,s}(\mathcal{C})\left(\int_{\mathcal{C}} \frac{|(1-\varphi_k)u_k|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \le \int_{\mathcal{C}}(1-\varphi_k)^2\left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2}u_k^2\right)dx + o(1)$$

as $k \to +\infty$. Therefore, we have that

$$\begin{aligned}
&\mu_{\gamma,s}(\mathcal{C})\left(\alpha^{\frac{2}{2^\star(s)}} + (1-\alpha)^{\frac{2}{2^\star(s)}} + o(1)\right)\\
&\le \mu_{\gamma,s}(\mathcal{C})\left(\left(\int_{\mathcal{C}} \frac{|\varphi_k u_k|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} + \left(\int_{\mathcal{C}} \frac{|(1-\varphi_k)u_k|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}\right)\\
&\le \int_{\mathcal{C}} (\varphi_k^2 + (1-\varphi_k)^2)\left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2}u_k^2\right)dx + o(1)\\
&\le \int_{\mathcal{C}} (1 - 2\varphi_k(1-\varphi_k))\left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2}u_k^2\right)dx + o(1)\\
&\le \mu_{\gamma,s}(\mathcal{C}) + 2\int_{\mathcal{C}} \varphi_k(1-\varphi_k)\frac{\gamma}{|x|^2}u_k^2\,dx + o(1)\\
&\le \mu_{\gamma,s}(\mathcal{C}) + O\left(R_k^{-2}\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} u_k^2\,dx\right) + o(1) \le \mu_{\gamma,s}(\mathcal{C}) + o(1)
\end{aligned}$$

as $k \to +\infty$. Therefore, $\alpha^{\frac{2}{2^\star(s)}} + (1-\alpha)^{\frac{2}{2^\star(s)}} \le 1$, which implies $\alpha = 1$ since $0 < \alpha \le 1$. This proves the claim in (12.6).

We now claim that there exists $u_\infty \in D^{1,2}(\mathcal{C})$ such that $u_k \rightharpoonup u_\infty$ weakly in $D^{1,2}(\mathcal{C})$ as $k \to +\infty$, and $x_0 \neq 0$ such that, in the sense of measures,

$$\text{either } \lim_{k\to+\infty} \tfrac{|u_k|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx = \tfrac{|u_\infty|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx \quad \text{and} \quad \int_{\mathcal{C}} \frac{|u_\infty|^{2^\star(s)}}{|x|^s}\,dx = 1 \tag{12.9}$$

$$\text{or } \lim_{k\to+\infty} \tfrac{|u_k|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx = \delta_{x_0} \qquad \text{and } u_\infty \equiv 0. \tag{12.10}$$

Arguing as above, we get that for all $x \in \mathbb{R}^n$, we have that

$$\lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\,dx = \alpha_x \in \{0,1\}.$$

It then follows from the second identity of (12.5) that $\alpha_0 \le 1/2$, and therefore $\alpha_0 = 0$. Moreover, it follows from the first identity of (12.5) that there exist at most one point $x_0 \in \mathbb{R}^n$ such that $\alpha_{x_0} = 1$. In particular $x_0 \neq 0$ since $\alpha_0 = 0$. Therefore, it follows from Lions's second concentration compactness lemma [40, 41] (see also Struwe [51] for an exposition in book form) that, up to a subsequence, there exists $u_\infty \in D^{1,2}(\mathcal{C})$, $x_0 \in \mathbb{R}^n \setminus \{0\}$ and $C \in \{0,1\}$ such that $u_k \rightharpoonup u_\infty$ weakly in $D^{1,2}(\mathcal{C})$ and

$$\lim_{k\to+\infty} \frac{|u_k|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx = \frac{|u_\infty|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx + C\delta_{x_0}.$$

In particular, due to (12.5) and the compactness (12.6), we have that

$$1 = \lim_{k\to+\infty}\int_{\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\,dx = \int_{\mathcal{C}} \frac{|u_\infty|^{2^\star(s)}}{|x|^s}\,dx + C.$$

Since $C \in \{0,1\}$, the claims in (12.9) and (12.10) follow.

We now assume that $u_\infty \not\equiv 0$, and we claim that $\lim_{k\to+\infty} u_k = u_\infty$ strongly in $D^{1,2}(\mathcal{C})$ and that u_∞ is an extremal for $\mu_{\gamma,s}(\mathcal{C})$.

Indeed, it follows from (12.9) that $\int_{\mathcal{C}} \frac{|u_\infty|^{2^\star(s)}}{|x|^s}\, dx = 1$. It then follows from the Hardy-Sobolev inequality (12.2) that

$$\mu_{\gamma,s}(\mathcal{C}) \leq \int_{\mathcal{C}} \left(|\nabla u_\infty|^2 - \frac{\gamma}{|x|^2} u_\infty^2 \right) dx.$$

Moreover, since $u_k \rightharpoonup u_\infty$ weakly as $k \to +\infty$, we have that

$$\int_{\mathcal{C}} \left(|\nabla u_\infty|^2 - \frac{\gamma}{|x|^2} u_\infty^2 \right) dx \leq \liminf_{k\to+\infty} \int_{\mathcal{C}} \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx = \mu_{\gamma,s}(\mathcal{C}).$$

Therefore, equality holds in this latest inequality, u_∞ is an extremal for $\mu_{\gamma,s}(\mathcal{C})$ and reflexivity yields convergence of (u_k) to u_∞ in $D^{1,2}(\mathcal{C})$. This proves the claim.

We now assume that $u_\infty \equiv 0$ and show that as $k \to +\infty$ in the sense of measures,

$$s = 0\ ,\ \lim_{k\to+\infty} \int_{\mathcal{C}} \frac{u_k^2}{|x|^2}\, dx = 0 \text{ and } |\nabla u_k|^2\, dx \rightharpoonup \mu_{\gamma,s}(\mathcal{C})\delta_{x_0}. \tag{12.11}$$

Indeed, since $u_k \rightharpoonup u_\infty \equiv 0$ weakly in $D^{1,2}(\mathcal{C})$ as $k \to +\infty$, then for any $1 \leq q < 2^\star := \frac{2n}{n-2}$, $u_k \to 0$ strongly in $L^q_{loc}(\mathcal{C})$ when $k \to +\infty$. Assume by contradiction that $s > 0$: then $2^\star(s) < 2^\star$ and therefore, since $x_0 \neq 0$, we have that

$$\lim_{k\to+\infty} \int_{B_\delta(x_0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 0$$

for $\delta > 0$ small enough, contradicting (12.10). Therefore $s = 0$ and the first part of the claim is proved.

For the rest, we let $f \in C^\infty(\mathbb{R}^n)$ be such that $f(x) = 0$ for $x \in B_\delta(x_0)$, $f(x) = 1$ for $x \in \mathbb{R}^n \setminus B_{2\delta}(x_0)$ and $0 \leq f \leq 1$. We define $\varphi := 1 - f^2$ and $\psi := f\sqrt{2 - f^2}$. Clearly $\varphi, \psi \in C^\infty(\mathbb{R}^n)$ and $\varphi^2 + \psi^2 = 1$. Inequality (12.2) yields

$$\mu_{\gamma,s}(\mathcal{C}) \left(\int_{\mathcal{C}} |\varphi u_k|^{2^\star}\, dx \right)^{\frac{2}{2^\star}} \leq \int_{\mathcal{C}} \left(|\nabla(\varphi u_k)|^2 - \frac{\gamma}{|x|^2} (\varphi u_k)^2 \right) dx.$$

Integrating by parts, using (12.10), using that $u_k \to 0$ strongly in $L^2_{loc}(\mathbb{R}^n)$ as $k \to +\infty$, and that $\varphi^2 = 1 - \psi^2$, we get that

$$\mu_{\gamma,s}(\mathcal{C}) \left(|\varphi(x_0)|^{2^\star} + o(1) \right)^{\frac{2}{2^\star}} \leq \int_{\mathcal{C}} \varphi^2 \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx + O\left(\int_{\text{Supp } \varphi\Delta\varphi} u_k^2\, dx \right)$$

$$\mu_{\gamma,s}(\mathcal{C}) + o(1) \leq \int_{\mathcal{C}} \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx - \int_{\mathcal{C}} \psi^2 \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx + o(1)$$

as $k \to +\infty$. Using again (12.4), we then get that

$$\int_{\mathcal{C}} \psi^2 \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx \leq o(1)$$

as $k \to +\infty$. Integrating again by parts and using the strong local convergence to 0, we get that

$$\int_{\mathcal{C}} \left(|\nabla(\psi u_k)|^2 - \frac{\gamma}{|x|^2} (\psi u_k)^2 \right) dx \leq o(1)$$

as $k \to +\infty$. The coercivity (12.2) then yields $\lim_{k\to+\infty} \|\nabla(\psi u_k)\|_2 = 0$. Therefore, the Hardy inequality yields convergence of $|x|^{-1}(\psi u_k)_k$ to 0 in $L^2(\mathcal{C})$. Therefore,

$$\lim_{k\to+\infty} \int_{(B_{2\delta}(x_0))^c\cap\mathcal{C}} \frac{u_k^2}{|x|^2}\, dx = 0.$$

Taking $\delta > 0$ small enough and combining this result with the strong convergence of $(u_k)_k$ in L^2_{loc} around $x_0 \neq 0$ yields

$$\lim_{k\to+\infty} \int_{\mathcal{C}} \frac{u_k^2}{|x|^2}\, dx = 0.$$

Combining this equality, $\lim_{k\to+\infty} \|\nabla(\psi u_k)\|_2 = 0$ and (12.4) yields the third part of the claim. This proves the claim.

We now show that if $u_\infty \equiv 0$, then $s = 0$ and

$$\mu_{\gamma,s}(\mathcal{C}) = \mu_{0,0}(\mathbb{R}^n) = \frac{1}{K(n,2)^2}.$$

Indeed, since $u_k \in D^{1,2}(\mathcal{C}) \subset D^{1,2}(\mathbb{R}^n)$, we have that

$$\mu_{0,0}(\mathbb{R}^n)\left(\int_{\mathbb{R}^n} |u_k|^{2^\star}\, dx\right)^{\frac{2}{2^\star}} \leq \int_{\mathbb{R}^n} |\nabla u_k|^2\, dx.$$

It then follows from (12.11, (12.4) and (12.5) that $\mu_{0,0}(\mathbb{R}^n) \leq \mu_{\gamma,s}(\mathcal{C})$. Conversely, the computations of Proposition 9.1 yield $\mu_{\gamma,s}(\mathcal{C}) \leq \mu_{0,0}(\mathbb{R}^n) = K(n,2)^{-1}$. These two inequalities prove the claim.

Note now that if $s = 0$, $\gamma > 0$ and $n \geq 4$, then necessarily

$$\mu_{\gamma,s}(\mathcal{C}) < \mu_{0,0}(\mathbb{R}^n) = \frac{1}{K(n,2)^2}. \tag{12.12}$$

Indeed, consider the family u_ε as in the proof of Proposition 9.1. Well known computations by Aubin [2] yield

$$J^{\mathcal{C}}_{\gamma,s}(u_\varepsilon) = K(n,2)^{-2} - \gamma|x_0|^{-2}c\theta_\varepsilon + o(\theta_\varepsilon) \text{ as } \varepsilon \to 0,$$

where $c > 0$, $\theta_\varepsilon = \varepsilon^2$ if $n \geq 5$ and $\theta_\varepsilon = \varepsilon^2 \ln \varepsilon^{-1}$ if $n = 4$. It follows that if $\gamma > 0$ and $n \geq 4$, then $\mu_{\gamma,s}(\mathcal{C}) < K(n,2)^{-1}$. This proves the claim.

As in Proposition 9.1, even if the cone is nonsmooth, it is easy to see that if $s = 0$ and $\gamma \leq 0$, then

$$\mu_{\gamma,s}(\mathcal{C}) = \mu_{0,0}(\mathbb{R}^n) = \frac{1}{K(n,2)^2}. \tag{12.13}$$

Moreover, is no extremal if $\gamma < 0$.

If now $s = 0$ and $\gamma = 0$, then

$$\mu_{\gamma,s}(\mathcal{C}) = \mu_{0,0}(\mathbb{R}^n) = \frac{1}{K(n,2)^2}, \tag{12.14}$$

and there are extremals iff there exists $z \in \mathbb{R}^n$ such that $(1+|x-z|^2)^{1-n/2} \in D^{1,2}(\mathcal{C})$ (in particular $\overline{\mathcal{C}} = \mathbb{R}^n$).

Again, the proof goes essentially as in Proposition 9.1, even if the cone is nonsmooth. The potential extremals for $\mu_{0,0}(\mathcal{C})$ are extremals for $\mu_{0,0}(\mathbb{R}^n)$, and therefore of the form $x \mapsto a(b + |x - z_0|^2)^{1-n/2}$ for some $a \neq 0$ and $b > 0$ (see Aubin [2] or Talenti [52]). Using the homothetic invariance of the cone, we then get that there is an extremal of the form $x \mapsto (1+|x-z|^2)^{1-n/2}$ for some $z \in \mathbb{R}^n$. Since an extremal has support in $\overline{\mathcal{C}}$, we then get that $\overline{\mathcal{C}} = \mathbb{R}^n$. This proves the claim.

Finally, assume that $s = 0$ and that there exists $z \in \mathbb{R}^n$ such that $x \mapsto (1+|x-z|^2)^{1-n/2} \in D^{1,2}(\mathcal{C})$. Then $\mu_{\gamma,0}(\mathcal{C}) < \frac{1}{K(n,2)^2}$ for all $\gamma > 0$. For that it suffices to consider $U(x) := (1+|x-z|^2)^{1-n/2}$ for all $x \in \mathbb{R}^n$, and to note that $J_{\gamma,0}^{\mathcal{C}}(U) = J_{\gamma,0}^{\mathbb{R}^n}(U) < J_{0,0}^{\mathbb{R}^n}(U) = K(n,2)^{-2}$.
This ends the proof of Theorem 12.1 and Corollaries 12.2, 12.3.

13. Appendix 2: Symmetry of the extremals for $\mu_{\gamma,s}(\mathbb{R}^n_+)$

The symmetry of the nonnegative solutions to the Euler-Lagrange equation for $\mu_{\gamma,s}(\mathbb{R}^n_+)$ is proved in Chern-Lin [10] for $\gamma < (n-2)^2/4$. The proof of the symmetry carried out by Ghoussoub-Robert [24] in the case $\gamma = 0$ extends immediately to the case $0 \leq \gamma < n^2/4$. For the convenience of the reader, we give here a general and complete proof inspired by Chern-Lin [10], which includes the case where $\gamma < 0$.

For $\gamma < n^2/4$, $s \in [0,2)$ and $2^\star(s) := \frac{2(n-s)}{n-2}$, we consider nontrivial solutions $u \in D^{1,2}(\mathbb{R}^n_+)$ to the problem

$$\left\{\begin{array}{ll} -\Delta u - \frac{\gamma}{|x|^2}u = \frac{u^{2^\star(s)-1}}{|x|^s} & \text{weakly in } D^{1,2}(\mathbb{R}^n_+) \\ u \geq 0 & \text{in } \mathbb{R}^n_+ \\ u = 0 & \text{on } \partial\mathbb{R}^n_+ \end{array}\right. \tag{13.1}$$

and prove the following.

Theorem 13.1. *If u is a solution to (13.1) in $D^{1,2}(\mathbb{R}^n_+)$, then $u \circ \sigma = u$ for all isometries of $\mathbb{R}^n$ such that $\sigma(\mathbb{R}^n_+) = \mathbb{R}^n_+$. In particular, there exists $v \in C^\infty((0,+\infty) \times \mathbb{R})$ such that for all $x_1 > 0$ and all $x' \in \mathbb{R}^{n-1}$, we have that $u(x_1, x') = v(x_1, |x'|)$.*

Remark: Unlike the case of the extremals for the full space $\mathbb{R}^n$, there is no symmetry-breaking phenomenon in the case of the half-space $\mathbb{R}^n_+$. However, the price to pay is that the best constant when restricted to the functions with best possible symmetry is unknown, contrary to the case of $\mathbb{R}^n$. We refer to the historical reference Catrina-Wang [5] and to Dolbeault-Esteban-Loss-Tarantello [14] for disussions and developments on the symmetry-breaking phenomenon.

We adapt the moving-plane method of Chern-Lin [10] that was made in the case $\gamma < \frac{(n-2)^2}{4}$. Given any $\theta \in [0, \frac{\pi}{2}]$, we define the hyperplane and the half space:

$$P_\theta := \{x \in \mathbb{R}^n / \, x_1 \cos\theta = x_2 \sin\theta\},$$
$$P_\theta^- := \{x \in \mathbb{R}^n / \, x_1 \cos\theta < x_2 \sin\theta\}.$$

We define $s_\theta : \mathbb{R}^n \to \mathbb{R}^n$ as the orthogonal symmetry with respect to P_θ. As one checks, we have that

$$s_\theta(x) = \begin{pmatrix} -x_1\cos(2\theta) + x_2 \sin(2\theta) \\ x_1 \sin(2\theta) + x_2\cos(2\theta) \\ x_i \; (i \geq 2) \end{pmatrix} \text{ and } s_\theta(x) - x = 2\left(x_2\sin\theta - x_1\cos\theta\right)\begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} \tag{13.2}$$

Note that it follows from Theorem 6.1 that there exists $K_1 > 0$ such that

$$u(x) \sim_{x \to 0} K_1 \frac{x_1}{|x|^{\alpha_-(\gamma)}}. \tag{13.3}$$

The proof of Theorem 13.1 relies on two main Lemmas:

Lemma 13.2. *For all $j = 1, ..., n$, we have that*

$$\lim_{x\to 0}\left(|x|^{\alpha_-(\gamma)}\partial_j u(x) - K_1\left(\delta_{j,1} - \alpha_-(\gamma)\frac{x_1 x_j}{|x|^2}\right)\right) = 0, \tag{13.4}$$

and

$$|x|^{\alpha_-(\gamma)+1}\|d^2 u_x\| \le C \text{ for all } x \in \mathbb{R}^n_+,\ |x| < 1. \tag{13.5}$$

Proof of Lemma 13.2: We proceed by contradiction and assume that there exists $(x_k)_k \in \mathbb{R}^n_+$ such that $x_k \to 0$ and

$$\left(|x_k|^{\alpha_-(\gamma)}\partial_j u(x_k) - K_1\left(\delta_{j,1} - \alpha_-(\gamma)\frac{x_{k,1}x_{k,j}}{|x_k|^2}\right)\right) \not\to 0 \tag{13.6}$$

as $k \to +\infty$. We define $u_k(x) := |x_k|^{\alpha_-(\gamma)-1}u(|x_k|x)$ for all $x \in \mathbb{R}^n_+$. It follows from (13.3) that

$$\lim_{k\to+\infty} u_k(x) = K_1\frac{x_1}{|x|^{\alpha_-(\gamma)}} \text{ for all } x \in \overline{\mathbb{R}^n_+}\setminus\{0\}. \tag{13.7}$$

Moreover, this convergence holds in $C^0_{loc}(\overline{\mathbb{R}^n_+}\setminus\{0\})$. Equation (13.1) rewrites as

$$-\Delta u_k - \frac{\gamma}{|x|^2}u_k = |x_k|^{(2^\star(s)-2)\left(\frac{n}{2}-\alpha_-(\gamma)\right)}\frac{u_k^{2^\star(s)-1}}{|x|^s} \text{ in } \mathbb{R}^n_+$$

for all k, and u_k vanishes on $\partial\mathbb{R}^n_+$. It then follows from elliptic theory that the convergence in (13.7) holds in $C^2_{loc}(\overline{\mathbb{R}^n_+}\setminus\{0\})$. Therefore,

$$\lim_{k\to+\infty}\partial_j u_k\left(\frac{x_k}{|x_k|}\right) = \partial_j(K_1 x_1|x|^{-\alpha_-(\gamma)})(X_\infty)$$

where $X_\infty := \lim_{k\to+\infty}\frac{x_k}{|x_k|}$. Coming back to u_k contradicts (13.6). This proves (13.4). The proof of (13.5) is similar. This ends the proof of Lemma 13.2.

The second Lemma is a general analysis of the difference $u(s_\theta(x)) - u(x)$.

Lemma 13.3. *We let $(\theta_i)_i \in \mathbb{R}$ and $(x_i) \in \mathbb{R}^n_+$ be such that $x_i \in \mathbb{R}^n_+ \cap P^-_{\theta_i}$ for all $i \in \mathbb{N}$. We assume that $\theta_i \to \theta_\infty$ and $x_i \to x_\infty$ as $i \to +\infty$, and that*

$$s_{\theta_i}(x_i) - x_i = o(|x_i|) \text{ as } i \to +\infty. \tag{13.8}$$

Then,

- *If $x_\infty \neq 0$ then*

$$\lim_{i\to+\infty}\frac{u(s_{\theta_i}(x_i)) - u(x_i)}{2(x_{i,2}\sin\theta_i - x_{i,1}\cos\theta_i)} = \cos(\theta_\infty)\partial_1 u(x_\infty) - \sin(\theta_\infty)\partial_2 u(x_\infty). \tag{13.9}$$

- *If $x_\infty = 0$, then*

$$\lim_{i\to+\infty}\frac{u(s_{\theta_i}(x_i)) - u(x_i)}{2(x_{i,2}\sin\theta_i - x_{i,1}\cos\theta_i)|x_i|^{-\alpha_-(\gamma)}} = K_1\cos(\theta_\infty). \tag{13.10}$$

Proof of Lemma 13.3: Taylor's formula yields

$$|u(s_{\theta_i}(x_i)) - u(x_i) - du_{x_i}(s_{\theta_i}(x_i) - x_i)| \le \|s_{\theta_i}(x_i) - x_i\|^2 \sup_{t\in[0,1]}\|d^2 u_{x_i+t(s_{\theta_i}(x_i)-x_i)}\| \tag{13.11}$$

for all i. It follows from (13.5) that

$$\begin{aligned}\sup_{t\in[0,1]}\|d^2u_{x_i+t(s_{\theta_i}(x_i)-x_i)}\| &\leq C\sup_{t\in[0,1]}\|x_i+t(s_{\theta_i}(x_i)-x_i)\|^{-(1+\alpha_-(\gamma))}\\ &= C\left\|\frac{x_i+s_{\theta_i}(x_i)}{2}\right\|^{-(1+\alpha_-(\gamma))},\end{aligned}$$

and therefore

$$u(s_{\theta_i}(x_i))=u(x_i)+du_{x_i}(s_{\theta_i}(x_i)-x_i)+O\left(\frac{\|s_{\theta_i}(x_i)-x_i\|^2}{\|x_i+s_{\theta_i}(x_i)\|^{1+\alpha_-(\gamma)}}\right),\tag{13.12}$$

and then, we get with (13.8) that

$$u(s_{\theta_i}(x_i))=u(x_i)+du_{x_i}(s_{\theta_i}(x_i)-x_i)+o\left(\frac{\|s_{\theta_i}(x_i)-x_i\|}{|x_i|^{\alpha_-(\gamma)}}\right),\tag{13.13}$$

as $i\to+\infty$. With the expression (13.2), we get that

$$\|s_{\theta_i}(x_i)-x_i\|=2(x_{i,2}\sin\theta_i-x_{i,1}\cos\theta_i)>0\tag{13.14}$$

and that

$$\frac{u(s_{\theta_i}(x_i))-u(x_i)}{2(x_{i,2}\sin\theta_i-x_{i,1}\cos\theta_i)}=\partial_1u(x_i)\cos\theta_i-\partial_2u(x_i)\sin\theta_i+o\left(|x_i|^{-\alpha_-(\gamma)}\right)\tag{13.15}$$

as $i\to+\infty$. If $x_\infty\neq0$, then we get (13.9) and we are done. We assume that $x_\infty=0$. It then follows from Lemma 13.2 that

$$\begin{aligned}\frac{|x_i|^{\alpha_-(\gamma)}(u(s_{\theta_i}(x_i))-u(x_i))}{\|s_{\theta_i}(x_i)-x_i\|} &= K_1\left(1-\alpha_-(\gamma)\left(\frac{x_{i,1}}{|x_i|}\right)^2\right)\cos\theta_i\\ &\quad+K_1\alpha_-(\gamma)\frac{x_{i,1}x_{i,2}}{|x_i|^2}\sin\theta_i+o(1)\\ &= K_1\left[\cos\theta_i+\alpha_-(\gamma)\frac{x_{i,1}}{|x_i|}\left(\frac{x_{i,2}\sin\theta_i-x_{i,1}\cos\theta_i}{|x_i|}\right)\right]\\ &\quad+o(1)\ \text{ as } i\to+\infty.\end{aligned}$$

Using (13.8) and (13.14), we get that

$$\lim_{i\to+\infty}\frac{|x_i|^{\alpha_-(\gamma)}(u(s_{\theta_i}(x_i))-u(x_i))}{\|s_{\theta_i}(x_i)-x_i\|}=K_1\cos\theta_\infty.\tag{13.16}$$

This ends the proof of Lemma 13.3. □

We are now in position to initiate the moving plane method.

Proposition 13.4. *There exists $\theta_0>0$ such that*

$$\text{for all }\theta\in(0,\theta_0),\ \text{ then } u(s_\theta(x))>u(x)\text{ for all } x\in P_\theta^-\cap\mathbb{R}^n_+\tag{13.17}$$

Proof of Proposition 13.4: We argue by contradiction and we assume that there exists $(\theta_i)_i\in(0,+\infty)$, there exists $x_i\in P_{\theta_i}^-\cap\mathbb{R}^n_+$ such that

$$\lim_{i\to+\infty}\theta_i=0\text{ and } u(s_{\theta_i}(x_i))\leq u(x_i)\text{ for all } i.\tag{13.18}$$

We first claim that without loss of generality, we can assume that $(x_i)_i$ is bounded in $\mathbb{R}^n$. For that we define the Kelvin transform $\tilde u(x):=|x|^{2-n}u(x/|x|^2)$ for all $x\in\mathbb{R}^n_+$. As one checks, $\tilde u\in D^{1,2}(\mathbb{R}^n_+)$ satisfies (13.1) and (13.18) rewrites $\tilde u(s_{\theta_i}(\tilde x_i))\leq\tilde u(\tilde x_i)$

for all i, where $\tilde{x}_i := x_i/|x_i|^2 \in P_{\theta_i}^- \cap \mathbb{R}^n_+$. Therefore, up to changing u into $\tilde{u}$, we can assume that $(x_i)_i$ is bounded. This proves the claim.

Now define $\lim_{i\to+\infty} x_i = x_\infty$. We claim that

$$x_{i,1} = o(x_{i,2}) \text{ as } i \to +\infty \text{ and } x_\infty \in \partial\mathbb{R}^n_+. \tag{13.19}$$

Indeed, since $x_i \in P_{\theta_i}^- \cap \mathbb{R}^n_+$, we have that $x_{i,1} > 0$ and $x_{i,1}\cos\theta_i < x_{i,2}\sin\theta_i$ for all i. Letting $i \to \infty$ yields $x_{i,1} = o(x_{i,2})$ as $i \to +\infty$, and therefore $x_\infty \in \partial\mathbb{R}^n_+$.

We now show that

$$s_{\theta_i}(x_i) - x_i = o(|x_i|) \text{ as } i \to +\infty. \tag{13.20}$$

Indeed, it suffices to note that the expression (13.14) and (13.19) yield $s_{\theta_i}(x_i) - x_i = o(|x_{i,2}|) = o(|x_i|)$ as $i \to +\infty$.

We now conclude the proof of Proposition 13.4. If $x_\infty = 0$, it follows from (13.10) that $u(s_{\theta_i}(x_i)) - u(x_i) > 0$ for $i \to +\infty$, contradicting (13.18). If $x_\infty \neq 0$, it follows from (13.9) and (13.18) that $\partial_1 u(x_\infty) \leq 0$: this contradicts Hopf's strong maximum principle since $x_\infty \in \partial\mathbb{R}^n_+$. This ends the proof of Proposition 13.4. □

Define now

$$\theta_0 := \sup\left\{0 < \theta \leq \frac{\pi}{2} / \, u(s_t(x)) > u(x) \text{ for all } x \in P_t^- \cap \mathbb{R}^n_+ \text{ and all } 0 < t < \theta\right\}$$

It follows from Proposition 13.4 that $\theta_0 > 0$ exists. Our objective is to prove that $\theta_0 = \frac{\pi}{2}$. We argue by contradiction and assume that

$$0 < \theta_0 < \frac{\pi}{2}. \tag{13.21}$$

For any $\theta \geq 0$, we define

$$v_\theta(x) := u(s_\theta(x)) - u(x)$$

for all $x \in P_\theta^- \cap \mathbb{R}^n_+$. Since s_θ is an isometry for all $\theta \geq 0$, we have that

$$-\Delta v_\theta - \frac{\gamma}{|x|^2} v_\theta = c_\theta(x) v_\theta \tag{13.22}$$

where $c_\theta(x) = |x|^{-s}\frac{u(s_\theta(x))^{2^\star(s)-1} - u(x)^{2^\star(s)-1}}{u(s_\theta(x)) - u(x)}$ if $u(s_\theta(x)) \neq u(x)$, and $c_\theta(x) = |x|^{-s}(2^\star(s)-1)u(x)^{2^\star(s)-2}$ otherwise. In particular, $c_\theta > 0$. It follows from the definition of θ_0 that $v_{\theta_0} \geq 0$. It then follows from (13.22) and Hopf's maximum principle that either $v_{\theta_0} > 0$ in $P_{\theta_0}^- \cap \mathbb{R}^n_+$ or $v_{\theta_0} \equiv 0$ in $P_{\theta_0}^- \cap \mathbb{R}^n_+$. In the latter case, taking points on $\partial\mathbb{R}^n_+$, we would get that $u(x) = 0$ on $P_{2\theta_0} \cap \mathbb{R}^n_+$: this is impossible since $\theta_0 < \frac{\pi}{2}$ and $u > 0$. Therefore

$$v_{\theta_0} > 0 \text{ in } P_{\theta_0}^- \cap \mathbb{R}^n_+. \tag{13.23}$$

It follows from the definition of θ_0 that there exists $(\theta_i)_i \in (\theta_0, +\infty)$ such that

$$\lim_{i\to+\infty} \theta_i = \theta_0 \text{ and } \forall i \text{ there exists } x_i \in P_{\theta_i}^- \cap \mathbb{R}^n_+ \text{ such that } v_{\theta_i}(x_i) \leq 0. \tag{13.24}$$

Arguing as in Step 1 of the proof of Proposition 13.4, we can assume with no loss of generality that $(x_i)_i$ is bounded, and, up to a subsequence, that there exists $x_\infty \in \mathbb{R}^n$ such that $\lim_{i\to+\infty} x_i = x_\infty$.

We claim that

$$x_\infty \in P_{\theta_0} \cap \overline{\mathbb{R}^n_+}. \tag{13.25}$$

Indeed, it follows from (13.24) that $x_\infty \in \overline{P_{\theta_0}^- \cap \mathbb{R}^n_+}$ and $v_{\theta_0}(x_\infty) \leq 0$. It then follows from (13.23) that $x_\infty \in \partial P_{\theta_0}^- \cap \mathbb{R}^n_+ = (P_{\theta_0} \cap \overline{\mathbb{R}^n_+}) \cup (\partial\mathbb{R}^n_+ \cap \overline{P_{\theta_0}^-}$ and $v_{\theta_0}(x_\infty) = 0$. We argue by contradiction and assume that (13.25) does not hold. Therefore, $x_\infty \in \partial\mathbb{R}^n_+$ and $u(s_{\theta_0}(x_\infty)) = v_{\theta_0}(x_\infty) = 0$, and then $s_{\theta_0}(x_\infty) \in \partial\mathbb{R}^n_+$. We then get with (13.2) and (13.21) that $s_{\theta_0}(x_\infty) = x_\infty$ and then $x_\infty \in P_{\theta_0}$, which contradicts our initial hypothesis. This proves (13.25) and therefore the claim.

We claim that

$$s_{\theta_i}(x_i) - x_i = o(|x_i|) \text{ as } i \to +\infty. \tag{13.26}$$

It follows from (13.25) that $s_{\theta_0}(x_\infty) = x_\infty$, and therefore (13.26) holds if $x_\infty \neq 0$. We now assume that $x_\infty = 0$. Dividing (13.26) by $|x_i|$ and passing to the limit $i \to +\infty$, one gets that (13.26) is equivalent to proving that $s_{\theta_0}(X_\infty) = X_\infty$ where $X_\infty := \lim_{i\to+\infty} \frac{x_i}{|x_i|}$. Since $x_i \in P_{\theta_i}^-$, we have that $x_{i,2}\sin\theta_i > x_{i,1}\cos\theta_i$ for all i. Dividing by $|x_i|$ and passing to the limit $i \to +\infty$ yields

$$X_{\infty,2}\sin\theta_0 \geq X_{\infty,1}\cos\theta_0. \tag{13.27}$$

Since $u(s_{\theta_i}(x_i)) \leq u(x_i)$, the asymptotic (13.7) yields

$$K_1 \frac{(s_{\theta_i}(x_i))_1}{|s_{\theta_i}(x_i)|^{\alpha_-(\gamma)}} \leq (1+o(1))K_1 \frac{x_{i,1}}{|x_i|^{\alpha_-(\gamma)}}$$

as $i \to +\infty$. Dividing by $|x_i|$ and passing to the limit, we get that

$$(s_{\theta_0}(X_\infty) - X_\infty)_1 \leq 0. \tag{13.28}$$

Plugging (13.27) and (13.28) into (13.2) yields $s_{\theta_0}(X_\infty) = X_\infty$. As already mentioned, this proves the claim.

Here goes the final argument. We apply Lemma 13.3. If $x_\infty = 0$, (13.10), (13.14) and (13.24) yield $K_1\cos(\theta_0) \leq 0$: a contradiction since $K_1 > 0$ and $0 < \theta_0 < \pi/2$. If $x_\infty \neq 0$, (13.9) and (13.24) yield

$$\partial_1 u(x_\infty)\cos(\theta_0) - \partial_2 u(x_\infty)\sin(\theta_0) \leq 0. \tag{13.29}$$

If $x_\infty \in \partial\mathbb{R}^n_+$, then $\partial_2 u(x_\infty) = 0$ and $\partial_1 u(x_\infty) > 0$ (Hopf's Lemma), contradicting (13.29). So $x_\infty \in P_{\theta_0} \setminus \partial\mathbb{R}^n_+$. It then follows from (13.22), (13.23), (13.25) and Hopf's Lemma that $\partial_{\vec{N}} v_{\theta_0}(x_\infty) < 0$ with $\vec{N} = (\cos\theta_0, -\sin\theta_0, 0)$. However, one can easily see that $\partial_{\vec{N}} v_{\theta_0}(x_\infty) = -2(\partial_1 u(x_\infty)\cos(\theta_0) - \partial_2 u(x_\infty)\sin(\theta_0))$, which again contradicts (13.29).

In all cases, we get a contradiction, and therefore (13.21) is not valid, which means that $\theta_0 = \frac{\pi}{2}$. It follows that

$$u(x_1, -x_2, ...) \geq u(x_1, x_2, ...) \text{ for all } x \in \mathbb{R}^n_+,\ x_2 > 0.$$

Since the equation satisfied by u is invariant under the action of isometries fixing $\partial\mathbb{R}^n_+$, we get the reverse inequality and therefore $u(x_1, x_2, ...) = u(x_1, -x_2, ...)$ for all $x \in \mathbb{R}^n_+$. So u is invariant under the action of the symmetry wrt $\{x_2 = 0\}$. This argument works for any hyperplane orthogonal to $\partial\mathbb{R}^n_+$: then u is invariant under the action of the symmetries fixing $\partial\mathbb{R}^n_+$. This completes the proof of Theorem 13.1.

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