SUBADDITIVE ERGODIC THEORY IN GENERAL FUNCTION SPACES

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I. Introduction: In this paper we try to give a systematic approach for the extensions of some ergodic and subergodic theorems form the classical L_p-setting to more general function spaces. Recently, Akcoglu and Sucheston [4] gave a method for doing so, based on the concept of "truncated limits." Our approach exploits the weak-star compactness of the double dual E** of the function space E involved and the existence of a projection from E** onto E whenever the latter is assumed to be weakly sequentially complete [17]. We shall first introduce the notions that will be relevant for the study of ergodic-type theorems on general function spaces.

Following Akcoglu-Sucheston [3] we shall say that a sequence (f_n) in a Banach lattice E converges stochastically to f in E if for every v in E_+, $\lim \| |f_n - f| \wedge v\|_E = 0$. If now E has an order continuous norm with a weak unit u, then E can be represented as a function space on some probability space $(\Omega,\mathcal{F},P)$. It is then easy to see that (f_n) converges stochastically to f if and only if (f_n) converges in probability to f on $(\Omega,\mathcal{F},P)$. Moreover (f_n) would norm-converge to f if it is also norm-uniformly integrable: that is $\lim_{\mu(A)\to 0} \sup_n \|f_n \chi_A\|_E = 0$.

Suppose now E is a Banach Lattice which is the range of a band projection Q in a Banach lattice G. We shall say that E is an L_p-ideal in G for some p $(1 \le p < \infty)$ if $\|Qx\|^p + \|(I-Q)x\|^p \le \|x\|^p$ for each x in G. Typical examples of spaces which are L_p-ideals in their second duals are

(a) reflexive Banach lattices (Q = Identity);

(b) p-concave Banach lattices with p-concavity constant equal to one [16].

A norm $\| \ \|$ on a Banach lattice E is said to be strictly monotone if $0 \le f < g$ implies $\|f\| < \|g\|$ for every f, g in E. Strictly convex norms are clearly strictly monotone and the results of [4] give that every order

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continuous Banach lattice has an equivalent stricIty monotone norm. The corresponding superproperty is the following: A norm $\| \ \|$ is said to be uniformly strictly monotone if for all $\varepsilon > 0$, there exists $\delta > 0$ so that if $0 \leq f < g$ and $\|g - f\| \geq \varepsilon$ with $\|f\| \leq 1$ then $\|g\| > \|f\| + \delta$.

If we denote by $\tilde{E}$ an ultraproduct of a Banach lattice E [16], the following lemma can be readily verified.

Lemma I.1: The following properties are equivalent:

a) E has a uniformly monotone norm.

b) $\tilde{E}$ has a uniformly monotone norm.

c) $\tilde{E}$ has a strictly monotone norm.

d) E in q-concave for some $1 \leq q < \infty$ with a concavity constant equal to one.

Let now T be a bounded linear operator on a Banach lattice G and let Q be a band projection from G onto a sub-ideal E. We shall say that T is Q-stable if T maps E into itself. In other words $(I-Q)TQ = 0$. We say that T is Q-consistent if for each x in G, $\lim_n \|Q[T(I-Q)]^n x\| = 0$). T is said to be uniformly Q-consistent if for each x in G, the series $\sum_{k=0}^{n} Q[T(I-Q)]^k x$ converges. In this case we will denote by Q_T the operator $\sum_{k=0}^{\infty} Q[T(I-Q)]^k$ from G onto E.

In the following proposition we give sufficient conditions on an operator T and a projection Q that ensure Q-stability and Q-consistency. The proof is left to the reader.

Proposition I.2 Let T be a positive operator on a Banach lattice G and let Q be a band projection from G onto a weakly sequentially complete ideal E of G.

(a) If T is Q-stable and (I-Q)-stable then T is uniformly Q-consistent.

(b) If there exists $\alpha\,(0<\alpha<1)$ such that for each x in G, $d(Qx + T(I-Q)x, E) \leq \alpha \|x\|$ then T is uniformly Q-consistent. ($d(y,E)$ denotes the distance from y to E and the condition holds in particular if $\|(I-Q)T(I-Q)x\| \leq \alpha \|x\|$)

(c) If E is an L_1-ideal in G, then every positive contraction on G is uniformly Q-consistent.

(d) If E is an L_p-ideal in G ($1<p<\infty$) then every positive contraction on G is Q-consistent.

(e) If T maps a weak unit of E into itself then T is Q-stable.

(f) If T is a lattice homomorphism Q on G that maps a weak unit of E into itself then T is Q and (I-Q)-stable.

(g) If T is Q-stable and invertible then T is (I-Q)-stable.

The following two lemmas illustrate the importance of the consistency conditions. The main idea is that under such hypothesis one can "project" the nice properties of some elements in the superspace G to the subspace E.

<u>Lemma I.3</u> Let Q be a band projection from a Banach lattice G onto a subideal E of G. Let T be a Q-stable and Q-consistent operator on G. If f is a fixed point for T in G_+ then Qf is a fixed point for T in E.

<u>Proof</u> Suppose f = Tf. We prove by induction that for each $k \geq 1$ we have:

$$(*) \qquad f = TQf + [T(I-Q)]^k f$$

The assertion is true for k = 1 since

$$f = Tf = TQf + T(I-Q)f.$$

Assume (*) is true up to k. We get by applying TQ

$$(**) \qquad TQf = TQTQf + TQ[T[I-Q)]^k f = T^2Qf + TQ[T(I-Q)]^k f$$

On the other hand, since f = Tf, (*) gives

$$f = Tf = T^2Qf + T[T(I-Q)^k f = T^2Qf + TQ[T(I-Q)]^k f + [T(I-Q)]^{k+1} f$$

Apply (**) to get $f = TQf + [T(I-Q)]^{k+1}f$. Hence (*) is true for any $k \geq 1$. Apply now Q: $Qf = TQf + Q[T(I-Q)]^k f$ for each $k \geq 1$. By the Q-consitency of T we get that $\lim_{k\to\infty} \|Q[(I-Q)]^k f\| = 0$ and $Qf = TQf$.

<u>Remark I.1</u>: We did not need the full strength of Q-consistency in the above proof. One actually need that $Q[T(I-Q)]^k f = 0$ whenever it is stationary. This is for instance assured if the norm on G is strictly monotone and T is a contraction since if $QT(I-Q)f \neq 0$

then $$(I-Q)T(I-Q)f < T(I-Q)f$$

and $$\|T[I-Q)]^k f\| < \|T(I-Q)f\|$$

which is a contradiction.

The following lemma is an extension of an idea of Brunel-Sucheston [6]

<u>Lemma I.4</u>: Let Q be a band projection a Banach lattice G onto a subideal E of G. Let T be a Q-stable and uniformly Q-consistent operator on G, then for any s, (s_n) in G_+ such that $0 \le s_n \le \sum_{i=0}^{n-1} T^i s$ for all $n \ge 1$ we have $0 \le Qs_n \le \sum_{i=0}^{n-1} T^i Q_T s$ for all $n \ge 1$.

<u>Proof</u>: Note that $s \ge s_1$ hence $Qs \ge Qs_1$. Also $s + Ts \ge s_2$ hence $s + TQs + T(I-Q)s \ge s_2$ and by applying Q we get $Qs + QTQs + QT(I-Q)s \ge Qs_2$ but $QTQ = TQ$ hence $Qs + TQs + QT(I-Q)s \ge Qs_2$. By induction on n, we get

$$\sum_{i=0}^{n-1} T^i Qs + \sum_{i=0}^{n-1} T^i Q[T(I-Q)]^k s + \sum_{i=0}^{n-k} T^i Q[T(I-Q)]^k s + Q[T(I-Q)]^n s \ge Qs_{n+1}.$$

But this implies

$$\sum_{i=0}^{n} T^i \Big(\sum_{k=0}^{n} Q[T(I-Q)]^k s \Big) \ge Qs_{n+1} \text{ for each } n \ge 0$$

and

$$\sum_{i=0}^{n} T^i Q_T s \ge Qs_{n+1} \quad \text{for each } n \ge 0.$$

The typical example of a superspace G is the double dual E** of E. If E is weakly sequentially complete, then E is the range of a projection band Q in E**. If now T is an operator on E, then T** is a Q-stable operator.

We shall say then that <u>T is consistent</u> (resp <u>uniformly consistent</u>) if T** is Q-consistent (resp Q-uniformly consistent).

<u>II ON THE NORM CONVERGENCE OF SUBADDITIVE PROCESSES</u>

Let E be a Banach space and let T be a contraction on E ($\|T\| \le 1$). We shall say that T is <u>mean ergodic on E</u> if it verifies one of the following equivalent conditions:

(1) For each x in E, $A_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$ converges strongly to an element Px in E

(2) $E = \mathrm{Ker}\,(I-T) \oplus \overline{\mathrm{Im}(I-T)}$

(3) For each x in E, the sequence $(A_n x)_n$ has a weak cluster point.

In this case P is a projection from E onto Ker(I-T) satisfying PT =

$TP = P$. For more details we refer to [8].

A positive contraction on a Banach lattice E is said to be <u>weak order contractive</u> if there exists a quasi-interior point u in E_+ such that $Tu \leq u$. If, moreover, there exists a strictly positive linear form μ on E such that $T^*\mu \leq \mu$, then T is said to be <u>order contractive</u>.

Order contractions are the natural extensions of the contractions on L_1 which are also contractions on L_∞. Actually, every order contraction on an order continuous Banach lattice induces a contraction on the L_1 and L_∞-spaces corresponding to u and μ [17].

We also recall that a positive operator T on E is said to be <u>irreducible</u> if {0} is a maximal T-stable ideal among all T-stable ideals different of E.

A sequence (s_n) in E is said to be <u>T-subadditive</u> (resp <u>T-superadditive</u>) of $s_{n+k} \leq s_n + T^n s_k$ (resp $s_{n+k} \geq s_n + T^n s_k$) for all integers n, k. Note that (s_n) is <u>T-additive</u> $(s_{n+k} = s_n + T^n s_k)$ if and only if $s_n = \sum_{i=0}^{n-1} T^i s_1$ for each $n \in N$.

<u>Theorem II.1</u> Every weak order contractive operator on a weakly sequentially complete Banach lattice is mean ergodic.

<u>Proof</u> We shall show that $E = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Im}(I-T)}$. Let $x = x^+ - x^-$ be an element in E. For each $k \in N$, the sequence $\{A_n(x^+ \wedge ku)\}_n$ (resp $\{A_n(x^- \wedge ku)\}_n$) is in the weakly compact order interval $[0,ku]$. A standard diagonalization argument gives a subsequence (n_j) and elements y^k, z^k such that for each k

(i) $\text{weak limit}_j \, A_{n_j}(x^+ \wedge ku) = y^k$

(ii) $\text{weak limit}_j \, A_{n_j}(x^- \wedge ku) = z^k$

(iii) $Ty^k = y^k$ and $Tz^k = z^k$.

Clearly the sequences (y^k) and (z^k) are increasing and norm bounded, hence they converge to y (resp z) such that $y^k = y \wedge k$, $z^k = z \wedge k$, $Ty = y$ and $Tz = z$.

We shall show that $x - (y-z)$ belongs to $\overline{\mathrm{Im}(I-T)}$. By the Hahn-Banach theorem, it is enough to prove that if $x^* \in E^*_+$ and $x^* = 0$ on $\overline{\mathrm{Im}(I-T)}$,

then $x^*[x - (y-z)] = 0$. But if $x^* = 0$ on $\overline{\mathrm{Im}(I-T)}$, this implies that $x^*(\omega) = x^*(T\omega)$ for each ω in E, hence for each $k>0$,
$x^*(x^+ \wedge ku) = x^*(T(x^+ \wedge ku)) = x^*(T^j(x^+ \wedge ku)) = x^*(A_{n_j}(x^+ \wedge ku))$ which converges to $x^*(y^k)$.

It follows that for each $k>0$, $x^*(x^+ \wedge ku) = x^*(y^k)$ and by letting k go to ∞ we get that $x^*(x^+) = x^*(y)$. Similarly $x^*(x^-) = x^*(z)$ from which follows that $x^*[x - (y-z)] = 0$.

Now, we can reprove the following result [17] without using the Riesz interpolation theorem.

<u>Corollary II.1</u> Every order contractive operator on an order continuous Banach lattice is mean ergodic.

<u>Proof</u> As noted above, T induces a contraction T_1 on $L_1(K,\mu)$ which is a contraction on $L_\infty(K,\mu)$, that is T_1 is a weak order contraction on the weakly sequentially complete Banach lattice $L_1(K,\mu)$. It follows by Theorem II.1 tht $A_n f$ converges in $L_1(K,\mu)$ for each f in E. If now $g \in L_\infty(K,\mu)$, then $(A_n g)_n$ is in the order interval $[-\|g\|_\infty u, \|g\|_\infty u]$ which is uniformly integrable, hence $A_n g$ converges strongly in E.

If now f is any element in E, then for each $\varepsilon>0$, there exists g in $L\infty(K,\mu)$ such that $\|f - g\|_E \le \varepsilon$. It follows that

$$\|A_n f - A_m f\|_E \le \|(A_n - A_m)g\| + \|(A_n - A_m)(f-g)\|$$
$$\le \|(A_n - A_m)g\| + 2\varepsilon$$

therefore lim sup $\|A_n f - A_m f\|_E \le 2\varepsilon$ and $A_n f$ converges strongly in E.

<u>Theorem II.2</u> If T is a positive mean ergodic contraction on an order continuous Banach lattice E, then for each $k>1$. T^k is mean ergodic. Moreover, if P_k is the associated projection on $\mathrm{Ker}(I-T^k)$, then $(P_k)_k$ is a "martingale" of projections $(P_m P_n = P_m$ if $m \le n)$ and for each $k \ge 1$ we have $(I+T+ \dots + T^{k-1})P_k = P$

<u>Proof</u> Fix $k>1$. It is easy to verify the following:

$$(4) \quad \sum_{i=0}^{k-1} T^i \sum_{i=0}^{n-1} (T^k)^i = \sum_{i=0}^{nk-1} T^i$$

If we denote by A_n^k the sequence $\frac{1}{n} \sum_{i=0}^{n-1} (T^k)^i$ then we get

(5) $\left(I + T + \dots + T^{k-1}\right) A_n^k = k\, A_{nk}$

By (3), it is enough to prove that for each x in E_+, the sequence $\left(A_n^k x\right)_n$ has a weak cluster point. For that note that (5) gives that $0 \leq A_n^k x \leq k\, A_{nk} x$. Hence $0 \leq kPx \leq \left(A_n^k x\right) \vee \left(kPx\right) \leq kA_{nk} x \vee kPx$. Since T is mean ergodic we get that $A_{nk} x$ converge to Px and $\lim_{n\to\infty} \left(A_n^k x\right) \vee kPx = kPx$. On the other hand, the sequence $\left\{\left(A_n^k x\right) \wedge \left(kPx\right)\right\}_n$ is in the weakly compact order interval $[0,kPx]$ hence there exists a subsequence (n_j) and $y \leq kPx$ such that $\left(A_{n_j}^k x\right) \wedge kPx$ goes weakly to y. Write now $A_n^k \wedge kPx + A_n^k \vee kPx = A_n^k + kPx$ to conclude that $\lim_{j\to\infty} A_{n_j}^k x = y$. If now P^k denotes the associated projection on $\mathrm{Ker}(I-T^k)$ we get from (5) that

$$\left(I + T + T^2 + \dots + T^{k-1}\right) P_k = kP, \text{ hence } PP_k = P.$$

Before proceeding to the study of subadditive processes we shall need the following concept: If T is a mean ergodic positive operator on a Banach lattice E and P is the associated projection on Ker(I-T), we shall say that T is <u>strongly positive</u> if P is strictly positive ($f \geq 0$ and $Pf = 0 \Rightarrow f = o$).

Suppose now T is a positive contraction on a weakly sequentially complete Banach lattice E and u is a positive fixed point of T, then $\overline{E}_u$ is T-stable by Proposition I.2.e and T is mean ergodic on $\overline{E}_u$ by Theorem II.1. We shall say that T is <u>locally strongly positive</u> if for every positive fixed element u in E_+, the restriction of T to $\overline{E}_u$ is strongly positive. Obviously, the two notions coincide if T has a fixed point which is a weak unit of E.

The following proposition gives sufficient conditions on the operator T and the space E to ensure strong positivity.

<u>Proposition II.1</u> a) If E has a strictly monotone norm then every positive contraction is locally strongly positive.

b) Every order contraction is locally strongly positive.

c) Every irreducible mean ergodic positive operator with non-zero fixed vectors is strongly positive.

Proof We can assume without loss of generality that E is a Banach lattice with a weak unit u that is T-invariant.
Let P be the associated projection on $\mathrm{Ker}(I-T)$. To prove then that T is strongly positive it is enough to show that if $0 \leq y \leq u = Pu$ and if $Py = Pu$ then $y = u$.

In case a) note that P is a contraction, hence $\|y\| = \|u\|$ since $\|y\| \leq \|u\| = \|Pu\| = \|Py\| \leq \|y\|$. The strict monotonicity of the norm gives then that $y = u$. For b) recall that T defines a positive contraction on the A-L space $L_1(K,\mu)$ associated to the functional μ in E^*. If now $0 \leq y \leq u = Pu$ and $Py = Pu$, the same reasoning as above applied to the L_1-norm $\|x\|_1 = \mu(|x|)$ gives that $\mu(Pu - y) = 0$ which implies that $y = Pu = u$ since μ is strictly positive. c) was proved in [17].

Now, we can prove the following;

Theorem II. 3 Let E be an order continuous Banach lattice. Let T be a locally strongly positive mean ergodic contraction, then for every positive T-subadditive sequence (s_n) in E, we have the strong convergence of $\frac{1}{n} s_n$ to $\inf \frac{1}{n} Ps_n = \inf \frac{1}{n} P_n s_n$ where P_n (resp P) are the projections on $\mathrm{Ker}(I-T^n)$ (resp $\mathrm{Ker}\ (I-T)$)

Proof Let P be the positive projection associated to the mean ergodic contraction T.
Since $s_{n+k} \leq s_n + T^n s_k$, we have since $PT = P$
$Ps_{n+k} \leq Ps_n + PT^n s_k = Ps_n + Ps_k$. It follows that (Ps_n) is a subadditive positive sequence and $(\frac{1}{k} Ps_k)$ converges strongly to $\inf\limits_k \frac{1}{k} Ps_k$ by the results of [12]. We can write $\inf\limits_k \frac{1}{k} Ps_k = Px$ for some x in E_+ since PE is closed.
We shall show that $\frac{1}{n} s_n$ converges to Px.

Fix $k > 0$ and suppose $n \geq k$. Let $N = N(n)$ be the integer part of $\frac{n}{k}$. By subadditivity we have

$$(6) \quad s_n \leq \sum_{n=1}^{N} T^{(r-1)k} s_k + T^{Nk} s_{n-Nk} \leq \sum_{n=1}^{N} T^{(r-1)k} s_k + \sum_{j=1}^{k-1} T^{Nk} s_j$$

Since T^k is mean ergodic for each k, we have

$$(7) \quad \lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} T^{(r-1)} s_k = P_k s_k.$$ On the other hand

(8) $\lim\limits_{N\to\infty} \frac{1}{N} \| \sum\limits_{j=1}^{k-1} T^{Nk} s_j \| \leq \lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{j=1}^{k-1} \|s_j\| = 0.$ Write now

(9) $\frac{1}{k} P_k s_k \leq \frac{1}{n} s_n \vee \frac{1}{k} P_k s_k \leq \left(\frac{1}{Nk} \sum\limits_{r=1}^{N} T^{(r-1)k} s_k + \frac{1}{Nk} \sum\limits_{j=1}^{k-1} T^{Nk} s_j \right) \vee \frac{1}{k} P_k s_k.$

Let $n\to\infty$, we get from (7) and (8)

(10) $\limsup\limits_{n} \left\| \frac{1}{n} s_n \vee \frac{1}{k} P_k s_k - \frac{1}{k} P_k s_k \right\| = 0$

On the other hand, for each $k > 0$, the sequence $\left(\frac{1}{n} s_n \wedge \frac{1}{k} P_k s_k\right)_n$ is in the order interval $\left[0, \frac{1}{k} P_k s_k\right]$. By diagonalization, there exists a subsequence (n_j) and vectors $y_k \leq \frac{1}{k} P_k s_k$ such that weak $\lim\limits_{j} \left(\frac{1}{n_j} s_{n_j} \wedge \frac{1}{k} P_k s_k\right) = y_k$ for each k.

By the identity

(11) $\frac{1}{n_j} s_{n_j} \wedge \frac{1}{k} P_k s_k + \frac{1}{n_j} s_{n_j} \vee \frac{1}{k} P_k s_k = \frac{1}{n_j} s_{n_j} + \frac{1}{k} P_k s_k.$

we get by letting $j\to\infty$, that all the y_k's are the same (say) y and that

$$\text{weak} \lim_{j} \frac{1}{n_j} s_{n_j} = y \leq \inf_{k} \frac{1}{k} P_k s_k \leq Ps_1$$

But $\frac{1}{n_j} Ps_{n_j}$ converges weakly to Py hence $Px = Py \leq P\left(\inf\limits_{k} \frac{1}{k} P_k s_k\right)$

$$\leq \inf_{k} \frac{1}{k} PP_k s_k = \inf \frac{1}{k} Ps_k = Px$$

Since T is locally strongly positive we deduce that

$y = \inf\limits_{k} \frac{1}{k} P_k s_k = \inf \frac{1}{k} Ps_k = Px.$

If follows that $\left(\frac{1}{n_j} s_{n_j} \wedge Px\right)_j$ goes weakly to Px, hence strongly by the results of [12]. Since the reasoning can be made for each subsequence (n_j) we get that $\left(\frac{1}{n} s_n\right) \wedge Px$ converges strongly to Px.

By rewriting (9) with Px replacing $\frac{1}{k} P_k S_k$ we get for each k

(12) $Px \leq \lim\limits_{n} \frac{1}{n} s_n \vee Px \leq \frac{1}{k} P_k s_k \vee Px = \frac{1}{k} P_k s_k$

hence $\lim\limits_{n} \frac{1}{n} s_n \vee Px = Px$ since $Px = \inf \frac{1}{k} P_k s_k$.

Use the identity (11) again to get $\lim_n \frac{1}{n} s_n = Px$

Corollary II.2 Suppose one of the following conditions are satisfied:

(a) T is a mean ergodic positive contraction on a Banach lattice E with an order continuous and strictly monotone norm.

(b) T is a mean ergodic irreducible positive contraction on an order continuous Banach lattice E.

(c) T is an order contraction on an order continuous Banach lattice E.

Then, for every positive T-subadditive process (s_n) in E, we have the strong convergence of $\frac{1}{n} s_n$.

Proof Follows from Theorem II.3, Corollary II.1 and Proposition II.1.

Now, we deal with non-necessarily positive subadditive processes. It is well known that this case reduces to the study of positive superadditive processes. Indeed, if (s_n) is a T-subadditive process then the process $s'_n = \sum_{i=0}^{n-1} T^i s_1 - s_n$ is a positive superadditive process. On the other hand, if one wants to reduce the problem to the case of positive subadditive processes in order to apply the above results, one need to investigate the possibility of finding an additive process above the given positive superadditive process. This idea was exploited first by Kingman [14] and extended by several authors (see [1], [6] and [12]. We shall give here an extension of an idea of Brunel-Sucheston [6] and it will cover all the cases known in the literature.

Let us say that a process (s_n) is of bounded T-variation if

(13) $\sup_m \frac{1}{m} \| \sum_{i=1}^{m} (s_i - Ts_{i-1}) \| = M < \infty$

Proposition II.2 Let T be a uniformly consistent positive operator on a weakly sequentially complete Banach lattice then, for every positive T-superadditive process of bounded T-variation (s_n) in E, there exists s in E_+ such that $0 \leq s_n \leq \sum_{i=0}^{n-1} T^i s$ for all $n \geq 1$.

Proof Let $\phi_m = \frac{1}{m} \sum_{i=1}^{m} (s_i - Ts_{i-1})$. A standard computational lemma [1] gives for $1 \leq m < n$

(14) $$\sum_{i=0}^{n-1} T^i \phi_m \geq \left(1 - \frac{n-1}{m}\right) s_n$$

Let now δ be a weak*-cluster point of (ϕ_m) in E^{**}. Since T^{**} is weak*-continuous we have

(15) $$\sum_{i=0}^{n-1} (T^{**})^i \delta \geq s_n \quad \text{for each } n \geq 1$$

Since T is uniformly consistent, Lemma I.4 applies to get $\sum_{i=0}^{n-1} T^i Q_{T^{**}} \delta \geq Qs_n = s_n$ for each $n \geq 1$. Note that $s = Q_{T^{**}}\delta$ belongs to E_+.

Corollary II.3 Let T be a uniformly consistent, locally strongly positive mean ergodic contraction on a weakly sequentially complete Banach lattice. Then for every T-subadditive process of bounded T-variation (s_n) we have the strong convergence of $\frac{1}{n} s_n$.

Proof The above remarks reduce the problem to positive superadditive sequences. The corollary follows from Theorem II.3 and Proposition II.2.

Corollary II.4 Suppose one of the following conditions is satisfied.

(a) T is a mean ergodic positive contraction on a Banach lattice E with a strictly monotone norm such that E is an L_1-ideal in its second dual.

(b) T is a positive contraction on a reflexive Banach lattice with a strictly monotone norm.

(c) T is a lattice homomorphism on a weakly sequentially complete Banach lattice with a strictly monotone norm, which maps a weak unit into itself. Then, for every T-subadditive process of bounded T-variation (s_n) we have the strong convergence of $\frac{1}{n} s_n$ in E.

Proof (a)(b) follow from Corollary II.3 and the fact shown in Proposition I.2.c that every positive contraction is uniformly consistent whenever E is an L_1-ideal in its second dual. (c) follows from Theorem II.1, Proposition I.3.b and Corollary II.3.

Problem II.1 Note that if T is order contractive and (s_n) is T-superadditive positive with bounded variation then the above applied to the contraction that T induces on $L_1(K,\mu)$ gives that $\frac{1}{n}s_n$ converges in the L_1-norm to $\sup_n \frac{1}{n} Ps_n \in E$. We do not know if the convergence holds in the norm of E.

III ON THE STOCHASTIC CONVERGENCE OF SUPERADDITIVE PROCESSES.

In this section, we are concerned with the stochastic convergence of Cesaro averages. We follow the ideas of Ackoglu-Sucheston in the case of L_1 [3] by noting that the restriction of a positive additive process to an absorbing set is positive superadditive. That is why we start our study of stochastic convergence with these processes.

First we recall the following renorming lemma due to Figiel-Johnson-Tzafriri [11].

Lemma III.1 Let P be a projection on a Banach lattice E. Let $|||\ \ |||$ be the semi-norm defined by $|||x||| = \sup\{\|Pg\|;\ |y| \leq |x|\}$, and let I be the ideal $= \{x\in E;\ |||x||| = 0\}$ then the completion of $(E/I,\ |||\ \ |||)$ is weakly sequentially complete if PE is. Moreover, the canonical map $Q:E\to(E/I,\ |||\ \ |||)$ is an isomorphism on PE.

In what follows the projection P will be positive, hence I will be the absolute Kernel of P that is $I = \{x\in E;\ P|x| = 0\}$ and the norm on E/I will be $|||x||| = \|P|x|\|$.

If T is a positive mean ergodic contraction on E and P is the associated projection, then T induces a canonical contraction on $(E/I,\ |||\ \ |||)$ since $|||Tx||| = \|P|Tx|\| \leq \|PT|x|\| = \|P|x|\| = |||x|||$.

If T is strongly positive, then P is strictly positive and we obtain then the completion of $(E,\ |||\ \ |||)$ for a weaker norm which is equivalent to the original one, on the invariant subspace of T.

Lemma III.2 Let T be a mean ergodic positive contraction on a weakly sequentially complete Banach lattice E such that the associated projection P verifies the following property:

(*) $\forall\varepsilon>0, \exists\delta>0$ such that if $\|f\| \geq \varepsilon$ and $f \geq 0$ then $\|Pf\| > \delta$ then for every positive T-superadditive process (s_n) verifying $\sup_n \frac{1}{n}\|s_n\| < \infty$, we have the strong convergence of $\frac{1}{n}s_n$.

Proof First note that the sequence (Ps_n) is superadditive and

$\sup_n \frac{1}{n} \|Ps_n\| < \infty$, hence it converges to $\sup \frac{1}{n} Ps_n = Px$ by the results of [12].

By superadditivity we get for a fixed $k > 0$, $n \geq k$ and $N = N(n)$ being the integer part of $\frac{n}{k}$

(16) $$0 \leq \sum_{r=1}^{N} T^{(r-1)k} s_k + T^{Nk} s_{n-NK} \leq s_n$$

As in Theorem I.3

$$\lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} T^{(r-1)k} s_k = P_k s_k \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} \|T^{Nk} s_{n-Nk}\| = 0$$

By writing

(17) $$0 \leq \left(\frac{1}{Nk} \sum_{r=1}^{N} T^{(r-1)k} s_k + T^{Nk} s_{n-Nk} \right) \wedge \frac{1}{k} P_k s_k \leq \left(\frac{1}{n} s_n\right) \wedge \left(\frac{1}{k} P_k s_k\right) \leq \frac{1}{k} P_k s_k.$$

we get for every $k > 0$

(18) $$\lim_{n\to\infty} \left\|\frac{1}{n} s_n \wedge \frac{1}{k} P_k s_k - \frac{1}{k} P_k s_k\right\| = 0$$

On the other hand, we have

(19) $$\lim_{k\to\infty} \limsup_{n\to\infty} \left\|\frac{1}{n} s_n - \left(\frac{1}{n} s_n\right) \wedge \frac{1}{k} P_k s_k\right\| = 0 .$$

Indeed, suppose not then there exists $\varepsilon > 0$ such that for any $k_0 > 0$ and any $\eta > 0$ we can find $k_1 > k_0$ such that

$$\left\|\frac{1}{k_1} s_{k_1} - \left(\frac{1}{k_1} s_{k_1}\right) \wedge \left(\frac{1}{k_0} P_{k_0} s_{k_0}\right)\right\| > \varepsilon$$

and in view of (18),

(20) $$\left\|\left(\frac{1}{n} s_n\right) \wedge \left(\frac{1}{k_0} P_{k_0} s_{k_0}\right) - \frac{1}{k_0} P_{k_0} s_{k_0}\right\| < \eta \qquad \text{for all } n \geq k_1$$

By (*) choose $\delta > 0$ such that $\|f\| \geq \varepsilon$ and $f \geq 0 \Rightarrow \|Pf\| > \delta$, and let $\eta = \frac{\delta}{2}$. Write now

(21) $$\frac{1}{k_1} s_{k_1} - \frac{1}{k_0} P_{k_0} s_{k_0} = \left(\frac{1}{k_1} s_{k_1} - \frac{1}{k_1} s_{k_1} \wedge \frac{1}{k_0} P_{k_0} s_{k_0}\right) - \left(\frac{1}{k_0} P_{k_0} s_{k_0} - \frac{1}{k_1} s_{k_1} \wedge \frac{1}{k_0} P_{k_0} s_{k_0}\right)$$

and apply P

$$(22)\quad \frac{1}{k_1} Ps_{k_1} - \frac{1}{k_0} Ps_{k_0} = P\Big(\frac{1}{k_1} s_{k_1} - \frac{1}{k_1} s_{k_1} \wedge \frac{1}{k_0}\, P_{k_0} s_{k_0}\Big) -$$
$$P\Big(\frac{1}{k_0} P_{k_0} s_{k_0} - \frac{1}{k_1} s_{k_1} \wedge \frac{1}{k_0} P_{k_0} s_{k_0}\Big).$$

We get $\|\frac{1}{k_1} Ps_{k_1} - \frac{1}{k_0} Ps_{k_0}\| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}$.

By repeating this procedure, we get a sequence (k_i) such that

$$(23)\qquad \|\frac{1}{k_i} Ps_{k_i} - \frac{1}{k_{i-1}} Ps_{k_{i-1}}\| \geq \frac{\delta}{2}$$

which contradicts the convergence of $\big(P\,\frac{1}{k}\, s_k\big)$, hence (19) holds.

To finish the proof write

$$\|\frac{s_n}{n} - \frac{s_m}{m}\| \leq \|\Big(\frac{s_n}{n} - \frac{s_n}{n} \wedge \frac{P_k s_k}{k}\Big) - \Big(\frac{s_m}{m} - \frac{s_m}{m} \wedge \frac{P_k s_k}{k}\Big)\|$$
$$+ \|\Big(\frac{s_n}{n} \wedge \frac{P_k s_k}{k} - \frac{P_k s_k}{k}\Big) - \Big(\frac{s_m}{m} \wedge \frac{P_k s_k}{k} - \frac{P_k s_k}{k}\Big)\|.$$

Use now (18) and (19) to deduce that $\big(\frac{1}{n}\, s_n\big)$ is a Cauchy sequence, hence it is convergent to an element $y \in E$. To identify the limit, notice that (18) gives that

$$y \wedge \frac{P_k s_k}{k} = \frac{P_k s_k}{k} \quad \text{hence} \quad \frac{P_k s_k}{k} \leq y.$$

(19) gives that $\lim\limits_{k\to\infty} \| y - y \wedge \frac{P_k s_k}{k} \| = 0$ from which follows that

$$y = \lim_k \frac{P_k s_k}{k} = \sup_k \frac{P_k s_k}{k}.$$

<u>Lemma III.3</u> Let T be a strongly positive mean ergodic contraction on a weakly sequentially complete Banach lattice. Let P be the associated projection. For every positive T-superadditive process (s_n) such that $\sup\limits_n \frac{1}{n} \|s_n\| < \infty$, we have the convergence of $\frac{s_n}{n}$ to $\sup\limits_k \frac{P_k s_k}{k}$ for the weaker norm $|||\ \ |||$.

Proof: Apply Lemma III.1 to deduce that the hypothesis of Lemma III.2 are satisfied for $(E, |||\ \ |||)$. Note that (*) is readily verified since $|||f||| = ||P|f|||$.

To come back to the original norm we need the following:

Lemma III.4 Let T be a strongly positive mean ergodic contraction on a weakly sequentially complete Banach lattice E. Let $|||\ \ |||$ be the norm associated to the projection P on Ker (I-T), then for every $u \in E_+$, the topologies induced by the norms $||\ \ ||$ and $|||\ \ |||$ are equivalent on [0,u].

Proof: Without loss we can assume that $0 \leq f_n \leq u$ and $||Pf_n|| \to 0$. Since [0,u] is weakly compact, there exists a subsequence (f_{m_k}) and f in [o,u] such that $f_{n_k} \to f$ weakly, hence $Pf_{n_k} \to Pf$ weakly and Pf = 0 which implies that $f_{n_k} \to 0$ weakly, hence strongly [2]. Since this is true for any subsequence, we get that $||f_n|| \to 0$.

Theorem III.1 If T is a strongly positive mean ergodic contraction on a weakly sequentially complete Banach lattice, then for every T-superadditive process (s_n) such that $\sup \frac{1}{n} ||s_n|| < \infty$, we have the stochastic convergence of $\frac{s_n}{n}$ to $\sup\limits_n \frac{P_n s_n}{n}$.

Proof: By Lemma III.3, $\frac{s_n}{n}$ converges to $\sup\limits_n \frac{1}{n} Ps_n = Px$ for the norm $|||\ \ |||$. That is for each $u \in E_+$, $\lim\limits_{n\to\infty} |||(\frac{1}{n} s_n - Px) \wedge u||| = 0$. Apply now Lemma III.4 to obtain that $\lim\limits_{n\to\infty} ||(\frac{1}{n} s_n - Px) \wedge u|| = 0$.

Lemma III.5 Let T be a positive operator on a Banach lattice G and let Q be a band projection from G onto a subideal E of G such that T is Q-stable. Then if (s_n) is a positive-T-superadditive process in G then (Qs_n) is a positive T-superadditive process in E.

Proof: Note that $(I-Q)T^nQ = 0$ for each $n \geq 0$, hence if $s_{n+k} \geq s_n + T^n s_k$ we get

$$Qs_{n+k} \geq Qs_n + QT^n s_k = Qs_n + T^n s_k - (I-Q)T^n s_k$$

$$= Qs_n + T^n s_k - (I-Q)T^n Qs_k - (I-Q)T^n(I-Q)\, s_k$$

$$= Qs_n + T^n s_k - (I-Q)T^n(I-Q)s_k \geq Qs_n + T^n s_k - T^n(I-Q)s = Qs_n + T^n Qs_k$$

<u>Theorem III.2</u> If T is a locally strongly positive contraction on a weakly sequentially complete Banach lattice and (s_n) is a positive T-superadditive process such that $\sup_n \frac{1}{n} \|s_n\| < \infty$ then for every T-invariant positive element u of E_+

1) $\frac{1}{n} Q_u(s_n)_n$ converges stochasticaly (Q_u is the projection on the support of u)

2) $\frac{s_n}{n} \wedge u$ converges strongly.

<u>Proof:</u> Note that u is a quasi-interior point for the ideal $\bar{E}_u$. Let $Q_u(x) = \sup_k (x \wedge ku)$ be the band projection on $\bar{E}_u$. Note that T is Q_u-stable by Proposition I.2. T is mean ergodic on $\bar{E}_u$ by Theorem II.1. Since T is locally strongly positive, T is strongly positive on $\bar{E}_u$. By Lemma III.5, $Q_u(s_n)$ is T-superadditive. Theorem III.1 applies and we get that $\frac{1}{n} Q_u s_n$ converges stochastically. For $(s_n \wedge u)_n$ note that $\frac{s_n}{n} \wedge u = \frac{1}{n} Q_u(s_n) \wedge u$ hence it is strongly convergent in view of the relation

$$\left| \frac{s_n}{n} \wedge u - \frac{s_m}{m} \wedge u \right| \leq Q_u\left(\frac{s_n}{n} - \frac{s_m}{m}\right) \wedge u.$$

<u>Corollary III.1</u> Let E be a weakly sequentially complete Banach lattice. Suppose one of the following conditions is verified

(a) T is a positive contraction and E has a strictly monotone norm,

(b) T is an order contraction,

then for every positive T-superadditive process (s_n) such that $\frac{1}{n} \sup \|s_n\| < \infty$, we have the stochastic convergence of $\frac{1}{n} Q_u s_n$ whenever Q_u is the projection on the support of a T-invariant element $u \in E_+$.

<u>Proof:</u> Follows immediately from Theorem III.2 and Proposition II.1 a) and b).

The following two lemmas are extensions of results of Akcoglu- Sucheston [3] [4].

Lemma III.6 Let E be a weakly sequentially complete Banach lattice and let (f_n) be a positive bounded sequence in E, such that $\lim_n \|f_n - Tf_n\| = 0$.

Then: either (f_n) converges stochastically to zero or there exists a non-zero g in E_+ such that $Tg = g$ for every consistent operator T on E. If (f_n) is contained in an ideal F of E then g is also contained in F. The same holds without any assumption on the contraction T if the norm on E** is strictly monotone.

Proof: Suppose that (f_n) does not converge stochastically to zero, that is $\operatorname{Lim\,sup} \|f_n \wedge u\| > 0$ for some quasi-interior point u of the ideal F generated by (f_n) . Let U be an ultrafilter on N. Note that $0 \leq f_n \wedge u \leq f_n$, hence if f_u is a U-weak-limit in E of $(f_n \wedge u)_n$ and f is a U- weak*-limit of (f_n) in E**, then $0 \leq f_u \leq f \wedge u$. Note that $f_u \neq 0$ since (f_n) does not converge stochastically to 0.

On the other hand, since $\lim_{n\to\infty} \|Tf_n - f_n\|$ and T** is weak*-continuous, we have $T^{**}f = f$, hence by the consistency condition and Lemma I .3, $TQf = Qf$ where Q is the band projection from E** onto E. Note now that $g = Qf = \sup_k f \wedge ku \geq f_u$ and g is different of 0.

Lemma III.7 If E be a weakly sequentially complete Banach lattice, then for every consistent positive contraction T, there exists a unique decomposition of E into the direct sum of two ideals $E = F \oplus F^{\perp}$ such that:

(i) F has a weak unit u so that $Tu = u$,

(ii) If (f_n) is a positive bounded sequence in E_+ such that $\lim_{b\to\infty} \|f_n - Tf_n\|$, then $(I-R)f_n$ converges stochastically to zero, where R is the band projection on F.

The same holds without any assumption on the contraction T if the norm on E** is supposed to be strictly monotone.

Proof: Let v be a weak unit of E. let $\mathcal{I}$ be the set of all closed ideals of E such that each I in $\mathcal{I}$ contains a weak unit u_I with $Tu_I = u_I$. For each I in $\mathcal{I}$ let v_I be the band component of v in I.
Since $T(u_{I_1} + u_{I_2}) = u_{I_1} + u_{I_2}$ which is a weak unit for $I_1 + I_2$ for any pair I_2, I_2 in $\mathcal{I}$, we get that $\{v_I;\ I \in \mathcal{I}\}$ is a directed subset of $[0,v]$;

hence $v_0 = \sup_I v_I$ exists and $v_0 = \sup_n v_{I_n}$ for some sequence (I_n) in $\mathfrak{I}$. It is clear that the ideal F generated by $u_0 = \sum_n \frac{u_{I_n}}{2^n \|u_n\|}$ is the largest ideal of E on which T induces a weak order contraction.

Suppose now (f_n) is a bounded sequence in E_+ such that $\lim \|Tf_n - f_n\| = 0$. Let $(I-R)T(I-R)$ be the induced operator on $F^\perp$. It is easy to see that $(I-R)T(I-R)$ is Q-consistent where Q is the band projection from $(F^\perp)^{**}$ onto $F^\perp$. Moreover, $(I-R)T(I-R)$ does not have any non-zero fixed point. Indeed, suppose $(I-R)T(I-R)\, z = z$ for some z in E_+, that is $z \in F^\perp$ and $(I-R)Tz = z$. It follows that $Tz \geq z$ and the sequence $(T^n z)$ is increasing and norm bounded since $\|T^n z\| \leq \|z\|$, hence it converges to w. Note that $Tw = w$ and $w \geq z$. But this implies that $T(u_0 + w) = u_0 + w$ which is a weak unit for the ideal generated by $u_0 + w$ which contradicts the maximality of F since $(I-R)(u_0 + w) = (I-R)w \geq (I-R)z = z$. It follows that $z = 0$.

On the other hand $TF \subseteq F$ hence $(I-R)TR = 0$. and

$$\|(I-R)f_n - (I-R)T(I-R)f_n\| = \|(I-R)f_n - (I-R)Tf_n + (I-R)TRf_n\|$$

$$\leq \|(I-R)(I-T)f_n\| \leq \|f_n - Tf_n\| .$$ It follows that $\lim \|(I-R)f_n - (I-R)T(I-R)f_n\| = 0$. Lemma III.7 applies to the operator $(I-R)T(I-R)$ to get that $(I-R)f_n$ converges stochastically to 0.

Following Ackoglu-Sucheston [3] we shall call F the positive part and $F^\perp$ the null part of T.

Now, we can prove the following:

<u>Theorem III.3</u> Let E be a weakly sequentially complete Banach lattice. Let T be a consistent and locally strongly positive contraction on E, then for each $f \in E$, the sequence $A_n f = \frac{1}{n}\sum_{k=0}^{n-1} T^i f$ converges stochastically for each f in E.

<u>Proof:</u> Since T is consistent, use Lemma III.7 to find the positive and null part of T(F and $F^\perp$ say). It follows that $(I-R)A_n f \to 0$ stochastically since $\lim_{n\to\infty} \|A_n - TA_n f\| = 0$. On the other hand $RA_n f$ is a positive T-superadditive process on F, hence it converges stochastically by Theorem III.2.

Corollary III.2 Suppose that either

a) E^{**} has a strictly monotone norm

or

b) E has a strictly monotone norm and E is an L_p-ideal in E^{**} for some p $(1 \le p < \infty)$, then every positive contraction on E is stochastically ergotic.

Proof: a) follows from Remark I .1 since it ensures that Qf is a fixed point for T whenever f is a fixed point for T^{**}. The fact that E has a strictly monotone norm insures that T is locally strongly positive.

b) follows from Theorem III.3 and Proposition I.2.

Corollary III.3 If E is an L_p-ideal in E^{**} for some p $(1 \le p < \infty)$, then every irreducible contraction is stochastically ergodic.

Proof: Since F is a T-stable ideal, then either $F = \{0\}$ which implies tht $(A_m x)$ goes to zero stochastically for each x in E. Otherwise $F = E$ and T is then mean ergodic. Note that if $f \in E_+$, then $RA_n f \le \sum_{i=0}^{n-1} T^i f$. If

$\left\| \sum_{n=0}^{\infty} R[T(I-R)]^k f \right\| = \|R_T f\| < \infty$, then we get $R_T f \in F$ and $0 \le RA_n f \le \sum_{i=0}^{n-1} T^i R_T f$

which in view of Theorem II.3 for positive-T subadditive processes, implies that $RA_n f$ actually converges strongly. Note that the condition $\|R_T f\| < \infty$ $f \in E_+$ is exactly the R-uniform consistency of T. In this case $A_n f$ can be decomposed into a strongly convergent part $RA_n f$ and another part $(I-R)A_n f$ which is stochastically convergent to zero. Here we summarize the cases where this holds.

Corollary III.4 Under the hypothesis of Corollary III.2, any of the following conditions imply that $\forall f \in E$, $A_n f$ is the sum of a norm converging part and a stochastically null part.

(a) T is a positive contraction and F is an L_1-ideal in E.

(b) T is a contractive lattice homomorphism.

(c) T is an irreducible positive contraction.

(d) T is an invertible positive contraction.

Proof: Any of these conditions imply that T is R-uniformly consistent (Proposition I.2)

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