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ON THE EXISTENCE OF HAMILTONIAN PATHS CONNECTING LAGRANGIAN SUBMANIFOLDS

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Abstract. We use a new variational method—based on the theory of anti-selfdual Lagrangians developed in [2] and [3]—to establish the existence of solutions of convex Hamiltonian systems that connect two given Lagrangian submanifolds in $\mathbb{R}^{2N}$. We also consider the case where the Hamiltonian is only semi-convex. A variational principle is also used to establish existence for the corresponding Cauchy problem.

Résumé. Une nouvelle méthode variationnelle—basée sur la théorie des Lagrangiens auto-adjoints developée récemment dans [2] et [3]—est utilisée pour établir l'existence de solutions de systèmes Hamiltoniens convexes, qui connectent deux sous-variétés Lagrangiennes données dans $\mathbb{R}^{2N}$. On considère aussi le cas des Hamiltoniens semi-convexes, ainsi que le problème de Cauchy correspondant.

1. Introduction.

We consider the following Hamiltonian System

$$\begin{cases} \dot{p}(t) \in \partial_2 H\big(p(t), q(t)\big) & t \in (0,T), \\ -\dot{q}(t) \in \partial_1 H\big(p(t), q(t)\big) & t \in (0,T), \end{cases} \tag{1}$$

where $H\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a convex and lower semi-continuous function and $T > 0$. We develop a new variational approach to establish existence of solutions satisfying two types of boundary conditions. The first one requires the path to connect two Lagrangian submanifolds associated to given convex lower semi-continuous functions ψ_1 and ψ_2 on $\mathbb{R}^N$, that is

$$q(0) \in \partial\psi_1\big(p(0)\big) \quad \text{and} \quad -p(T) \in \partial\psi_2\big(q(T)\big). \tag{2}$$

In other words, the Hamiltonian path must connect the graph of $\partial\psi_1$ to the graph of $-\partial\psi_2$. The second is simply an initial value problem of the form

$$p(0) = p_0, \quad q(0) = q_0 \tag{3}$$

where p_0 and q_0 are two given vectors in $\mathbb{R}^N$.

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Note that the graphs of $\partial\psi_1$ and $-\partial\psi_2$ are typical Lagrangian submanifolds in $\mathbb{R}^{2N}$, and the first problem can be seen as a Lagrangian intersection problem. The solutions will be obtained from a novel variational principle developed in full generality in a series of papers [2], [3] and [6]. It is based on the concept of anti-selfdual Lagrangians to which one associates action functionals whose infimum is necessarily equal to zero. The equations are then derived from the limiting case in Legendre–Fenchel duality as opposed to standard Euler–Lagrange theory.

In the next section, we start with the case of convex Hamiltonian systems connecting Lagrangian submanifolds. This is then extended to the semi-convex case in Section 4. The corresponding Cauchy problem is studied in Section 3. The case of periodic solutions was considered in [5].

2. Connecting Lagrangian submanifolds.

Given a time $T > 0$, we let $X = W^{1,2}(0,T;\mathbb{R}^N)$ be the one-dimensional Sobolev space endowed with the norm $\|u\| = (\|u\|^2_{L^2} + \|\dot{u}\|^2_{L^2})^{\frac{1}{2}}$ where $\|u\|_{L^2} = (\int_0^T |u|^2\,dt)^{\frac{1}{2}}$ stands for the norm on $L^2 := L^2(0,T;\mathbb{R}^N)$. For every $p,q \in \mathbb{R}^N$, $p\cdot q$ denotes the inner product in $\mathbb{R}^N$ and $(p,q)\cdot(r,s)$ denotes the inner product in $\mathbb{R}^N\times\mathbb{R}^N$ defined by $(p,q)\cdot(r,s) = p\cdot r + q\cdot s$.

Say that a Hamiltonian H on $\mathbb{R}^{2N}$ is β-subquadratic for $\beta > 0$, if for some positive constants α,γ, we have,

$$-\alpha \le H(p,q) \le \frac{\beta}{2}(|p|^2+|q|^2)+\gamma \quad \text{for all } (p,q)\in\mathbb{R}^{2N}. \tag{4}$$

We shall prove the following result.

Theorem 1. *Suppose $H\colon \mathbb{R}^{2N} \to \mathbb{R}$ is a convex lower semi-continuous β-subquadratic Hamiltonian with*

$$\beta < \frac{1}{2\max(2T^2,1)}. \tag{5}$$

Let ψ_1 and ψ_2 be two convex lower semi-continuous and coercive functions on R^N such that one of them satisfies the following condition:

$$\liminf_{|p|\to+\infty} \frac{\psi_i(p)}{|p|^2} > 2T \quad \textit{for } i = 1 \textit{ or } 2. \tag{6}$$

Then the minimum of the functional

$$\begin{aligned} I(p,q) := \int_0^T &\big[H\big(p(t),q(t)\big) + H^*\big(-\dot{q}(t),\dot{p}(t)\big) + 2\dot{q}(t)\cdot p(t)\big]\,dt \\ &+ \psi_2\big(q(T)\big) + \psi_2^*\big(-p(T)\big) + \psi_1\big(p(0)\big) + \psi_1^*\big(q(0)\big) \end{aligned}$$

on $Y = X \times X$ is zero and is attained at a solution of

$$
\begin{cases}
\dot{p}(t) \in \partial_2 H\big(p(t), q(t)\big) & t \in (0, T), \\
-\dot{q}(t) \in \partial_1 H\big(p(t), q(t)\big) & t \in (0, T), \\
q(0) \in \partial\psi_1\big(p(0)\big), & -p(T) \in \partial\psi_2\big(q(T)\big).
\end{cases} \tag{7}
$$

Before we proceed with the proof, we note that condition (5) is satisfied as soon as we have

$$
-\alpha \leq H(p, q) \leq \beta(|p|^r + |q|^r + 1) \quad (1 < r < 2)
$$

where α, β are any positive constants.

The proof requires a few preliminary lemmas, but first, and anticipating that at some point of the proof the conjugate H^* of H needs to be finite everywhere (*i.e.*, H coercive), we start by replacing H with the following perturbed Hamiltonian $H_\epsilon(p, q) = \frac{\epsilon}{2}(|p|^2 + |q|^2) + H(p, q)$ for some $\epsilon > 0$. It is then clear that

$$
\frac{1}{2(\beta + \epsilon)}(|p|^2 + |q|^2) - \gamma \leq H_\epsilon^*(p, q) \leq \frac{1}{2\epsilon}(|p|^2 + |q|^2) + \alpha. \tag{8}
$$

Lemma 1. *For any convex Hamiltonian H, and convex lower semi-continuous functions ψ_1, ψ_2, we have that $I(p, q) \geq 0$ for every $(p, q) \in X \times X$.*

Proof. Use that

$$
2\int_0^T \dot{q} \cdot p \, dt = \int_0^T \dot{q} \cdot p \, dt - \int_0^T q \cdot \dot{p} \, dt + q(T) \cdot p(T) - q(0) \cdot p(0)
$$

to write

$$
\begin{aligned}
I(p, q) :&= \int_0^T \big[H\big(p(t), q(t)\big) + H^*\big(-\dot{q}(t), \dot{p}(t)\big) + 2\dot{q}(t) \cdot p(t)\big] \, dt \\
&\quad + \psi_2\big(p(T)\big) + \psi_2^*\big(-q(T)\big) + \psi_1\big(p(0)\big) + \psi_1^*\big(q(0)\big) \\
&= \int_0^T \big[H\big(p(t), q(t)\big) + H^*\big(-\dot{q}(t), \dot{p}(t)\big) + \dot{q}(t) \cdot p(t) - \dot{p}(t) \cdot q(t)\big] \, dt \\
&\quad + \big[\psi_2\big(q(T)\big) + \psi_2^*\big(-p(T)\big) + p(T) \cdot q(T)\big] \\
&\quad + \big[\psi_1\big(p(0)\big) + \psi_1^*\big(q(0)\big) - p(0) \cdot q(0)\big] \\
&\geq 0
\end{aligned}
$$

by applying the Legendre–Fenchel inequality three times. $\square$

Lemma 2. *To a convex lower semi-continuous Hamiltonian H on $\mathbb{R}^{2N}$, we associate the "Functional Lagrangian" $L\colon Y\times Y\to\mathbb{R}\cup\{+\infty\}$ defined as*

$$\begin{aligned} L(r,s;p,q) := \int_0^T &[(\dot r,-\dot s)\cdot(q,p)+H^*(-\dot q,\dot p)-H^*(-\dot s,\dot r)+2\dot q\cdot p]\,dt\\ &-p(T)\cdot s(T)+\psi_2\big(q(T)\big)-\psi_2\big(s(T)\big)+r(0)\cdot q(0)\\ &+\psi_1\big(p(0)\big)-\psi_1\big(r(0)\big). \end{aligned}$$

Then we have $I(p,q)\le\sup_{(r,s)\in X\times X}L(r,s;p,q)$ for every $(p,q)\in Y$.

Proof. Indeed, set

$$A:=\Big\{r\in X: r(t)=\int_0^t f(\alpha)\,d\alpha+y,\ \text{for some } y\in\mathbb{R}^N \text{ and } f\in L_2(0,T;\mathbb{R}^N)\Big\}$$

and

$$B:=\Big\{s\in X: s(t)=-\int_t^T g(\alpha)\,d\alpha+x,\ \text{for some } x\in\mathbb{R}^N \text{ and } g\in L_2(0,T;\mathbb{R}^N)\Big\}$$

and note that for every $(p,q)\in X\times X$ we have

$$\begin{aligned} &\sup_{X\times X}L(r,s;p,q)\\ &\quad\ge\sup_{(r,s)\in A\times B}L(r,s;p,q)\\ &\quad=\sup_{\substack{f,g\in L_2\\ x,y\in\mathbb{R}^N}}\Big\{\int_0^T[(f,-g)\cdot(q,p)+H^*(-\dot q,\dot p)-H^*(-g,f)+2\dot q\cdot p]\,dt\Big\}\\ &\qquad\quad -x\cdot p(T)+\psi_2\big(q(T)\big)-\psi_2(x)+y\cdot q(0)+\psi_1\big(p(0)\big)-\psi_1(y)\}\\ &\quad=\sup_{f,g\in L^2}\Big\{\int_0^T[(f,-g)\cdot(q,p)+H^*(-\dot q,\dot p)-H^*(-g,f)+2\dot q\cdot p]\,dt\Big\}\\ &\qquad\quad+\sup_{x\in\mathbb{R}^N}\big\{-x\cdot p(T)\\ &\qquad\quad+\psi_2\big(q(T)\big)-\psi_2(x)\big\}+\sup_{y\in\mathbb{R}^N}\big\{y\cdot q(0)+\psi_1\big(p(0)\big)-\psi_1(y)\big\}\\ &\quad=\int_0^T[H^*(-\dot q,\dot p)+H(p,q)+2\dot q\cdot p]\,dt\\ &\qquad\quad+\big\{\psi_2\big(q(T)\big)+\psi_2^*\big(-p(T)\big)\big\}+\big\{\psi_1\big(p(0)\big)+\psi_1^*\big(q(0)\big)\big\} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T [H^*(-\dot q, \dot p) + H(p,q) + \dot q \cdot p - q \cdot \dot p]\, dt \\
&\qquad + \big[\psi_2\big(q(T)\big) + \psi_2^*\big(-p(T)\big) + p(T)\cdot q(T)\big] \\
&\qquad + \big[\psi_1\big(p(0)\big) + \psi_1^*\big(q(0)\big) - p(0)\cdot q(0)\big] \\
&= I(p,q).
\end{aligned}
$$

□

Lemma 3. *Under the above conditions, and assuming that ϵ is small enough so that $\beta+\epsilon < \frac{1}{2\max(2T^2,1)}$, we have the following coercivity property:*

$$L_\epsilon(0,0;p,q) \to +\infty \text{ as } \|p\| + \|q\| \to +\infty,$$

where L_ϵ is the functional Lagrangian associated to the perturbed Hamiltonian H_ϵ.

Proof. Without loss of generality we assume ψ_1 satisfies (6). An easy calculation shows that

$$\|p\|_{L^2} \le T\|\dot p\|_{L^2} + \sqrt{T}|p(0)|, \quad \|q\|_{L^2} \le T\|\dot q\|_{L^2} + \sqrt{T}|q(T)|. \tag{9}$$

Also note that $\frac{1}{2(\beta+\epsilon)}(|p|^2+|q|^2) - \gamma \le H_\epsilon^*(p,q)$, hence modulo a constant we have

$$L(0,0,p,q) \ge \frac{1}{2(\beta+\epsilon)} \int_0^T (|\dot q|^2 + |\dot p|^2)\, dt + 2\int_0^T \dot q \cdot p\, dt + \psi_2\big(q(T)\big) + \psi_1\big(p(0)\big). \tag{10}$$

Holder's inequality and inequality (9) for the second term on the right hand side of (10) imply

$$\left|\int_0^T \dot q \cdot p\, dt\right| \le \frac{1}{2}\int |p|^2\, dt + \frac{1}{2}\int |\dot q|^2\, dt \le T^2 \int |\dot p|^2\, dt + T|p(0)|^2 + \frac{1}{2}\int |\dot q|^2\, dt. \tag{11}$$

From (10) and (11), we get

$$
\begin{aligned}
L(0,0,p,q) \ge \frac{1}{2(\beta+\epsilon)} \int_0^T (|\dot q|^2 + |\dot p|^2)\, dt - \max(2T^2,1)\int_0^T (|\dot q|^2 + |\dot p|^2)\, dt \\
+ \psi_2\big(q(T)\big) + \psi_1\big(p(0)\big) - 2T|p(0)|^2
\end{aligned}
$$

which together with the coercivity condition on ψ_1 and ψ_2 and the fact that $\beta+\epsilon < \frac{1}{2\max(2T^2,1)}$ imply the claimed coercivity for L. □

The theorem is now a consequence of the following Ky Fan type min-max theorem which is essentially due to Brezis–Nirenberg–Stampachia (see [1]).

LEMMA 4. *Let D be an unbounded closed convex subset of a reflexive Banach space Y, and let $L(x,y)$ be a real valued function on $D \times D$ that satisfies the following conditions:*
(1) $L(x,x) \leq 0$ *for every* $x \in D$.
(2) *For each* $x \in D$, *the function* $y \to L(x,y)$ *is concave.*
(3) *For each* $y \in D$, *the function* $x \to L(x,y)$ *is weakly lower semi-continuous.*
(4) *The set* $D_0 = \{x \in D; L(x,0) \leq 0\}$ *is bounded in* Y.
Then there exists $x_0 \in D$ *such that* $\sup_{y \in D} L(x_0, y) \leq 0$.

PROOF OF THEOREM 1. It is easy to see that $\tilde{L}_\epsilon$, defined by

$$\begin{aligned}\tilde{L}_\epsilon(p,q;r,s) = L_\epsilon(r,s;p,q)& \\ = \int_0^T [(\dot{r}, -\dot{s}) \cdot (q,p) &+ H_\epsilon^*(-\dot{q}, \dot{p}) - H_\epsilon^*(-\dot{s}, \dot{r}) + 2\dot{q} \cdot p] \, dt \\ &- p(T) \cdot s(T) + \psi_2\big(q(T)\big) - \psi_2\big(s(T)\big) + r(0) \cdot q(0) \\ &+ \psi_1\big(p(0)\big) - \psi_1\big(r(0)\big),\end{aligned}$$

satisfies all the hypothesis of Lemma 4 on the space $Y = X \times X$. Indeed, from (8) it is clear that $L_\epsilon(p,q;p,q) = 0$, and Lemma 3 gives that the set $Y_0 = \{(p,q) \in Y; L_\epsilon(0,0;p,q) \leq 0\}$ is bounded in Y. The function $(r,s) \to L_\epsilon(r,s;p,q)$ is concave for every (p,q) while $(p,q) \to L_\epsilon(r,s;p,q)$ is weakly lower semi-continuous for every $(r,s) \in Y$. It follows that there exists (p_ϵ, q_ϵ) such that $\sup_{(r,s) \in X \times X} L_\epsilon(r,s;p_\epsilon,q_\epsilon) \leq 0$, so that by Lemma 2 we have

$$I(p_\epsilon, q_\epsilon) \leq \sup_{(r,s) \in X \times X} L_\epsilon(r,s;p_\epsilon,q_\epsilon) \leq 0.$$

On the other hand by Lemma 1 we have that $I_\epsilon(p_\epsilon, q_\epsilon) \geq 0$ which means that the latter is zero.

Now let $0 < \delta < \frac{1}{2\max(2T^2,1)} - \beta$. For each $0 < \epsilon < \delta$ there exist $(p_\epsilon, q_\epsilon) \in X \times X$ such that

(12)
$$\begin{aligned}I_\epsilon(p_\epsilon, q_\epsilon) := \int_0^T \big[H_\epsilon\big(p_\epsilon(t), q_\epsilon(t)\big) &+ H_\epsilon^*\big(-\dot{q}_\epsilon(t), \dot{p}_\epsilon(t)\big) + 2\dot{q}_\epsilon(t) \cdot p_\epsilon(t)\big] \, dt \\ &+ \psi_2\big(q_\epsilon(T)\big) + \psi_2^*\big(-p_\epsilon(T)\big) + \psi_1\big(p_\epsilon(0)\big) + \psi_1^*\big(q_\epsilon(0)\big) = 0.\end{aligned}$$

We shall show that (p_ϵ, q_ϵ) is bounded in $X \times X$. Indeed, similar to the proof of Lemma 3, we get

$$\left|2\int_0^T \dot{q}_\epsilon \cdot p_\epsilon \, dt\right| \leq \max(2T^2,1) \int_0^T (|\dot{q}_\epsilon|^2 + |\dot{p}_\epsilon|^2) \, dt + 2T|p_\epsilon(0)|^2. \tag{13}$$

Combining (12) and (13), we obtain

(14)
$$\int_0^T \big[H_\epsilon\big(p_\epsilon(t), q_\epsilon(t)\big) + H_\epsilon^*\big(-\dot{q}_\epsilon(t), \dot{p}_\epsilon(t)\big)\big] - \max(2T^2, 1) \int_0^T (|\dot{q}_\epsilon|^2 + |\dot{p}_\epsilon|^2)\, dt$$
$$+ \psi_2\big(q_\epsilon(T)\big) + \psi_2^*\big(-p_\epsilon(T)\big) + \psi_1\big(p_\epsilon(0)\big) + \psi_1^*\big(q_\epsilon(0)\big) - 2T|p_\epsilon(0)|^2 \le 0.$$

This inequality and the fact that H and ψ_i^*, $i = 1, 2$, are bounded from below guarantee the existence of a constant $C > 0$ independent of ϵ such that

$$\Big(\frac{1}{2(\beta+\delta)} - \max(2T^2, 1)\Big) \int_0^T (|\dot{q}_\epsilon|^2 + |\dot{p}_\epsilon|^2)\, dt$$
$$+ \psi_2\big(q_\epsilon(T)\big) + \psi_1\big(p_\epsilon(0)\big) - 2T|p_\epsilon(0)|^2 \le C.$$

The coercivity of ψ_1 and ψ_2 together with the fact that $\frac{1}{\beta+\delta} - 2\max(2T^2, 1) > 0$ then implies the boundedness of (p_ϵ, q_ϵ) in $X \times X$. Therefore, $(p_\epsilon, q_\epsilon) \rightharpoonup (\bar{p}, \bar{q})$ in $X \times X$, up to a subsequence.

Now we show that

$$I(\bar{p}, \bar{q}) \le \liminf_{\epsilon \to 0} I_\epsilon(p_\epsilon, q_\epsilon) = 0. \tag{15}$$

Indeed, first note that

$$\int_0^T H_\epsilon^*(\dot{p}_\epsilon, \dot{q}_\epsilon)\, dt := \inf_{u,v \in L^2(0,T;\mathbb{R}^N)} \int_0^T \Big[H^*(u, v) + \frac{\|\dot{p}_\epsilon - u\|^2}{2\epsilon} + \frac{\|\dot{q}_\epsilon - v\|^2}{2\epsilon}\Big]\, dt,$$

and since H^* is convex and lower semi-continuous, there exists $u_\epsilon, v_\epsilon \in L^2(0, T; \mathbb{R}^N)$ such that this infimum is attained at (u_ϵ, v_ϵ), *i.e.*,

$$H_\epsilon^*(\dot{p}_\epsilon, \dot{q}_\epsilon) = \int_0^T \Big[H^*(u_\epsilon, v_\epsilon) + \frac{\|\dot{p}_\epsilon - u_\epsilon\|^2}{2\epsilon} + \frac{\|\dot{q}_\epsilon - v_\epsilon\|^2}{2\epsilon}\Big]\, dt.$$

It follows from (14) and the boundedness of (p_ϵ, q_ϵ) in $X \times X$ that there exists a constant $C > 0$ not dependent on ϵ such that

$$H_\epsilon^*(\dot{p}_\epsilon, \dot{q}_\epsilon) = \int_0^T \Big[H^*(u_\epsilon, v_\epsilon) + \frac{\|\dot{p}_\epsilon - u_\epsilon\|^2}{2\epsilon} + \frac{\|\dot{q}_\epsilon - v_\epsilon\|^2}{2\epsilon}\Big]\, dt < C.$$

Since H^* is bounded from below, we have $\int_0^T [\|\dot{p}_\epsilon - u_\epsilon\|^2 + \|\dot{q}_\epsilon - v_\epsilon\|^2]\, dt < 4C\epsilon$ which means that $(u_\epsilon, v_\epsilon) \rightharpoonup (\dot{\bar{p}}, \dot{\bar{q}})$ in $L^2 \times L^2$. Hence

$$\begin{aligned}
\int_0^T H^*(\dot{\bar{p}}, \dot{\bar{q}})\, dt &\le \inf_{\epsilon \to 0} \int_0^T H^*(u_\epsilon, v_\epsilon)\, dt \\
&\le \inf_{\epsilon \to 0} \int_0^T \Big[H^*(u_\epsilon, v_\epsilon) + \frac{\|\dot{p}_\epsilon - u_\epsilon\|^2}{2\epsilon} + \frac{\|\dot{q}_\epsilon - v_\epsilon\|^2}{2\epsilon}\Big]\, dt \\
&= \inf_{\epsilon \to 0} \int_0^T H_\epsilon^*(\dot{p}_\epsilon, \dot{q}_\epsilon)\, dt.
\end{aligned}$$

We also have

$$\begin{aligned}\int_0^T H(\bar p,\bar q)\,dt &\le \inf_{\epsilon\to 0}\int_0^T H\big(p_\epsilon(t),q_\epsilon(t)\big)\,dt\\ &\le \inf_{\epsilon\to 0}\int_0^T \big[H\big(p_\epsilon(t),q_\epsilon(t)\big)+\epsilon\big(|p_\epsilon(t)|^2+|q_\epsilon(t)|^2\big)\big]\,dt\\ &= \inf_{\epsilon\to 0}\int_0^T H_\epsilon\big(p_\epsilon(t),q_\epsilon(t)\big)\,dt.\end{aligned}$$

Moreover, $\dot q_\epsilon \rightharpoonup \dot{\bar q}$ weakly and $p_\epsilon \to \bar p$ strongly in L^2, thus $\lim_{\epsilon\to 0}\int_0^T \dot q_\epsilon\cdot p_\epsilon\,dt = \int_0^T \dot{\bar q}\cdot\bar p\,dt$. Therefore

$$\begin{aligned}I(\bar p,\bar q) \le \liminf_{\epsilon\to 0}\int_0^T &\big[H_\epsilon\big(p_\epsilon(t),q_\epsilon(t)\big)+H_\epsilon^*\big(-\dot q_\epsilon(t),\dot p_\epsilon(t)\big)+2\dot q_\epsilon(t)\cdot p_\epsilon(t)\big]\,dt\\ &+\psi_2\big(q_\epsilon(T)\big)+\psi_2^*\big(-p_\epsilon(T)\big)+\psi_1\big(p_\epsilon(0)\big)+\psi_1^*\big(q_\epsilon(0)\big)=0.\end{aligned}$$

Since by Lemma 1, $I(\bar p,\bar q)\ge 0$, the latter is therefore zero, and it follows that

$$\begin{aligned}0&=I(\bar p,\bar q)\\ &=\int_0^T\big[H\big(\bar p(t),\bar q(t)\big)+H^*\big(-\dot{\bar q}(t),\dot{\bar p}(t)\big)+2\dot{\bar q}(t)\cdot\bar p(t)\big]\,dt\\ &\quad+\psi_2\big(\bar q(T)\big)+\psi_2^*\big(-\bar p(T)\big)+\psi_1\big(\bar p(0)\big)+\psi_1^*\big(\bar q(0)\big)\\ &=\int_0^T\big[H\big(\bar p(t),\bar q(t)\big)+H^*\big(-\dot{\bar q}(t),\dot{\bar p}(t)\big)+\dot{\bar q}(t)\cdot\bar p(t)-\dot{\bar p}(t)\cdot\bar q(t)\big]\,dt\\ &\quad+\big[\psi_2\big(\bar q(T)\big)+\psi_2^*\big(-\bar p(T)\big)+\bar p(T)\cdot\bar q(T)\big]\\ &\quad+\big[\psi_1\big(\bar p(0)\big)+\psi_1^*\big(\bar q(0)\big)-\bar p(0)\cdot\bar q(0)\big].\end{aligned}$$

The result is now obtained from the following three identities and from the limiting case in Legendre–Fenchel duality:

$$\begin{aligned}H\big(\bar p(t),\bar q(t)\big)+H^*\big(-\dot{\bar q}(t),\dot{\bar p}(t)\big)+\dot{\bar q}(t)\cdot\bar p(t)-\dot{\bar p}(t)\cdot\bar q(t)&=0,\\ \psi_2\big(\bar q(T)\big)+\psi_2^*\big(-\bar p(T)\big)+\bar p(T)\cdot\bar q(T)&=0,\\ \psi_1\big(\bar p(0)\big)+\psi_1^*\big(\bar q(0)\big)-\bar p(0)\cdot\bar q(0)&=0.\end{aligned}$$

□

3. The Cauchy problem for Hamiltonian systems. Here is our result for the corresponding Cauchy problem.

THEOREM 2. *Suppose $H\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a proper convex lower semi-continuous function such that $H(p,q) \to \infty$ as $|p|+|q| \to \infty$. Assume that*

$$-\alpha \le H(p,q) \le \beta(|p|^r + |q|^r + 1) \quad (1 < r < \infty) \tag{16}$$

where α, β are positive constants. Then the infimum of the functional

$$J(p,q) := \int_0^T \big[H\big(p(t),q(t)\big) + H^*\big(-\dot q(t), \dot p(t)\big) + \dot q(t)\cdot p(t) - \dot p(t)\cdot q(t)\big]\,dt$$

on the set $D := \{(p,q) \in X \times X; p(0) = p_0, q(0) = q_0\}$ is equal to zero and is attained at a solution of

$$\begin{cases} \dot p(t) \in \partial_2 H\big(p(t),q(t)\big) & t \in (0,T), \\ -\dot q(t) \in \partial_1 H\big(p(t),q(t)\big) & t \in (0,T) \\ \big(p(0),q(0)\big) = (p_0,q_0). \end{cases} \tag{17}$$

To prove Theorem 2, we first consider the subquadratic case $(1 < r < 2)$.

PROPOSITION 1. *Assume H is a proper convex and lower semi-continuous Hamiltonian that is subquadratic on $\mathbb{R}^N \times \mathbb{R}^N$. Then the infimum of the functional*

$$J(p,q) := \int_0^T \big[H\big(p(t),q(t)\big) + H^*\big(-\dot q(t), \dot p(t)\big) + \dot q(t)\cdot p(t) - \dot p(t)\cdot q(t)\big]\,dt \tag{18}$$

on $D := \{(p,q) \in X \times X, p(0) = p_0, q(0) = q_0\}$ is zero and is attained at a solution of (17).

PROOF OF PROPOSITION 1. As in the proof of Theorem 1, it is clear that $J(p,q) \ge 0$ for every $(p,q) \in X \times X$. For the reverse inequality, we may as in Section 1 consider a perturbed Hamiltonian H_ϵ to insure coercivity, and then pass to a limit when $\epsilon \to 0$. We therefore can and shall assume that H is coercive. We then introduce the following Hamiltonian

$$L(r,s;p,q) := \int_0^T \big[(\dot r, -\dot s)\cdot(q,p) + H^*(-\dot q, \dot p) - H^*(-\dot s, \dot r) + \dot q(t)\cdot p(t) - \dot p(t)\cdot q(t)\big]\,dt,$$

and we show that $I(p,q) \le \sup_{(r,s)\in D} L(r,s;p,q)$. Indeed, setting

$$A := \Big\{(r,s) \in D : r(t) = \int_0^t f(\alpha)d\alpha + p_0, s(t) = \int_0^t g(\alpha)\,d\alpha + q_0, \\ \text{for some } f,g \in L_2(0,T;\mathbb{R}^N)\Big\}.$$

We have

$$
\begin{aligned}
&\sup_{(r,s)\in D} L(r,s;p,q)\\
&\quad \geq \sup_{(r,s)\in A} L(r,s;p,q)\\
&\quad = \sup_{f,g\in L^2}\left\{\int_0^T [(f,-g)\cdot(q,p) + H^*(-\dot{q},\dot{p}) - H^*(-g,f) + \dot{q}p - \dot{p}\cdot q]\,dt\right\}\\
&\quad = \int_0^T [H^*(-\dot{q},\dot{p}) + H(p,q) + \dot{q}\cdot p - \dot{p}\cdot q]\,dt\\
&\quad = I(p,q).
\end{aligned}
$$

The rest follows in the same way as in the proof of Theorem 1; that is, the subquadraticity of H gives the right coercivity for L, and we are able to apply Ky Fan's min-max principle as in Theorem 1 to find $(\bar{p},\bar{q})\in D$ such that $J(\bar{p},\bar{q})=0$. □

Now, we deal with the general case, that is when (16) holds with $r>2$. For that we shall use an unusual variation of the standard inf-convolution procedure to reduce the problem to the subquadratic case where Proposition 1 applies. For every $\lambda>0$, define

$$
H_\lambda(p,q) := \inf_{u,v\in\mathbb{R}^N}\left\{H(u,v) + \frac{\|p-u\|_s^s}{s\lambda^s} + \frac{\|q-v\|_s^s}{s\lambda^s}\right\} \tag{19}
$$

where $s=\frac{r}{r-1}$. Obviously, $1<s<2$, and since H is convex and lower semi-continuous, the infimum in (19) is attained, so that for every $p,q\in\mathbb{R}^N$, there exist unique points $i(p), j(q)\in\mathbb{R}^N$ such that

$$
H_\lambda(p,q) = H\big(i(p),j(q)\big) + \frac{\|p-i(p)\|_s^s}{s\lambda^s} + \frac{\|q-j(q)\|_s^s}{s\lambda^s}. \tag{20}
$$

LEMMA 5. *The regularized Hamiltonian H_λ satisfies the following properties:*

(i) $H_\lambda(p,q)\to H(p,q)$ *as* $\lambda\to 0^+$.

(ii) $H_\lambda(p,q) \leq H(0,0) + \dfrac{\|q\|_s^s + \|p\|_s^s}{s\lambda^s}$.

(iii) $H_\lambda^*(p,q) = H^*(p,q) + \dfrac{\lambda^r}{r}(\|p\|_r^r + \|q\|_r^r)$.

Proof. (i) and (ii) are easy. For (iii), we have

$$\begin{aligned}
H_\lambda^*(p,q) &= \sup_{u,v\in\mathbb{R}^N}\{u\cdot p+v\cdot q-H_\lambda(u,v)\}\\
&= \sup_{u,v\in\mathbb{R}^N}\left\{u\cdot p+v\cdot q-\inf_{z,w\in\mathbb{R}^N}\left\{H(z,w)+\frac{\|z-u\|_s^s+\|w-v\|_s^s}{s\lambda^s}\right\}\right\}\\
&= \sup_{u,v\in\mathbb{R}^N}\sup_{z,w\in\mathbb{R}^N}\left\{u\cdot p+v\cdot q-H(z,w)-\frac{\|z-u\|_s^s}{s\lambda^s}-\frac{\|w-v\|_s^s}{s_{\lambda_s}}\right\}\\
&= \sup_{z,w\in\mathbb{R}^N}\sup_{u,v\in\mathbb{R}^N}\Big\{(u-z)\cdot p+(v-w)\cdot q+z\cdot p+w\cdot q\\
&\qquad\qquad -H(z,w)-\frac{\|z-u\|_s^s+\|w-v\|_s^s}{s\lambda^s}\Big\}\\
&= \sup_{z,w\in\mathbb{R}^N}\sup_{u_1,v_1\in\mathbb{R}^N}\Big\{u_1\cdot p+v_1\cdot q-\frac{\|u_1\|_s^s}{s\lambda^s}-\frac{\|v_1\|_s^s}{s\lambda^s}\\
&\qquad\qquad +z\cdot p+w\cdot q-H(z,w)\Big\}\\
&= \sup_{u_1,v_1\in\mathbb{R}^N}\left\{u_1\cdot p+v_1\cdot q-\frac{\|u_1\|_s^s}{s\lambda^s}-\frac{\|v_1\|_s^s}{s\lambda^s}\right\}\\
&\quad+\sup_{z,w\in\mathbb{R}^N}\{z\cdot p+w\cdot q-H(z,w)\}\\
&= \frac{\lambda^r}{r}(\|p\|_r^r+\|q\|_r^r)+H^*(p,q).
\end{aligned}$$

□

Now consider the Cauchy problem associated to H_λ. By Proposition 1, there exists $(p_\lambda,q_\lambda)\in X\times X$ such that $p_\lambda(0)=p_0, q_\lambda(0)=q_0$ and

$$0=I(p_\lambda,q_\lambda)=\int_0^T[H_\lambda(p_\lambda,q_\lambda)+H_\lambda^*(-\dot q_\lambda,p_\lambda)+\dot q_\lambda\cdot p_\lambda-\dot p_\lambda\cdot q_\lambda]\,dt \tag{21}$$

yielding

$$\begin{cases}\dot p_\lambda\in\partial_2H_\lambda(p_\lambda,q_\lambda)\\ -\dot q_\lambda\in\partial_1H_\lambda(p_\lambda,q_\lambda)\end{cases}$$
$$p_\lambda(0)=p_0,\ q_\lambda(0)=q_0.$$

From (20), we have

$$H_\lambda(p_\lambda,q_\lambda)=H\big(i_\lambda(p_\lambda),j_\lambda(q_\lambda)\big)+\frac{\|p_\lambda-i_\lambda(p_\lambda)\|_s^s}{s\lambda^s}+\frac{\|q_\lambda-j_\lambda(q_\lambda)\|_s^s}{s\lambda^s}. \tag{22}$$

We now relate (p_λ,q_λ) to the original Hamiltonian.

Lemma 6. *For every $\lambda > 0$, we have*

$$\begin{cases} \dot{p}_\lambda \in \partial_2 H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big) \\ -\dot{q}_\lambda \in \partial_1 H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big). \end{cases}$$

Proof. From (21) and the definition of Legendre–Fenchel duality we can write

$$H_\lambda(p_\lambda, q_\lambda) + H^*_\lambda(-\dot{q}_\lambda, p_\lambda) + \dot{q}_\lambda \cdot p_\lambda - \dot{p}_\lambda \cdot q_\lambda = 0 \quad \forall t \in (0, T). \tag{23}$$

Part (iii) of Lemma 5, together with (22) and (23), give

$$\begin{aligned} 0 = H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big) &+ \frac{\|p_\lambda - i_\lambda(q_\lambda)\|_s^s + \|q_\lambda - j_\lambda(q_\lambda)\|_s^s}{s\lambda^s} \\ &+ H^*(-\dot{q}_\lambda, \dot{p}_\lambda) + \frac{\lambda^r}{r}\big(\|\dot{p}_\lambda\|_r^r + \|\dot{q}_\lambda\|_r^r\big) + \dot{q}_\lambda \cdot p_\lambda - \dot{p}_\lambda \cdot q_\lambda. \end{aligned} \tag{24}$$

Note that

$$p_\lambda \cdot \dot{q}_\lambda = \big(p_\lambda - i_\lambda(p_\lambda)\big) \cdot \dot{q}_\lambda + i_\lambda(p_\lambda) \cdot \dot{q}_\lambda \quad \text{and} \quad \dot{p}_\lambda \cdot q_\lambda = \big(q_\lambda - j_\lambda(q_\lambda)\big) \cdot \dot{p}_\lambda + \big(\dot{p}_\lambda \cdot j_\lambda(q_\lambda)\big). \tag{25}$$

By Young's inequality, we have

$$\big|\big(p_\lambda - i_\lambda(p_\lambda)\big) \cdot \dot{q}_\lambda\big| \le \frac{\|p_\lambda - i_\lambda(p_\lambda)\|_s^s}{s\lambda^s} + \frac{\lambda^r}{r}\|\dot{q}_\lambda\|_r^r \tag{26}$$

$$\big|\big(q_\lambda - j_\lambda(q_\lambda)\big) \cdot \dot{p}_\lambda\big| \le \frac{\|q_\lambda - j_\lambda(q_\lambda)\|_s^s}{s\lambda^s} + \frac{\lambda^r}{r}\|\dot{p}_\lambda\|_r^r. \tag{27}$$

Combining (24)–(27) gives

$$\begin{aligned} 0 &= H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big) + \frac{\|p_\lambda - i_\lambda(q_\lambda)\|_s^s + \|q_\lambda - j_\lambda(q_\lambda)\|_s^s}{s\lambda^s} \\ &\quad + H^*(-\dot{q}_\lambda, \dot{p}_\lambda) + \frac{\lambda^r}{r}(\|\dot{p}_\lambda\|_r^r + \|\dot{q}_\lambda\|_r^r) + \big(p_\lambda - i_\lambda(p_\lambda)\big) \cdot \dot{q}_\lambda \\ &\quad + i_\lambda(p_\lambda) \cdot \dot{q}_\lambda - \big(q_\lambda - j_\lambda(q_\lambda)\big) \cdot \dot{p}_\lambda - \dot{p}_\lambda \cdot j_\lambda(q_\lambda) \\ &\ge H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big) + \frac{\|p_\lambda - i_\lambda(q_\lambda)\|_s^s + \|q_\lambda - j_\lambda(q_\lambda)\|_s^s}{s\lambda^s} \\ &\quad + H^*(-\dot{q}_\lambda, \dot{p}_\lambda) + \frac{\lambda^r}{r}(\|\dot{p}_\lambda\|_r^r + \|\dot{q}_\lambda\|_r^r) + i_\lambda(p_\lambda) \cdot \dot{q}_\lambda - \dot{p}_\lambda \cdot j_\lambda(q_\lambda) \\ &\quad - \frac{\lambda^r}{r}(\|\dot{p}_\lambda\|_r^r + \|\dot{q}_\lambda\|_r^r) - \frac{\|p_\lambda - i_\lambda(p_\lambda)\|_s^s + \|q_\lambda - j_\lambda(q_\lambda)\|_s^s}{s\lambda^s} \\ &= H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big) + H^*(-\dot{q}_\lambda, \dot{p}_\lambda) + i_\lambda(p_\lambda) \cdot \dot{q}_\lambda - \dot{p}_\lambda \cdot j_\lambda(q_\lambda). \end{aligned}$$

On the other hand, by the definition of Fenchel–Legendre duality

$$H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big) + H^*(-\dot q_\lambda, \dot p_\lambda) + i_\lambda(p_\lambda)\cdot \dot q_\lambda - \dot p_\lambda \cdot j_\lambda(q_\lambda) \geq 0 \tag{28}$$

which means we have equality in (28), so that

$$\begin{cases} \dot p_\lambda \in \partial_2 H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big), \\ -\dot q_\lambda \in \partial_1 H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big). \end{cases}$$

□

Lemma 7. *With the above notation we have:*
(1) $\sup_{t\in(0,T)} |q_\lambda - j_\lambda(q_\lambda)| + |p_\lambda - i_\lambda(p_\lambda)| \leq c\lambda$, *where c is a constant.*
(2) *If $H(p,q)\to\infty$ as $|p|+|q|\to\infty$ then*

$$\sup_{t\in(0,T),\lambda>0} |q_\lambda| + |j_\lambda(q_\lambda)| + |p_\lambda| + |i_\lambda(p_\lambda)| < \infty.$$

Proof. For every $\lambda>0$ and $t\in(0,T)$, we have

$$\begin{cases} \dot p_\lambda = \partial_2 h_\lambda(p_\lambda, q_\lambda), \\ -\dot q_\lambda = \partial_1 h_\lambda(p_\lambda, q_\lambda). \end{cases}$$

Multiplying the first equation by $\dot q_\lambda$ and the second one by $\dot p_\lambda$ gives

$$\begin{cases} \dot p_\lambda \dot q_\lambda = \dot q_\lambda \partial_2 H_\lambda(p_\lambda, q_\lambda), \\ -\dot q_\lambda p_\lambda = \dot p_\lambda \partial_1 H_\lambda(p_\lambda, q_\lambda). \end{cases}$$

So $\frac{d}{dt}H_\lambda(p_\lambda,q_\lambda) = 0$ and $H_\lambda\big(p_\lambda(t), q_\lambda(t)\big) = H_\lambda\big(p(0),q(0)\big) \leq H\big(p(0),q(0)\big) := c < +\infty$. Hence, it follows from (22) that

$$H\Big(i_\lambda\big(p_\lambda(t)\big), j_\lambda\big(q_\lambda(t)\big)\Big) + \frac{\|p_\lambda - i_\lambda(p_\lambda)\|_s^s + \|q_\lambda - j_\lambda(q_\lambda)\|_s^s}{s\lambda^s} \leq c$$

which yields

$$\sup_{t\in(0,T]} |q_\lambda - j_\lambda(q_\lambda)| + |p_\lambda - i_\lambda p_\lambda| \leq c\lambda$$

and

$$\sup_{t\in(0,T]} H\Big(i_\lambda\big(p_\lambda(t)\big), j_\lambda\big(q_\lambda(t)\big)\Big) < +\infty.$$

Since H is coercive the last equation gives $\sup_{t\in(0,T)\lambda} |j_\lambda(q_\lambda)| + |i_\lambda p_\lambda| < \infty$, which together with part (i) prove the lemma.

□

LEMMA 8. *We have the following estimate:*

$$\sup_{t\in[0,T],\lambda>0} |\dot{p}_\lambda(t)| + |\dot{q}_\lambda(t)| < +\infty.$$

PROOF. Since $-\alpha < H(p,q) \leq \beta|p|^r + \beta|q|^r + \beta$ where $r > 2$, an easy calculation shows that if $(p^*, q^*) \in \partial H(p,q)$ then

$$|p^*| + |q^*| \leq \Big\{ s(2\beta)^{\frac{r}{s}} (|p| + |q| + \alpha + \beta) + 1 \Big\}^{r-1}. \tag{29}$$

Since by Lemma 6 we have $(\dot{p}_\lambda, -\dot{q}_\lambda) = \partial H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big)$, it follows from (29) that

$$|\dot{p}_\lambda| + |\dot{q}_\lambda| \leq \big\{ s(2\beta)^{\frac{r}{s}} \big(|i_\lambda(p_\lambda)| + |j_\lambda(q_\lambda)| + \alpha + \beta\big) + 1 \big\}^{r-1}$$

which together with Lemma 7 prove the desired result. □

END OF PROOF OF THEOREM 2. From Lemma 6, we have

$$\int_0^T \Big[H\big(i_\lambda(p_\lambda), j_\lambda(q_\lambda)\big) + H^*(-\dot{q}_\lambda, \dot{p}_\lambda) + \dot{q}_\lambda \cdot i_\lambda(p_\lambda) - \dot{p}_\lambda \cdot j_\lambda(q_\lambda) \Big]\, dt = 0, \tag{30}$$

while $p_\lambda(0) = p_0$ and $q_\lambda(0) = q_0$. By Lemma 8, $\dot{p}_\lambda$ and $\dot{q}_\lambda$ are bounded in $L^2(0,T;\mathbb{R}^N)$, so there exists $(p,q) \in X \times X$ such that $\dot{p}_\lambda \rightharpoonup \dot{p}$ and $\dot{q}_\lambda \rightharpoonup \dot{q}$ weakly in $L^2(0,T;\mathbb{R}^N)$ and $p_\lambda \to p$ and $q_\lambda \to q$ strongly in $L_\infty(0,T;\mathbb{R}^N)$. So by Lemma 7, $i_\lambda(p_\lambda) \to p$ and $j_\lambda(q_\lambda) \to q$ strongly in $L_\infty(0,T;\mathbb{R}^N)$. Hence by letting $\lambda \to 0$ in (30), we get

$$\int_0^T [H(p,q) + H^*(-\dot{q}, \dot{p}) + \dot{q} \cdot p - \dot{p} \cdot q]\, dt \leq 0,$$

which means $p(0) = p_0$ and $q(0) = q_0$ and

$$\begin{cases} \dot{p}(t) = \partial_2 H\big(p(t), q(t)\big) \\ -\dot{q}(t) = \partial_1 H\big(p(t), q(t)\big). \end{cases}$$

□

4. **Semi-convex Hamiltonian systems.**

In this section, we consider the following system:

$$\begin{cases} \dot{p}(t) \in \partial_2 H\big(p(t), q(t)\big) + \delta_1 q(t) & t \in (0,T) \\ -\dot{q}(t) \in \partial_1 H\big(p(t), q(t)\big) + \delta_2 p(t) & t \in (0,T) \\ q(0) \in \partial\psi_1\big(p(0)\big) & -p(T) \in \partial\psi_2\big(q(T)\big) \end{cases} \tag{31}$$

where $\delta_1, \delta_2 \in \mathbb{R}$. Note that if $\delta_i \geq 0$ then the problem reduces to the one studied in Section 1 with a new convex Hamiltonian $\tilde{H}(p,q) = H(p,q) + \frac{\delta_1}{2}|q|^2 + \frac{\delta_2}{2}|p|^2$. The case that concerns us here is when $\delta_i < 0$.

THEOREM 3. *Suppose $H\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a proper convex lower semi-continuous Hamiltonian that is β-subquadratic with*

$$\beta < \frac{1}{4}\min\Big\{\frac{1-4T^2|\delta_1|^2}{\max(2T^2,1)-2\delta_1 T^2}, \frac{1-4T^2|\delta_2|^2}{\max(2T^2,1)-2\delta_2 T^2}\Big\}. \tag{32}$$

Assume

$$|\delta_i| < \frac{1}{2T} \quad \text{for } i=1,2, \tag{33}$$

and let ψ_1 and ψ_2 be convex lower semi-continuous functions on $\mathbb{R}^N$ satisfying
(34)

$$\liminf_{|p|\to+\infty} \frac{\psi_1(p)}{|p|^2} > \frac{T|\delta_2|^2}{\beta} + 2T(1-\delta_2) \quad \text{and} \quad \liminf_{|p|\to+\infty} \frac{\psi_2(p)}{|p|^2} > \frac{T|\delta_1|^2}{\beta} - 2T\delta_1.$$

Then the minimum of the functional

$$\begin{aligned} I(p,q) := \int_0^T &\big[H\big(p(t),q(t)\big) + H^*\big(-\dot q(t) - \delta_2 p(t), \dot p(t) - \delta_1 q(t)\big) \\ &+ \delta_1|q|^2 + \delta_2|p|^2 + 2\dot q(t)\cdot p(t)\big]\,dt \\ &+ \psi_2\big(q(T)\big) + \psi_2^*\big(-p(T)\big) + \psi_1\big(p(0)\big) + \psi_1^*\big(q(0)\big) \end{aligned}$$

on $Y = X \times X$ is equal to zero and is attained at a solution of (31).

By considering a perturbed Hamiltonian H_ϵ, then passing to a limit when $\epsilon \to 0$ as in Section 1, we can and shall assume that H is coercive. Also, note that for every $(p,q)\in Y$,

$$\begin{aligned} I(p,q) = \int_0^T &\big[H\big(p(t),q(t)\big) + H^*\big(-\dot q(t) - \delta_2 p(t), \dot p(t) - \delta_1 q(t)\big) \\ &+ \delta_1|q|^2 + \delta_2|p|^2 + \dot q(t)\cdot p(t) - \dot p(t)\cdot q(t)\big]\,dt \\ &+ \big[\psi_2\big(p(T)\big) + \psi_2^*\big(-q(T)\big) + p(T)\cdot q(T)\big] \\ &+ \big[\psi_1\big(p(0)\big) + \psi_1^*\big(q(0)\big) - p(0)\cdot q(0)\big] \\ \geq 0,& \end{aligned}$$

by three applications of Legendre inequality.

For the reverse inequality, we introduce the following functional Lagrangian $L\colon Y\times Y \to \mathbb{R}$ defined by

$$\begin{aligned} L(r,s;p,q) := \int_0^T &[H^*(-\dot q - \delta_2 p, \dot p - \delta_1 q) - H^*(-\dot s - \delta_2 r, \dot r - \delta_1 s) \\ &+ (\dot r - \delta_1 s, -\dot s - \delta_2 r)\cdot(q,p) + \delta_1|q|^2 + \delta_2|p|^2 + 2\dot q\cdot p]\,dt \\ &- p(T)\cdot s(T) + \psi_2\big(q(T)\big) - \psi_2\big(s(T)\big) \\ &+ r(0)\cdot q(0) + \psi_1\big(p(0)\big) - \psi_1\big(r(0)\big). \end{aligned}$$

In order to apply the anti-selfduality argument, we need the following lemma.

LEMMA 9. *For any $f, g \in L^2(0,T;\mathbb{R}^N)$ and $x, y \in \mathbb{R}^N$, there exists $(r,s) \in X \times X$ such that*

$$\begin{cases} \dot r(t) = \delta_2 s(t) + f(t) \\ -\dot s(t) = \delta_1 r(t) + g(t) \\ r(0) = x \\ s(T) = y. \end{cases} \tag{35}$$

PROOF. This is standard and is essentially a linear system of ordinary differential equations. Also, one can rewrite the problem as follows.

$$\begin{cases} -\dot r(t) + f(t) = -\partial_2 G\big(r(t), s(t)\big) \\ \dot s(t) + g(t) = -\partial_1 G\big(r(t), s(t)\big) \\ r(0) = x \\ s(T) = y \end{cases}$$

where $G\big(r(t), s(t)\big) = -\frac{\delta_1}{2}\int_0^T |r(t)|^2\,dt - \frac{\delta_2}{2}\int_0^T |s(t)|^2\,dt$. Hence

$$G^*\big(\dot s(t) + g(t), -\dot r(t) + f(t)\big) = -\frac{1}{2\delta_2}\int |\dot s(t) + g(t)|^2 - \frac{1}{2\delta_1}\int |-\dot r(t) + f(t)|^2\,dt.$$

One can show as in Theorem 1 that whenever $|\delta_i| < \frac{1}{2T}$, coercivity holds and the following infimum is achieved at a solution of (35):

$$\begin{aligned} 0 = \inf_{(r,s)\in D\subseteq X\times X} & G^*\big(\dot s(t) + g(t), -\dot r(t) + f(t)\big) + G\big(r(t), s(t)\big) \\ & + \int_0^T \dot r(t)\cdot s(t)\,dt - \int_0^T \dot s(t)\cdot r(t)\,dt - \int_0^T \big(f(t)\cdot s(t) + r(t)\cdot g(t)\big)\,dt, \end{aligned}$$

where $D = \{(r,s) \in X \times X \mid r(0) = x, s(T) = y\}$. $\square$

LEMMA 10. *For every $(p,q) \in X \times X$, we have*

$$I(p,q) \le \sup_{(r,s)\in X\times X} L(r,s;p,q).$$

PROOF. Use the above lemma to write

$$\begin{aligned}
&\sup_{(r,s)\in X\times X} L(r,s;p,q)\\
&\quad = \sup_{f,g\in L^2}\sup_{x,y\in\mathbb{R}^N}\int_0^T \Big[(f,g)\cdot(q,p) + H^*(-\dot q - \delta_2 p, \dot p - \delta_1 q)\\
&\qquad\qquad\qquad - H^*(g,f) + \delta_1|q|^2 + \delta_2|p|^2 + 2\dot q\cdot p\Big]\,dt\\
&\qquad - p(T)\cdot y + \psi_2\big(q(T)\big) - \psi_2(y) + x\cdot q(0) + \psi_1\big(p(0)\big) - \psi_1(x)\\
&\quad = \int_0^T \Big[H^*(-\dot q - \delta_2 p, \dot p - \delta_1 q) + H(q,p) + \delta_1|q|^2 + \delta_2|p|^2 + 2\dot q\cdot p\Big]\,dt\\
&\qquad + \psi_2\big(q(T)\big) + \psi_2^*\big(-p(T)\big) + \psi_1\big(p(0)\big) + \psi_1^*\big(q(0)\big)\\
&\quad = I(p,q).
\end{aligned}$$

□

In order to again apply Ky Fan's lemma, it remains to establish the following coercivity property.

LEMMA 11. *Under the above hypothesis, we have*

$$L(0,0;p,q) \to +\infty \quad \textit{as} \quad \|p\| + \|q\| \to +\infty.$$

PROOF. Since H, ψ_1^*, ψ_2^* are bounded from below and $\frac{1}{2(\beta+\epsilon)}(|p|^2+|q|^2)-\gamma \le H_\epsilon^*(p,q)$, modulo a constant we have,

$$\begin{aligned}
L(0,0;p,q) \ge \frac{1}{2(\beta+\epsilon)}&\int_0^T (|\dot q + \delta_2 p|^2 + |\dot p - \delta_1 q|^2)\,dt\\
&+ \int_0^T (\delta_1|q|^2 + \delta_2|p|^2 + 2\dot q\cdot p)\,dt\\
&+ \psi_2\big(q(T)\big) + \psi_1\big(p(0)\big).
\end{aligned} \tag{36}$$

It is easily seen that

$$|\dot q + \delta_2 p|^2 \ge \frac{1}{2}|\dot q|^2 - |\delta_2|^2\,|p|^2, \quad \text{and} \quad |\dot p - \delta_1 q|^2 \ge \frac{1}{2}|\dot p|^2 - |\delta_1|^2\,|q|^2. \tag{37}$$

It follows from (9) that

$$\begin{aligned}
&\int_0^T |p(t)|^2\,dt \le 2\Big(T^2\int_0^T |\dot p|^2\,dt + T|p(0)|^2\Big) \quad \text{and}\\
&\int_0^T |q(t)|^2\,dt \le 2\Big(T^2\int_0^T |\dot q|^2\,dt + T|q(T)|^2\Big).
\end{aligned} \tag{38}$$

Combining (36) and (37) gives

$$
\begin{aligned}
(39)\qquad & \int_0^T [|\dot{q} + \delta_2 p|^2 + |\dot{p} - \delta_1 q|^2]\,dt \\
&\geq \int_0^T \Big[\frac{1}{2}(|\dot{q}|^2 + |\dot{p}|^2) - |\delta_2|^2\,|p|^2 - |\delta_1|^2\,|q|^2\Big]\,dt \\
&\geq \int_0^T \Big[\frac{1}{2}(|\dot{q}|^2 + |\dot{p}|^2) - 2T^2(|\delta_2|^2\,|\dot{p}|^2 + |\delta_1|^2\,|\dot{q}|^2)\Big]\,dt \\
&\quad - 2T(|\delta_2|^2\,|p(0)|^2 + |\delta_1|^2\,|q(T)|^2) \\
&\geq \int_0^T \frac{1}{2}[(1 - 4T^2|\delta_2|^2)|\dot{p}|^2 + (1 - 4T^2|\delta_1|^2)|\dot{q}|^2]\,dt \\
&\quad - 2T\big(|\delta_2|^2\,|p(0)|^2 + |\delta_1|^2\,|q(T)|^2\big) \\
&= \int_0^T \frac{1}{2}[\epsilon_1|\dot{q}|^2 + \epsilon_2|\dot{p}|^2] - 2T\big(|\delta_2|^2\,|p(0)|^2 + |\delta_1|^2\,|q(T)|^2\big)
\end{aligned}
$$

where $\epsilon_i := 1 - 4T^2|\delta_i|^2 > 0$ since $|\delta_i| < \frac{1}{2T}$. Also, similarly to the proof of Theorem 1, we get

$$
(40)\qquad \left|\int_0^T 2\dot{q}\cdot p\,dt\right| \leq \max\big(2T^2, 1\big)\int_0^T \big(|\dot{q}|^2 + |\dot{p}|^2\big)\,dt + 2T|p(0)|^2.
$$

Hence, combining (36)–(39) yields

$$
\begin{aligned}
L(0,0;p,q) \geq{}& \frac{1}{4(\beta+\epsilon)}\int_0^T [\epsilon_1|\dot{q}|^2 + \epsilon_2|\dot{p}|^2] - A(\delta_1,T)\int_0^T |\dot{q}|^2\,dt \\
&- A(\delta_2,T)\int_0^T |\dot{p}|^2\,dt + \psi_2\big(q(T)\big) - \frac{T|\delta_1|^2}{\beta+\epsilon}|q(T)|^2 + 2\delta_1 T|q(T)|^2 \\
&+ \psi_1\big(p(0)\big) - \frac{T|\delta_2|^2}{\beta+\epsilon}|p(0)|^s - 2T(1-\delta_2)|p(0)|^2
\end{aligned}
$$

where $A(\delta_i, T) = \max(2T^2, 1) - 2\delta_i T^2$. This inequality together with the coercivity condition on ψ_1 and ψ_2, and the fact that $\beta < \frac{1}{4}\min\{\frac{\epsilon_1}{A(\delta_1,T)}, \frac{\epsilon_2}{A(\delta_2,T)}\}$ yields the claimed result. $\square$

References

1. H. Brézis, L. Nirenberg, and G. Stampacchia, *A remark on Ky Fan's Minimax Principle.* Boll. Un. Mat. Ital. **6** (1972), 293–300.
2. N. Ghoussoub, *Anti-self-dual Lagrangians: variational resolutions of non-self-adjoint equations and dissipative evolutions.* Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 2, 171–205.

3. ———, *Antisymmetric Hamiltonians: variational resolutions for Navier–Stokes and other nonlinear evolutions.* Comm. Pure Appl. Math. **60** (2007), no. 5, 619–653.

4. ———, *Self-dual partial differential systems and their variational principles.* Springer Monographs in Mathematics, Springer-Verlag, 2009.

5. N. Ghoussoub and A. Moameni, *Selfdual variational principles for periodic solutions of Hamiltonian and other dynamical systems.* Comm. Partial Differential Equations **32** (2007), no. 4-6, 771–795.

6. N. Ghoussoub and L. Tzou, *A variational principle for gradient flows.* Math. Ann. **330** (2004), no. 3, 519–549.

7. ———, *Anti-selfdual Lagrangians II. Unbounded non self-adjoint operators and evolution equations.* Ann. Mat. Pura Appl. **187** (2008), no. 2, 323–352.

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