

# Homogenization of maximal monotone vector fields via selfdual variational calculus

Nassif Ghoussoub \*

Department of Mathematics, University of British Columbia,  
Vancouver BC Canada V6T 1Z2  
nassif@math.ubc.ca

Abbas Moameni<sup>†</sup>

Department of Mathematics and Statistics, Queen's University,  
Kingston ON Canada K7L 3N6  
momeni@mast.queensu.ca

Ramón Zárate Sáiz <sup>‡</sup>

Department of Mathematics, University of British Columbia,  
Vancouver BC Canada V6T 1Z2  
ramon@math.ubc.ca

## Abstract

We use the theory of selfdual Lagrangians to give a variational approach to the homogenization of equations in divergence form, that are driven by a periodic family of maximal monotone vector fields. The approach has the advantage of using  $\Gamma$ -convergence methods for corresponding functionals just as in the classical case of convex potentials, as opposed to the graph convergence methods used in the absence of potentials. A new variational formulation for the homogenized equation is also given.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries on selfdual Lagrangians</b>	<b>5</b>
2.1	A variational principle for selfdual Lagrangians . . . . .	5
2.2	Mosco and $\Gamma$ -convergence of selfdual functionals . . . . .	6
2.3	Continuity of $L \rightarrow \bar{\partial}L$ for the $\Gamma$ -convergence of selfdual Lagrangians . . . . .	8
<b>3</b>	<b>A selfdual variational approach to existence theory</b>	<b>9</b>
3.1	Selfdual Lagrangians associated to maximal monotone operators . . . . .	9
3.2	Self-dual Lagrangians on $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ . . . . .	11
3.3	Variational formula for the homogenized maximal monotone vector field . . . . .	13

---

\*Partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

<sup>†</sup>Research supported by a Coleman fellowship at Queen's University.

<sup>‡</sup>Partially supported by a graduate fellowship from CONACYT (Mexico) and is part of a PhD thesis under the supervision of N. Ghoussoub

<b>4</b>	<b>A variational approach to homogenization</b>	<b>17</b>
4.1	The homogenization of general Lagrangians on $W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$ . . . . .	17
4.2	Variational homogenization of maximal monotone operators on $W_0^{1,p}(\Omega)$ . . . . .	25
<b>5</b>	<b>Appendix</b>	<b>28</b>

# 1 Introduction

We consider the homogenization of the problem

$$\begin{cases} \tau_n(x) \in \mathfrak{B}(\frac{x}{\varepsilon_n}, \nabla u_n(x)) & x \in \Omega, \\ -\operatorname{div}(\tau_n(x)) = u_n^*(x) & x \in \Omega, \\ u_n(x) = 0 & x \in \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  and  $\mathfrak{B} : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a measurable map on  $\Omega \times \mathbb{R}^N$  such that  $\mathfrak{B}(x, \cdot)$  is maximal monotone on  $\mathbb{R}^N$  for almost all  $x \in \Omega$ , and such that  $\mathfrak{B}(\cdot, \xi)$  is  $Q$ -periodic for an open non-degenerate parallelogram  $Q$  in  $\mathbb{R}^p$ . This problem has been investigated in recent years by many authors. We refer the interested reader to [1, 5, 6, 8, 11, 14, 15, 24] for related results.

The particular case where the maximal monotone operator is a subdifferential of the form

$$\mathfrak{B}(x, \xi) = \partial_\xi \psi(x, \xi), \quad (2)$$

with  $\psi : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  being a convex function in the second variable is particularly appealing and completely understood. Indeed, under appropriate boundedness and coercivity conditions on  $\psi$ , say

$$C_0(|\xi|^p - 1) \leq \psi(x, \xi) \leq C_1(|\xi|^p + 1) \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^N,$$

where  $1 < p < \infty$  and  $C_0, C_1$  are positive constants, one can then use a variational approach to identify for a given  $u^* \in W^{-1,p}(\Omega)$ , the solution  $(u, \tau)$  of (1) as the respective minima of the problems

$$\inf \left\{ \int_\Omega \psi(x, \nabla u(x)) dx - \int_\Omega u^*(x)u(x) dx; u \in W_0^{1,p}(\Omega) \right\}, \quad (3)$$

and

$$\inf \left\{ \int_\Omega \psi^*(x, \tau(x)) dx; \operatorname{div}(\tau) = u^* \right\}, \quad (4)$$

where  $\psi^*$  is the Fenchel-Legendre dual (in the second variable) of  $\psi$ . In this case, the classical concept of  $\Gamma$ -convergence –introduced by DeGiorgi– can be used to show that if  $u_n^* \rightarrow u^*$  strongly in  $W^{-1,q}(\Omega)$  with  $q = \frac{p}{p-1}$ , then up to a subsequence  $u_n \rightarrow u$  weakly in  $W_0^{1,p}(\Omega)$  and  $\tau_n \rightarrow \tau$  weakly in  $L^q(\Omega; \mathbb{R}^N)$ , where  $u$  is a solution and  $\tau$  is a momentum of the homogenized problem

$$\begin{cases} \tau(x) \in \mathfrak{B}_{hom}(\nabla u(x)) & a.e. x \in \Omega, \\ -\operatorname{div}(\tau(x)) = u^*(x) & a.e. x \in \Omega. \end{cases} \quad (5)$$

Here  $\mathfrak{B}_{hom}$  can be defined variationally as follows: for  $\xi \in \mathbb{R}^N$ ,  $\mathfrak{B}_{hom}(\xi) = \partial\psi_{hom}(\xi)$ , where

$$\psi_{hom}(\xi) := \min_{\varphi \in W_{\#}^{1,p}(Q)} \frac{1}{|Q|} \int_Q \psi(x, \xi + \nabla\varphi(x)) dx, \quad (6)$$

and

$$W_{\#}^{1,p}(Q) = \{u \in W^{1,p}(Q); \int_Q u(x) dx = 0 \text{ and } u \text{ is } Q\text{-periodic}\}. \quad (7)$$

As mentioned above, a similar result can be obtained for general maximal monotone maps  $\mathfrak{B} : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  with appropriate boundedness conditions (see below), by using the more cumbersome graph convergence (or  $G$ -convergence) methods. In this case,  $\mathfrak{B}_{hom}$  is defined by the following non-variational formula

$$\mathfrak{B}_{hom}(\xi) = \left\{ \int_Q g(y) dy \in \mathbb{R}^N; g \in L_{\#}^q(Q; \mathbb{R}^N), g(y) \in \mathfrak{B}(y, \xi + \nabla\psi(y)) \text{ a.e. in } Q \text{ for some } \psi \in W_{\#}^{1,p}(Q) \right\}, \quad (8)$$

where

$$L_{\#}^q(Q; \mathbb{R}^N) := \left\{ g \in L^q(Q; \mathbb{R}^N); \int_Q \langle g(y), \nabla \varphi(y) \rangle_{\mathbb{R}^N} dy = 0 \text{ for every } \varphi \in W_{\#}^{1,p}(Q) \right\}. \quad (9)$$

More recently, the first-named author proposed a variational approach to deal with general maximal monotone operators, including corresponding equations of the form (1) via the theory of selfdual Lagrangians on phase space [17, 18]. Our goal here is to describe how this approach is particularly well suited to deal with the homogenization of such equations, first by showing that –just as in the case of a convex potential (2)– the limiting process can be handled again through  $\Gamma$ -convergence of associated selfdual Lagrangians, and secondly by giving a variational characterization for the limiting vector field (8) in the same spirit as in (6).

We first recall that a *selfdual Lagrangian*  $L$  on a reflexive Banach space  $X$ , is any convex lower semi-continuous function on phase space  $L : X \times X^* \rightarrow \mathbb{R} \cup \{+\infty\}$  that satisfy the following duality property:

$$L^*(u^*, u) = L(u, u^*) \quad \text{for all } (u, u^*) \in X \times X^*, \quad (10)$$

where  $X^*$  is the Banach space dual to  $X$ , and  $L^*$  is the Fenchel-Legendre dual of  $L$  in both variables, i.e.,

$$L^*(u^*, u) = \sup\{\langle v, u^* \rangle + \langle u, v^* \rangle - L(v, v^*) : (v, v^*) \in X \times X^*\}.$$

Such Lagrangians satisfy the following basic property:

$$L(u, u^*) - \langle u, u^* \rangle \geq 0 \text{ for every } (u, u^*) \in X \times X^*.$$

We then consider the corresponding –possibly multivalued– *selfdual vector field*  $\bar{\partial}L : X \rightarrow 2^{X^*}$  defined for each  $u \in X$  as the –possibly empty– subset  $\bar{\partial}L(u)$  of  $X^*$  given by

$$\bar{\partial}L(u) := \{u^* \in X^*; L(u, u^*) - \langle u, u^* \rangle = 0\} = \{u^* \in X^*; (u^*, u) \in \partial L(u, u^*)\}. \quad (11)$$

Here  $\partial L$  is the subdifferential of the convex function  $L$  on  $X \times X^*$ , which should not be confused with  $\bar{\partial}L$ . Before going further, let us note that *selfdual vector fields* are natural and far reaching extensions of subdifferentials of convex lower semi-continuous functions. Indeed, the most basic selfdual Lagrangians are of the form  $L(u, u^*) = \varphi(u) + \varphi^*(u^*)$  where  $\varphi$  is a convex function in  $X$ , and  $\varphi^*$  is its Fenchel dual on  $X^*$  (i.e.,  $\varphi^*(u^*) = \sup\{\langle u, u^* \rangle - \varphi(u); u \in X\}$ , in which case

$$\bar{\partial}L(u) = \partial\varphi(u).$$

More interesting examples of selfdual Lagrangians are of the form  $L(u, u^*) = \varphi(u) + \varphi^*(-\Gamma u + u^*)$  where  $\varphi$  is a convex and lower semi-continuous function on  $X$ , and  $\Gamma : X \rightarrow X^*$  is a skew adjoint operator. The corresponding *selfdual vector field* is then

$$\bar{\partial}L(u) = \Gamma u + \partial\varphi(u).$$

Actually, both  $\partial\varphi$  and  $\partial\varphi + \Gamma$  are particular examples of the so-called *maximal monotone operators*, which are set-valued maps  $\beta : X \rightarrow 2^{X^*}$  whose graph in  $X \times X^*$  are maximal (for set inclusion) among all monotone subsets  $G$  of  $X \times X^*$ , i.e., those  $G$  satisfying

$$\langle x - y, p - q \rangle \geq 0 \text{ for every } (x, p) \text{ and } (y, q) \text{ in } G. \quad (12)$$

It turned out that the class of maximal monotone operators and the one of selfdual vector fields coincide. Indeed, the following was proved in [17].

**Theorem 1.1** *If  $\beta : D(\beta) \subset X \rightarrow 2^{X^*}$  is a maximal monotone operator with a non-empty domain  $D(\beta)$ , then there exists a selfdual Lagrangian on  $X \times X^*$  such that  $\beta = \bar{\partial}L$ .*

*Conversely, if  $L$  is a proper selfdual Lagrangian on  $X \times X^*$ , then the vector field  $u \rightarrow \bar{\partial}L(u)$  is maximal monotone.*

This means that selfdual Lagrangians can be seen as the *potentials* for maximal monotone operators, in the same way as convex lower semi-continuous energies are the potentials of their own subdifferential, leading to a variational formulation and resolution of most equations involving maximal monotone operators such

as the one in (1). This was indeed done in [17] in the case where  $\beta$  does not depend on the state  $x \in \Omega$ . We shall however need to consider in this paper measurable families  $\mathfrak{B}(x, \cdot) : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  of maximal monotone operators with suitable boundedness and coercivity conditions, and the possibility of associating to them measurable families  $L(x, \cdot, \cdot) : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  of selfdual Lagrangians on  $\mathbb{R}^N \rightarrow \mathbb{R}^N$  that reflect these conditions. For that we recall the definition of the class  $M_{\Omega,p}(\mathbb{R}^N)$  introduced in [8].

**Definition 1.1** For a domain  $\Omega$  in  $\mathbb{R}^N$ ,  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , we denote by  $M_{\Omega,p}(\mathbb{R}^N)$  the class of all possibly multi-valued functions  $\mathfrak{B} : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  with closed values, which satisfy the following conditions:  
(i)  $\mathfrak{B}$  is measurable with respect to  $\mathcal{L}(\Omega) \times \mathcal{B}(\mathbb{R}^N)$  and  $\mathcal{B}(\mathbb{R}^N)$  where  $\mathcal{L}(\Omega)$  is the  $\sigma$ -field of all measurable subsets of  $\Omega$  and  $\mathcal{B}(\mathbb{R}^N)$  is the  $\sigma$ -field of all Borel subsets of  $\mathbb{R}^N$ .  
(ii) For a.e.  $x \in \Omega$ , the map  $\mathfrak{B}(x, \cdot) : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is maximal monotone.  
(iii) There exist non-negative constants  $m_1, m_2, c_1$  and  $c_2$  such that for every  $\xi \in \mathbb{R}^N$  and  $\eta \in \mathfrak{B}(\xi)$ ,

$$\langle \xi, \eta \rangle_{\mathbb{R}^N} \geq \max \left\{ \frac{c_1}{p} |\xi|^p - m_1, \frac{c_2}{q} |\eta|^q - m_2 \right\}, \quad (13)$$

holds, where  $\langle \cdot, \cdot \rangle_{\mathbb{R}^N}$  is the inner product in  $\mathbb{R}^N$ .

The following is the main application of the results in this paper.

**Theorem 1.2** Let  $\Omega$  be a domain in  $\mathbb{R}^N$ ,  $q, p > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , and assume  $u_n^* \rightarrow u^*$  strongly in  $W^{-1,q}(\Omega)$ . Let  $u_n$  (resp.,  $\tau_n$ ) be (weak) solutions in  $W_0^{1,p}(\Omega)$  (resp., momenta in  $L^q(\Omega; \mathbb{R}^N)$ ) for the Dirichlet boundary value problems (1), where  $\mathfrak{B} : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  belongs to  $M_{\Omega,p}(\mathbb{R}^N)$ . If  $\mathfrak{B}(\cdot, \xi)$  is  $Q$ -periodic for an open non-degenerate parallelogram  $Q$  in  $\mathbb{R}^n$  then, up to a subsequence

$$\begin{aligned} u_n &\rightarrow u && \text{weakly in } W_0^{1,p}(\Omega), \\ \tau_n &\rightarrow \tau && \text{weakly in } L^q(\Omega; \mathbb{R}^N), \end{aligned}$$

where  $u$  is a solution and  $\tau$  is a momentum of the homogenized problem

$$\begin{cases} \tau(x) \in \mathfrak{B}_{hom}(\nabla u(x)) & \text{a.e. } x \in \Omega, \\ -\operatorname{div}(\tau(x)) = u^*(x) & \text{a.e. } x \in \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases} \quad (14)$$

Here  $\mathfrak{B}_{hom} = \bar{\partial}L_{hom}$ , with  $L_{hom}$  being a selfdual Lagrangian on  $\mathbb{R}^N \times \mathbb{R}^N$  defined by

$$L_{hom}(a, b) := \min_{\substack{\varphi \in W_{\#}^{1,p}(Q) \\ g \in L_{\#}^q(Q; \mathbb{R}^N)}} \frac{1}{|Q|} \int_Q L(x, a + D\varphi(x), b + g(x)) \, dx, \quad (15)$$

where for each  $x \in \Omega$ ,  $L(x, \cdot, \cdot)$  is a selfdual Lagrangian on  $\mathbb{R}^N \times \mathbb{R}^N$  such that

$$\mathfrak{B}(x, \cdot) = \bar{\partial}L(x, \cdot). \quad (16)$$

The above theorem will be a byproduct of several results which have their own interest. In section 2, we consider various topologies on the class of selfdual Lagrangians that are relevant for homogenization. It turns out that the standard concept of  $\Gamma$ -convergence is equivalent to the stronger notion of *Mosco-convergence* in the context of selfdual Lagrangians. This has a direct implication on the corresponding maximal monotone operators. We also extend to selfdual Lagrangians one of the most attractive properties of the Mosco convergence of convex functions, which is that it implies the convergence of the graphs of their corresponding subdifferentials in the topology of Kuratowski-Painlevé on sets. We shall show in section 2 that similarly, the map  $L \rightarrow \bar{\partial}L$  is continuous when we equip the class of selfdual Lagrangians with the topology of  $\Gamma$ -convergence and the class of maximal monotone operators with the topology of  $G$ -convergence. In section 3, we start by extending Theorem 1.1 above by establishing a correspondence between state-dependent measurable maximal monotone operators in  $M_{\Omega,p}(\mathbb{R}^N)$  and the following class of  $\Omega$ -dependent selfdual Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$ .

**Definition 1.2** An  $(\Omega, p)$ -dependent selfdual Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$  is a measurable function  $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that

1. For any  $x \in \Omega$ , the map  $(a, b) \rightarrow L(x, a, b)$  is a selfdual Lagrangian on  $\mathbb{R}^N \times \mathbb{R}^N$ .
2. There exist non-negative constants  $C_0$  and  $C_1$  and  $n_0, n_1 \in L^1(\Omega)$  such that

$$C_0(|a|^p + |b|^q - n_0(x)) \leq L(x, a, b) \leq C_1(|a|^p + |b|^q + n_1(x)) \text{ for all } a, b \in \mathbb{R}^N. \quad (17)$$

As in Theorem 1.1, any map  $\beta : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  in  $M_{\Omega, p}(\mathbb{R}^N)$  can be seen as a potential of an  $\Omega$ -dependent selfdual Lagrangian  $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ , that is  $\bar{\partial}L(x, a) = \beta(x, a)$  for almost all  $x \in \Omega$ .

We then proceed to use the above representation of  $\beta$  to give a variational resolution for the problem

$$\begin{cases} f \in \beta(x, \nabla u(x)) & \text{a.e. } x \in \Omega, \\ -\operatorname{div}(f) = u^*, \\ u \in W_0^{1,p}(\Omega), \end{cases} \quad (18)$$

by “lifting” the corresponding  $\Omega$ -dependent selfdual Lagrangian  $L$  on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$  to a selfdual Lagrangian on the function space  $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$  via the formula:

$$F(u, u^*) := \inf \left\{ \int_{\Omega} L(x, \nabla u(x), f(x)) dx; f \in L^q(\Omega; \mathbb{R}^N), -\operatorname{div}(f) = u^* \right\}. \quad (19)$$

A solution can then be obtained by simply minimizing for a given  $u^* \in W^{-1,q}(\Omega)$  the non-negative functional

$$I(u) = \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ -\operatorname{div}(f) = u^*}} \int_{\Omega} [L(x, \nabla u(x), f(x)) - \langle u(x), u^*(x) \rangle_{\mathbb{R}^N}] dx,$$

on  $W_0^{1,p}(\Omega)$ . We end the section by showing that if  $\bar{\partial}L(x, \cdot) = \beta(x, \cdot)$ , then

$$\beta_{hom} = \bar{\partial}L_{hom}, \quad (20)$$

where  $\beta_{hom}$  is defined in (8) and  $L_{hom}$  is as in (15).

We start section 4 by a homogenization result via  $\Gamma$ -convergence for general  $Q$ -periodic Lagrangians which are not necessarily selfdual. This is then applied to obtain the result claimed in Theorem 1.2 above in the case of selfdual Lagrangians. The last section is an appendix meant for auxiliary results that are needed throughout the paper.

## 2 Preliminaries on selfdual Lagrangians

We first recall the needed notions and results from the theory of selfdual Lagrangians developed in the book [18]. We shall also establish new ones, in particular those regarding the convergence properties in suitable topologies of selfdual Lagrangians and their associated maximal monotone vector fields.  $X$  will denote a real reflexive Banach space and  $X^*$  its dual.

### 2.1 A variational principle for selfdual Lagrangians

As mentioned in the introduction, maximal monotone operators  $\beta$  can be written as  $\beta = \bar{\partial}L$ , where  $L$  is a selfdual Lagrangian on  $X \times X^*$ , in such a way that solving the equation

$$u^* \in \beta(u), \quad (21)$$

amounts to minimizing the non-negative functional  $I(u) := L(u, u^*) - \langle u, u^* \rangle$ . The following existence result is essential for the sequel. It gives sufficient conditions for the infimum of selfdual Lagrangians to be attained, and –as importantly– to be zero.

**Theorem 2.1** *Let  $L$  be a selfdual Lagrangian on a reflexive Banach space  $X \times X^*$ , let  $u^* \in X^*$  be such that  $(0, u^*) \in \text{Dom}(L)$ , and consider the functional  $I(u) := L(u, u^*) - \langle u, u^* \rangle$ . Then*

$$\inf_{u \in X} I(u) = 0, \quad (22)$$

and in particular, if the functional  $I$  is coercive on  $X$ , then there exists  $\bar{u} \in X$  such that

$$I(\bar{u}) = \min_{u \in X} I(u) = 0 \text{ and } u^* \in \bar{\partial}L(\bar{u}). \quad (23)$$

Note that since  $L_{u^*}(u, v^*) := L(u, u^* + v^*) - \langle u, u^* \rangle$  is a selfdual Lagrangian whenever  $L$  is, it suffices to assume that  $u^* = 0$ . The above theorem is then a consequence of the following result originally established in [16] (see also [18]) under a slightly stronger coercivity condition.

**Theorem 2.2** *Let  $L$  be a selfdual functional on a reflexive Banach space  $X \times X^*$  such that for some  $u_0 \in X$ , the functional  $v^* \rightarrow L(u_0, v^*)$  is bounded above on a neighborhood of the origin in  $X^*$ . Then there exists  $\bar{u} \in X$  such that  $I(\bar{u}) = \min_{u \in X} I(u) = 0$ .*

**Proof of Theorem 2.1:** Since  $L$  is a selfdual Lagrangian on  $X \times X^*$ , so is its  $\lambda$ -regularization,

$$L_\lambda(u, u^*) = \inf \left\{ L(v, v^*) + \frac{1}{2\lambda} \|u - v\|^2 + \frac{\lambda}{2} \|v\|^2 + \frac{1}{2\lambda} \|u^* - v^*\|^2 + \frac{\lambda}{2} \|v^*\|^2; v \in X, v^* \in X^* \right\},$$

for each  $\lambda > 0$ , by virtue of Lemma 3.2 in Chapter 2 of [18]. Note that the Lagrangian  $L_\lambda$  satisfies the boundedness condition of Theorem 2.2. It then follows that  $\min_{u \in X} L_\lambda(u, 0) = 0$ . On the other hand, because of the properties of Yoshida regularization for convex functions, for each  $(u, u^*) \in \text{Dom}(L)$  we have  $\liminf_{\lambda \rightarrow 0} L_\lambda(u, u^*) = L(u, u^*)$ . It follows that

$$\inf_{u \in X} L(u, 0) = \inf_{u \in X} \liminf_{\lambda \rightarrow 0} L_\lambda(u, 0) = \liminf_{\lambda \rightarrow 0} \inf_{u \in X} L_\lambda(u, 0) = 0.$$

Therefore  $\inf_{u \in X} I(u) = 0$ . Now if  $I$  is coercive then the minimum is attained for some  $\bar{u} \in X$ , i.e.,  $I(\bar{u}) = L(\bar{u}, 0) = 0$  and consequently  $\bar{u}$  is a solution of  $0 \in \bar{\partial}L(\bar{u})$ .  $\square$

## 2.2 Mosco and $\Gamma$ -convergence of selfdual functionals

We first recall the main definitions and statements in the theory of variational convergence for functionals, as well as the graph convergence for possibly multi-valued operators. A complete study relating the various modes of convergence of convex functions and their subdifferentials can be found in [10].

**Definition 2.1** *Let  $F_n$  and  $F$  be functionals on a reflexive Banach space  $X$ . The sequence  $\{F_n\}$  is said to  $\Gamma$ -converge (resp., Mosco-converge) to  $F$ , if the following two conditions are satisfied:*

1. *For any sequence  $\{u_n\} \subset X$  such that  $u_n \rightarrow u$  strongly (resp.,  $u_n \rightharpoonup u$  weakly) in  $X$  to some  $u \in X$ , one has*

$$F(u) \leq \liminf_{n \rightarrow \infty} F_n(u_n).$$

2. *For any  $u \in X$ , there exists a sequence  $\{u_n\} \subset X$  such that  $u_n \rightarrow u$  strongly in  $X$  and*

$$\lim_{n \rightarrow \infty} F_n(u_n) = F(u).$$

The following is a fundamental property of Mosco-convergence.

**Lemma 2.1** *Let  $F_n, F$  be a proper convex lower semi-continuous functionals, then  $\{F_n\}$  Mosco-converge to  $F$  if and only their Fenchel-Legendre duals  $\{F_n^*\}$  Mosco-converge to  $F^*$ .*

In the following we note that this property implies the agreeable fact that Mosco and  $\Gamma$ -convergence are actually equivalent for a sequence of selfdual Lagrangians  $\{L_n\}$ , as long as the limiting Lagrangian  $L$  is itself selfdual.

**Theorem 2.3** Let  $\{L_n\}$  be a family of selfdual Lagrangians on  $X \times X^*$ , where  $X$  is a reflexive Banach space, and let  $L$  be a Lagrangian on  $X \times X^*$ . The following statements are then equivalent:

1.  $\{L_n\}$  Mosco-converges to  $L$ .
2.  $L$  is selfdual and  $\{L_n\}$   $\Gamma$ -converges to  $F$ .
3.  $L$  is selfdual and for any  $(u, u^*) \in X \times X^*$ , there exists a sequence  $(u_n, u_n^*)$  converging strongly to  $(u, u^*)$  in  $X \times X^*$  such that

$$\limsup_n L_n(u_n, u_n^*) \leq L(u, u^*).$$

**Proof.** For (1)  $\rightarrow$  (2) we just need to prove that  $L$  is selfdual since Mosco convergence clearly implies  $\Gamma$ -convergence. Since  $L$  is the Mosco limit of  $L_n$ , it follows from Lemma 2.1 that  $L^*$  is a Mosco limit of  $L_n^*$ . Denoting

$$L_n^T(u^*, u) := L_n(u, u^*) \text{ and } L^T(u^*, u) := L(u, u^*),$$

it follows that  $L^T$  is a Mosco-limit of  $L_n^T$  on  $X^* \times X$ . On the other hand, by selfduality of  $L_n$  we have that  $L_n^T = L_n^*$  from which we obtain that  $L^T = \lim_n L_n^T = \lim_n L_n^* = L^*$ , and therefore  $L^T = L^*$ , and  $L$  is therefore selfdual.

(2) $\rightarrow$ (3) follows from the definition of  $\Gamma$ -convergence.

For (3) $\rightarrow$ (1) we let  $(u^*, u) \in X^* \times X$  and consider a sequence  $\{(u_n^*, u_n)\} \subset X^* \times X$  such that  $(u_n^*, u_n) \rightarrow (u^*, u)$  weakly in  $X^* \times X$ . By the definition of Fenchel-Legendre duality we have

$$\liminf_n L_n^*(u_n^*, u_n) = \liminf_n \sup_{(v, v^*) \in X \times X^*} \{\langle u_n, v^* \rangle + \langle v, u_n^* \rangle - L_n(v, v^*)\}. \quad (24)$$

Consider now an arbitrary pair  $(\tilde{u}, \tilde{u}^*)$  and let  $\{(\tilde{u}_n, \tilde{u}_n^*)\}$  be the recovery sequence given in item (3). It follows from (24) that

$$\liminf_n L_n(u_n, u_n^*) \geq \liminf_n \left( \langle u_n, \tilde{u}_n^* \rangle + \langle \tilde{u}_n, u_n^* \rangle - L_n(\tilde{u}_n, \tilde{u}_n^*) \right) = \langle u, \tilde{u}^* \rangle + \langle \tilde{u}, u^* \rangle - L(\tilde{u}, \tilde{u}^*).$$

Since  $(\tilde{u}, \tilde{u}^*)$  is arbitrary, taking the supremum over all  $(\tilde{u}, \tilde{u}^*)$  yields

$$\liminf_n L_n^*(u_n^*, u_n) \geq L^*(u^*, u).$$

Since both  $L_n$  and  $L$  are selfdual, this implies that

$$\liminf_n L_n(u_n, u_n^*) \geq L(u, u^*),$$

and therefore that  $L$  is a Mosco-limit of  $L_n$ . □

**Remark 2.1** Note that while the Mosco convergence of selfdual Lagrangians automatically implies that the limiting Lagrangian  $L$  is itself selfdual, this fails for  $\Gamma$ -convergence as shown in the following example.

Let  $H$  be an infinite dimensional Hilbert space. Consider a set  $\{e_n\}$  with  $\|e_n\| = 1$  and  $e_n \rightarrow 0$  (For example, the orthonormal basis of the space). Define

$$L_n(u, u^*) := \frac{1}{2} \|u - e_n\|^2 + \frac{1}{2} \|u^*\|^2 + \langle u^*, e_n \rangle,$$

in such a way that  $L_n$  is selfdual. It can be checked directly that for any  $(u_n, u_n^*) \rightarrow (u, u^*)$  in  $H \times H$  we have  $\lim_n L_n(u_n, u_n^*) = L(u, u^*)$ , where

$$L(u, u^*) := \frac{1}{2} \|u\|^2 + \frac{1}{2} \|u^*\|^2 + \frac{1}{2}.$$

This means that  $L$  is a  $\Gamma$ -limit of  $L_n$ . On the other hand, it is easily seen that  $L$  is not selfdual and therefore we do not have Mosco convergence.

### 2.3 Continuity of $L \rightarrow \bar{\partial}L$ for the $\Gamma$ -convergence of selfdual Lagrangians

One of the most attractive properties of Mosco convergence is the fact that for convex functions it implies the graph convergence (or *Kuratowski-Painlevé convergence*) of their corresponding subdifferentials [2, Theorem 4.2]. We shall extend this result to selfdual Lagrangians by showing that their Mosco (or  $\Gamma$ -convergence) also yield the graph convergence of their derived vector fields (i.e., their corresponding maximal monotone operators).

Considering a sequence of sets  $\{A_n\}$  in  $X$ , the corresponding sequential lower and upper limit sets are respectively given by

$$Li_X(A_n) = \{u \in X : \exists u_n \rightarrow u, u_n \in A_n\},$$

and

$$Ls_X(A_n) = \{u \in X : \exists k(n) \rightarrow \infty, \exists u_{n(k)} \rightarrow u, u_{n(k)} \in A_k\}.$$

In other words,  $Li(A_n)$  corresponds to the collection of all *limit* points of the sequence  $\{A_n\}$  and  $Ls(A_n)$  is the collection of all *cluster* points of the sequence  $\{A_n\}$ . We clearly have  $Li_X(A_n) \subseteq Ls_X(A_n)$ .

**Definition 2.2** *A sequence of subsets  $\{A_n\}$  of  $X$  is said to converge to  $A \subset X$ , in the sense of Kuratowski-Painlevé, if  $Ls_X(A_n) = A = Li_X(A_n)$ .*

This definition, when  $X$  is replaced by the phase space  $X \times X^*$  and when the subsets  $A_n$  are graphs of maps from  $X$  to  $X^*$ , is also referred to as *graph-convergence* (see Definition 3.5 on [8]).

Recall that for a selfdual Lagrangian  $F$  on  $X \times X^*$ , its associated vector field at  $u \in X$  is denoted by  $\bar{\partial}F(u)$  and given by  $\bar{\partial}F(u) = \{u^* \in X^*; F(u, u^*) = \langle u, u^* \rangle\}$ . We shall therefore also denote by  $\bar{\partial}F$  the graph of  $\bar{\partial}F$  in  $X \times X^*$ , i.e.,

$$(u, u^*) \in \bar{\partial}F \text{ if and only if } u^* \in \bar{\partial}F(u).$$

The following is the main result of this section.

**Theorem 2.4** *Let  $X$  be a reflexive Banach space and suppose  $\{F_n\}$  is a family of selfdual Lagrangians on  $X \times X^*$ . If  $F : X \times X^* \rightarrow \mathbb{R} \cup \{+\infty\}$  is a selfdual Lagrangian that is a  $\Gamma$ -limit of  $\{F_n\}$ , then the graph of  $\bar{\partial}F_n$  converge to the graph of  $\bar{\partial}F$  in the sense of Kuratowski-Painlevé.*

For the proof, we shall make use of the following theorem that can be seen as the counterpart of the *Brøndsted-Rockafellar* result for convex functions [23].

**Lemma 2.2** *Let  $L : X \times X^* \rightarrow \mathbb{R} \cup \{+\infty\}$  be a selfdual Lagrangian and assume that for a pair  $(u_0, u_0^*) \in X \times X^*$ , we have  $L(u_0, u_0^*) - \langle u_0, u_0^* \rangle \leq \varepsilon$ . Then, there exists a pair  $(u_\varepsilon, u_\varepsilon^*) \in \bar{\partial}L$  such that*

1.  $\|u_\varepsilon - u_0\| \leq \sqrt{\varepsilon}$ ,
2.  $\|u_\varepsilon^* - u_0^*\|_* \leq \sqrt{\varepsilon}$ ,
3.  $|L(u_\varepsilon, u_\varepsilon^*) - L(u_0, u_0^*)| \leq 2\varepsilon + \sqrt{\varepsilon}(\|u_0\| + \|u_0^*\|_*)$ .

**Proof:** First assume that  $M$  is a selfdual Lagrangian such that  $M(0, 0) \leq \varepsilon$ . We claim that there exists then a pair  $(v_\varepsilon, v_\varepsilon^*) \in \bar{\partial}M$  such that

1.  $\|v_\varepsilon\| \leq \sqrt{\varepsilon}$ ,
2.  $\|v_\varepsilon^*\|_* \leq \sqrt{\varepsilon}$ ,
3.  $|M(v_\varepsilon, v_\varepsilon^*)| \leq \varepsilon$ .

Indeed, denote by  $J$  the *duality mapping* from  $X$  to  $X^*$  and use the fact that  $\bar{\partial}M$  is a maximal monotone operator to find  $\tilde{u} \in X$  such that

$$-J\tilde{u} \in \bar{\partial}M(\tilde{u}).$$

It follows that  $M(\tilde{u}, -J\tilde{u}) = \langle \tilde{u}, -J\tilde{u} \rangle = -\|\tilde{u}\|^2$ . Now, since  $M$  is selfdual, we have

$$M(0, 0) = M^*(0, 0) = \sup_{(u, u^*) \in X \times X^*} -M(u, u^*) \geq -M(\tilde{u}, -J\tilde{u}) = \|\tilde{u}\|^2,$$



from which we obtain that  $\|\tilde{u}\|^2 \leq \varepsilon$ . Since  $\|\tilde{u}\| = \|J\tilde{u}\|_*$ , it suffices to set  $v_\varepsilon := \tilde{u}$  and  $v_\varepsilon^* := J\tilde{u}$ , to obtain that  $\|v_\varepsilon\| = \|v_\varepsilon^*\|_* \leq \sqrt{\varepsilon}$  and  $|M(v_\varepsilon, v_\varepsilon^*)| = \|v_\varepsilon\|^2 \leq \varepsilon$ .

To complete the proof of Theorem 2.2, we set

$$M(u, u^*) := L(u + u_0, u^* + u_0^*) - \langle u, u_0^* \rangle - \langle u_0, u^* \rangle - \langle u_0, u_0^* \rangle,$$

which is a selfdual Lagrangian on  $X \times X^*$ . The hypothesis yields that

$$M(0, 0) = L(u_0, u_0^*) - \langle u_0, u_0^* \rangle \leq \varepsilon.$$

It then follows from the above that there exists a pair  $(v_\varepsilon, v_\varepsilon^*) \in \bar{\partial}M$  such that  $\|v_\varepsilon\| \leq \sqrt{\varepsilon}$ ,  $\|v_\varepsilon^*\|_* \leq \sqrt{\varepsilon}$  and  $|M(v_\varepsilon, v_\varepsilon^*)| \leq \varepsilon$ . Setting  $u_\varepsilon := v_\varepsilon + u_0$  and  $u_\varepsilon^* := v_\varepsilon^* + u_0^*$ , and since  $M(v_\varepsilon, v_\varepsilon^*) = \langle v_\varepsilon, v_\varepsilon^* \rangle$ , we have  $L(u_\varepsilon, u_\varepsilon^*) = \langle u_\varepsilon, u_\varepsilon^* \rangle$ , and therefore  $(u_\varepsilon, u_\varepsilon^*) \in \bar{\partial}L$ . Note also that  $\|u_\varepsilon - u_0\| \leq \sqrt{\varepsilon}$  and  $\|u_\varepsilon^* - u_0^*\|_* \leq \sqrt{\varepsilon}$ . Finally, we have

$$L(u_\varepsilon, u_\varepsilon^*) - L(u_0, u_0^*) = M(v_\varepsilon, v_\varepsilon^*) + \langle v_\varepsilon, u_0^* \rangle + \langle u_0, v_\varepsilon^* \rangle - M(0, 0),$$

which together with  $|M(v_\varepsilon, v_\varepsilon^*)| \leq \varepsilon$ , yields that

$$|L(u_\varepsilon, u_\varepsilon^*) - L(u_0, u_0^*)| \leq 2\varepsilon + \sqrt{\varepsilon}(\|u_0\| + \|u_0^*\|).$$

□

**Proof of Theorem 2.4.** Fix  $(u, u^*) \in \bar{\partial}F$ . There exists then in view of the  $\Gamma$ -convergence, a sequence  $(u_n, u_n^*)$  converging strongly to  $(u, u^*)$  in  $X \times X^*$  such that  $F_n(u_n, u_n^*) \rightarrow F(u, u^*)$ . We then have  $F(u, u^*) = \langle u, u^* \rangle = \lim_n \langle u_n, u_n^* \rangle$ , and therefore if we define  $\varepsilon_n := F_n(u_n, u_n^*) - \langle u_n, u_n^* \rangle$ , we obtain that  $\lim_n \varepsilon_n = 0$ . Hence, by Lemma 2.2, we have the existence of a pair  $(\tilde{u}_n, \tilde{u}_n^*) \in \bar{\partial}F_n$  such that  $\|u_n - \tilde{u}_n\| < \sqrt{\varepsilon_n}$  and  $\|u_n^* - \tilde{u}_n^*\|_* < \sqrt{\varepsilon_n}$ . Clearly  $\tilde{u}_n \rightarrow u$  and  $\tilde{u}_n^* \rightarrow u^*$  as  $\varepsilon_n \rightarrow 0$ . This shows that  $\bar{\partial}F \subset Li(\bar{\partial}F_n)$ .

To complete the proof, we just need to show that  $Ls(\bar{\partial}F_n) \subset \bar{\partial}F$ . Letting  $(v, v^*) \in Ls(\bar{\partial}F_n)$ , there exists some sequence  $(v_{n_k}, v_{n_k}^*) \in \bar{\partial}F_{n_k}$  such that  $(v_{n(k)}, u_{n(k)}) \rightarrow (v, v^*)$ . Now take an arbitrary  $(u, u^*) \in \bar{\partial}F$ . From what we have shown, there exists a sequence  $(u_n, u_n^*) \in \bar{\partial}F_n$  such that  $(u_n, u_n^*) \rightarrow (u, u^*)$ . For each  $k$  we have  $\langle u_{n(k)} - v_{n(k)}, u_{n(k)}^* - v_{n(k)}^* \rangle \geq 0$ , and as  $k \rightarrow \infty$  we get  $\langle u - v, u^* - v^* \rangle \geq 0$ . The above holds for all  $(u, u^*) \in \bar{\partial}F$  and so by the maximality of  $\bar{\partial}F$  we obtain that  $(v, v^*) \in \bar{\partial}F$ , which completes the proof. □

### 3 A selfdual variational approach to existence theory

In this section, we first establish a correspondence between maximal monotone maps in  $M_{\Omega, p}(\mathbb{R}^N)$  and a class of  $\Omega$ -dependent selfdual Lagrangians. We then proceed to give a variational formulation and resolution to equation (1) even in the case where the maximal monotone operator  $\beta$  is nor derived from the potential of a convex function.

#### 3.1 Selfdual Lagrangians associated to maximal monotone operators

**Definition 3.1** Let  $\Omega$  be a domain in  $\mathbb{R}^N$ .

(i) A function  $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$  is said to be an  $\Omega$ -dependent Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$ , if it is measurable with respect to the  $\sigma$ -field generated by the products of Lebesgue sets in  $\Omega$  and Borel sets in  $\mathbb{R}^N \times \mathbb{R}^N$ .

(ii) Such a Lagrangian  $L$  is said to be *selfdual on*  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$  if for any  $x \in \Omega$ , the map  $L_x : (a, b) \rightarrow L(x, a, b)$  is a selfdual Lagrangian on  $\mathbb{R}^N \times \mathbb{R}^N$ , i.e., if  $L^*(x, b, a) = L(x, a, b)$  for all  $a, b \in \mathbb{R}^N$  where

$$L^*(x, b, a) = \sup\{\langle b, \xi \rangle_{\mathbb{R}^N} + \langle a, \eta \rangle_{\mathbb{R}^N} - L(x, \xi, \eta) : (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N\}.$$

The following was proved in [17] for a single maximal monotone operator.

**Proposition 3.1** *If  $\beta \in M_{\Omega, p}(\mathbb{R}^N)$  for some  $p > 1$ , then there exists an  $\Omega$ -dependent selfdual Lagrangian  $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $\beta(x, \cdot) = \bar{\partial}L(x, \cdot)$  for a.e.  $x \in \Omega$  and*

$$C_0(|a|^p + |b|^q - n_0(x)) \leq L(x, a, b) \leq C_1(|a|^p + |b|^q + n_1(x)) \quad \text{for all } a, b \in \mathbb{R}^N. \quad (25)$$

where  $C_0$  and  $C_1$  are two positive constants and  $n_0, n_1 \in L^1(\Omega)$ .

Conversely, if  $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  is an  $\Omega$ -dependent selfdual Lagrangian satisfying (25), then  $\bar{\partial}L(x, \cdot) \in M_{\Omega, p}(\mathbb{R}^N)$ .

**Proof.** Let  $N : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$  be the Fitzpatrick function [13] associated to  $\beta$ , i.e.,

$$N(x, a, b) := \sup\{\langle b, \xi \rangle_{\mathbb{R}^N} + \langle a - \xi, \eta \rangle_{\mathbb{R}^N}; \eta \in \beta(x, \xi)\}.$$

Note that measurability assumptions on  $\beta$  ensures that  $N$  is a normal integrand. Also, by the properties of the Fitzpatrick function [18], it follows that

$$N^*(x, b, a) \geq N(x, a, b) \geq \langle a, b \rangle_{\mathbb{R}^N} \text{ for a.e. } x \in \Omega \text{ and for all } a, b \in \mathbb{R}^N.$$

Moreover,

$$\eta \in \beta(x, \xi) \text{ if and only if } N^*(x, \eta, \xi) = N(x, \xi, \eta) = \langle \eta, \xi \rangle_{\mathbb{R}^N} \text{ a.e. } x \in \Omega. \quad (26)$$

Define  $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  by

$$L(x, a, b) = \inf\left\{\frac{1}{2}N(x, a_1, b_1) + \frac{1}{2}N^*(x, b_2, a_2) + \frac{1}{4p}|a_1 - a_2|^p + \frac{1}{4q}|b_1 - b_2|^q; (a, b) = \frac{1}{2}(a_1, b_1) + \frac{1}{2}(a_2, b_2)\right\}.$$

We shall show that  $L$  is  $\Omega$ -dependent selfdual Lagrangian such that

$$N^*(x, b, a) \geq L(x, a, b) \geq N(x, a, b) \text{ for a.e. } x \in \Omega \text{ and for all } a, b \in \mathbb{R}^N. \quad (27)$$

Fix  $a, b \in \mathbb{R}^N$ . We have

$$\begin{aligned} L^*(x, b, a) &= \sup_{\xi, \eta \in \mathbb{R}^N} \{\langle \xi, b \rangle_{\mathbb{R}^N} + \langle a, \eta \rangle_{\mathbb{R}^N} - L(x, \xi, \eta)\} \\ &= \sup_{\xi, \eta \in \mathbb{R}^N} \left\{ \langle \xi, b \rangle_{\mathbb{R}^N} + \langle a, \eta \rangle_{\mathbb{R}^N} - \frac{1}{2}N(x, \xi_1, \eta_1) - \frac{1}{2}N^*(x, \xi_2, \eta_2) \right. \\ &\quad \left. - \frac{1}{4p}|\xi_1 - \xi_2|^p - \frac{1}{4q}|\eta_1 - \eta_2|^q; (\xi, \eta) = \frac{1}{2}(\xi_1, \eta_1) + \frac{1}{2}(\xi_2, \eta_2) \right\} \\ &= \frac{1}{2} \sup_{\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}^N} \left\{ \langle \xi_1 + \xi_2, b \rangle_{\mathbb{R}^N} + \langle a, \eta_1 + \eta_2 \rangle_{\mathbb{R}^N} - N(x, \xi_1, \eta_1) - N^*(x, \xi_2, \eta_2) \right. \\ &\quad \left. - \frac{1}{2p}|\xi_1 - \xi_2|^p - \frac{1}{2q}|\eta_1 - \eta_2|^q \right\}. \end{aligned}$$

Using the fact that the Fenchel dual of some of two functions is their inf-convolution, we obtain

$$L^*(x, b, a) = \frac{1}{2} \inf_{a_1, b_1 \in \mathbb{R}^N} \left\{ N^*(x, b_1, a_1) + N(x, 2a - a_1, 2b - b_1) + \frac{2^{q-1}}{q}|b - b_1|^q + \frac{2^{p-1}}{2p}|a - a_1|^p \right\}.$$

Setting  $a_2 = 2a - a_1$  and  $b_2 = 2b - b_1$  we have  $a = \frac{a_1 + a_2}{2}$  and  $b = \frac{b_1 + b_2}{2}$ . It then follows that

$$\begin{aligned} L^*(x, b, a) &= \frac{1}{2} \inf_{a_1, b_1, a_2, b_2 \in \mathbb{R}^N} \left\{ N^*(x, b_1, a_1) + N(x, a_2, b_2) \right. \\ &\quad \left. + \frac{2^{q-1}}{q} \left| \frac{b_1 - b_2}{2} \right|^q + \frac{2^{p-1}}{2p} \left| \frac{a_1 - a_2}{2} \right|^p; (a, b) = \frac{1}{2}(a_1, b_1) + \frac{1}{2}(a_2, b_2) \right\} \\ &= \inf \left\{ \frac{1}{2}N^*(x, b_1, a_1) + \frac{1}{2}N(x, a_2, b_2) + \frac{1}{4q}|b_1 - b_2|^q + \frac{1}{4p}|a_1 - a_2|^p; \right. \\ &\quad \left. (a, b) = \frac{1}{2}(a_1, b_1) + \frac{1}{2}(a_2, b_2) \right\} \\ &= L(x, a, b). \end{aligned}$$

Thus,  $L$  is a  $\Omega$ -dependent selfdual Lagrangian. Inequalities (27) simply follow from the definition and selfduality of  $L$ . We shall now prove that  $L$  satisfies the estimate (25). Note first that for all  $\eta \in \beta(x, \xi)$  we have

$$\frac{1}{p}|\xi|^p + \frac{1}{q}|\eta|^q \leq m_1 + m_2 + (c_1 + c_2)\langle \xi, \eta \rangle_{\mathbb{R}^N}.$$

It follows from the definition of the Fitzpatrick function  $N$  that

$$\begin{aligned}
N(x, a, b) &= \sup\{\langle b, \xi \rangle_{\mathbb{R}^N} + \langle a - \xi, \eta \rangle_{\mathbb{R}^N}; \eta \in \mathfrak{B}(x, \xi)\} \\
&\leq \sup\left\{\langle b, \xi \rangle_{\mathbb{R}^N} + \langle a, \eta \rangle_{\mathbb{R}^N} - \frac{1}{p(c_1 + c_2)}|\xi|^p - \frac{1}{q(c_1 + c_2)}|\eta|^q - \frac{m_1 + m_2}{c_1 + c_2}; \eta \in \mathfrak{B}(x, \xi)\right\} \\
&\leq \sup_{\xi, \eta \in \mathbb{R}^N} \left\{\langle b, \xi \rangle_{\mathbb{R}^N} + \langle a, \eta \rangle_{\mathbb{R}^N} - \frac{1}{p(c_1 + c_2)}|\xi|^p - \frac{1}{q(c_1 + c_2)}|\eta|^q - \frac{m_1 + m_2}{c_1 + c_2}\right\} \\
&= \frac{(c_1 + c_2)^{p-1}}{p}|a|^p + \frac{(c_1 + c_2)^{q-1}}{q}|b|^q + \frac{m_1 + m_2}{c_1 + c_2}.
\end{aligned} \tag{28}$$

Let  $\eta_0(x) \in \mathfrak{B}(x, 0)$ . By assumption  $|\eta_0(x)|^q \leq m_2 + \langle 0, \eta_0(x) \rangle = m_2$  for a.e.  $x \in \Omega$ , from which we get  $\eta_0 \in L^q(\Omega)$ . It also follows from (26) that  $N^*(x, \eta_0(x), 0) = 0$  for a.e.  $x \in \Omega$ . From the definition of  $L$  and (28), we get that

$$\begin{aligned}
L(x, a, b) &\leq \frac{1}{2}N(x, 2a - \eta_0(x), 2b) + \frac{1}{2}N^*(x, \eta_0(x), 0) + \frac{2^q}{4q}|b|^q + \frac{2^p}{4p}|a - \eta_0(x)|^p \\
&\leq C_1(|a|^p + |b|^q + n_1(x)) \text{ a.e. } x \in \Omega,
\end{aligned}$$

where  $C_1$  is a positive constant and  $n_1 \in L^1(\Omega)$ . The reverse inequality follows from the selfduality of  $L$ .

Conversely, let  $L$  be a  $\Omega$ -dependent selfdual Lagrangian satisfying (25). If  $\eta \in \bar{\partial}L(x, \xi)$  then

$$\langle \xi, \eta \rangle = L(x, \xi, \eta) \geq C_0(|\xi|^p + |\eta|^q - n_0(x)),$$

from which we conclude that  $\bar{\partial}L(x, \cdot) \in M_{\Omega, p}(\mathbb{R}^N)$ .  $\square$

### 3.2 Self-dual Lagrangians on $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$

We now show how one can “lift” an  $\Omega$ -dependent selfdual Lagrangian to a selfdual Lagrangian on the phase space  $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ . This will allow us to give a variational formulation and resolution –via Theorem 2.1– of equations involving maximal monotone operators in divergence form. The following extends a result in [17].

**Theorem 3.1** *Let  $\mathfrak{B} \in M_{\Omega, p}(\mathbb{R}^N)$  for some  $p > 1$ , then for every  $u^* \in W^{-1,q}(\Omega)$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , there exist  $\bar{u} \in W_0^{1,p}(\Omega)$  and  $\bar{f}(x) \in L^q(\Omega; \mathbb{R}^N)$  such that*

$$\begin{cases} \bar{f} \in \mathfrak{B}(x, \nabla \bar{u}(x)) & \text{a.e. } x \in \Omega \\ -\operatorname{div}(\bar{f}) = u^*. \end{cases} \tag{29}$$

It is obtained by minimizing the functional

$$I(u) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ -\operatorname{div}(f) = u^*}} \int_{\Omega} [L(x, \nabla u(x), f(x)) - \langle u(x), u^*(x) \rangle_{\mathbb{R}^N}] dx$$

on  $W^{1,p}(\Omega)$ , where  $L$  is an  $\Omega$ -dependent selfdual Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$  associated to  $\mathfrak{B}$  in such a way that  $\bar{\partial}L(x, \cdot) = \mathfrak{B}(x, \cdot)$  for a.e.  $x \in \Omega$ .

The above theorem will follow from the representation of a maximal monotone map in  $M_{\Omega, p}(\mathbb{R}^N)$  by an  $\Omega$ -dependent selfdual Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$  (Proposition 3.1) combined with the following two propositions.

**Proposition 3.2** *Suppose  $L$  is an  $\Omega$ -dependent selfdual Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$  such that  $L(\cdot, 0, 0) \in L^1(\Omega)$ , then the Lagrangian defined on  $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$  by*

$$F(u, u^*) := \inf \left\{ \int_{\Omega} L(x, \nabla u(x), f(x)) dx; f \in L^q(\Omega; \mathbb{R}^N), -\operatorname{div}(f) = u^* \right\}, \tag{30}$$

is selfdual.

**Proof:** Denote  $W_0^{1,p}(\Omega)$  by  $X$  and its dual  $W^{-1,q}(\Omega)$  by  $X^*$ . For a fixed  $(v^*, v) \in X^* \times X$ , we have

$$\begin{aligned}
F^*(v^*, v) &= \sup\{\langle u, v^* \rangle + \langle u^*, v \rangle - F(u, u^*); u \in X, u^* \in X^*\} \\
&= \sup_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ -\operatorname{div}(f) = u^* \\ (u, u^*) \in X \times X^*}} \left\{ \langle u, v^* \rangle + \langle u^*, v \rangle - \int_{\Omega} L(x, \nabla u(x), f(x)) \, dx \right\} \\
&= \sup\{\langle u, v^* \rangle - \langle \operatorname{div}(f), v \rangle - \int_{\Omega} L(x, \nabla u(x), f(x)) \, dx; u \in X, f \in L^q(\Omega; \mathbb{R}^N)\} \\
&= \sup\{\langle u, v^* \rangle + \langle f, \nabla v \rangle - \int_{\Omega} L(x, \nabla u(x), f(x)) \, dx; u \in X, f \in L^q(\Omega; \mathbb{R}^N)\}.
\end{aligned}$$

Now set  $E := \{g \in L^p(\Omega; \mathbb{R}^N); g = \nabla u, u \in X\}$  and let  $\chi_E$  be the indicator function in  $L^p(\Omega; \mathbb{R}^N)$ , e.g.,

$$\chi_E(g) = \begin{cases} 0 & g \in E, \\ +\infty & \text{elsewhere.} \end{cases}$$

An easy computation shows that

$$\chi_E^*(f) = \begin{cases} 0 & \operatorname{div}(f) = 0, \\ +\infty & \text{elsewhere.} \end{cases}$$

Fix  $f_0 \in L^q(\Omega; \mathbb{R}^N)$  with  $-\operatorname{div}(f_0) = v^*$ . It follows that

$$\begin{aligned}
F^*(v^*, v) &= \sup\{\langle g, f_0 \rangle + \langle f, \nabla v \rangle - \int_{\Omega} L(x, g(x), f(x)) \, dx - \chi_E(g); g \in L^p(\Omega; \mathbb{R}^N), f \in L^q(\Omega; \mathbb{R}^N)\} \\
&= \inf\left\{ \int_{\Omega} L^*(x, f_0 - f, \nabla v) \, dx + \chi_E^*(f); f \in L^q(\Omega; \mathbb{R}^N) \right\}.
\end{aligned}$$

Note that we have used the fact that  $(\int_{\Omega} L(x, \cdot, \cdot) \, dx)^*(g, f) = \int_{\Omega} L^*(x, f(x), g(x)) \, dx$  that holds since  $L(\cdot, 0, 0) \in L^1(\Omega)$ . We finally get

$$\begin{aligned}
F^*(v^*, v) &= \inf\left\{ \int_{\Omega} L^*(x, f_0 - f, \nabla v) \, dx; f \in L^q(\Omega; \mathbb{R}^N), \operatorname{div}(f) = 0 \right\} \\
&= \inf\left\{ \int_{\Omega} L(x, \nabla v, f_0 - f) \, dx; f \in L^q(\Omega; \mathbb{R}^N), \operatorname{div}(f) = 0 \right\} \\
&= \inf\left\{ \int_{\Omega} L(x, \nabla v, f) \, dx; f \in L^q(\Omega; \mathbb{R}^N), -\operatorname{div}(f) = v^* \right\} \\
&= F(v, v^*).
\end{aligned}$$

□

Here is our variational resolution for equation (29).

**Proposition 3.3** *Suppose  $L$  is  $\Omega$ -dependent selfdual Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$ . Assume the following coercivity condition:*

$$L(x, a, b) \geq m(x) + C(|a|^p + |b|^q) \text{ for all } a, b \in \mathbb{R}^N, \quad (31)$$

where  $m \in L^1(\Omega)$  and  $C$  is a positive constant. Then for every  $u^* \in W^{-1,q}(\Omega)$  the functional

$$I(u) = \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ -\operatorname{div}(f) = u^*}} \int_{\Omega} [L(x, \nabla u(x), f(x)) - \langle u(x), u^*(x) \rangle_{\mathbb{R}^N}] \, dx$$

attains its minimum at some  $\bar{u} \in W_0^{1,p}(\Omega)$  such that  $I(\bar{u}) = 0$ , and there exists  $\bar{f} \in L^q(\Omega; \mathbb{R}^N)$  such that

$$\begin{cases} \bar{f}(x) \in \bar{\partial}L(x, \nabla \bar{u}(x)) & \text{a.e. } x \in \Omega \\ -\operatorname{div}(\bar{f}) = u^*. \end{cases}$$

**Proof.** Take  $f_0 \in L^q(\Omega; \mathbb{R}^N)$  with  $-\operatorname{div}(f_0(x)) = u^*(x)$ . Since  $L$  is an  $\Omega$ -dependent selfdual Lagrangian,  $M(x, a, b) := L(x, a, b + f_0(x)) - \langle a, f_0(x) \rangle$  is also an  $\Omega$ -dependent selfdual Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$ . It follows from the above proposition that

$$F(v, v^*) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ -\operatorname{div}(f) = v^*}} \int_{\Omega} M(x, \nabla v(x), f(x)) dx$$

is a selfdual Lagrangian on  $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ . In view of the coercivity condition, Theorem 2.1 applies and there exists  $\bar{u} \in W_0^{1,p}(\Omega)$  such that

$$F(\bar{u}, 0) = \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ -\operatorname{div}(f) = 0}} \int_{\Omega} M(x, \nabla \bar{u}(x), f(x)) dx = 0.$$

Using again the coercivity condition, we get that the above infimum is attained at some  $f_1 \in L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div}(f_1) = 0$ . Thus,

$$\begin{aligned} 0 = F(\bar{u}, 0) &= \int_{\Omega} M(x, \nabla \bar{u}(x), f_1(x)) dx \\ &= \int_{\Omega} [L(x, \nabla \bar{u}(x), f_1(x) + f_0(x)) - \langle \nabla \bar{u}(x), f_0(x) \rangle_{\mathbb{R}^N}] dx \\ &= \int_{\Omega} [L(x, \nabla \bar{u}(x), f_1(x) + f_0(x)) - \langle \nabla \bar{u}(x), f_1(x) + f_0(x) \rangle_{\mathbb{R}^N}] dx. \end{aligned}$$

Taking into consideration that  $L(x, \nabla \bar{u}(x), f_1(x) + f_0(x)) - \langle \nabla \bar{u}(x), f_1(x) + f_0(x) \rangle_{\mathbb{R}^N} \geq 0$ , we obtain that the latter is indeed zero, i.e.,

$$L(x, \nabla \bar{u}(x), f_1(x) + f_0(x)) - \langle \nabla \bar{u}(x), f_1(x) + f_0(x) \rangle_{\mathbb{R}^N} = 0 \quad \text{for a.e. } x \in \Omega.$$

Setting  $\bar{f} := f_1 + f_0$ , we finally get that  $\bar{f}(x) \in \bar{\partial}L(x, \nabla \bar{u}(x))$  for a.e.  $x \in \Omega$  and that  $-\operatorname{div}(\bar{f}) = u^*$ .  $\square$

### 3.3 Variational formula for the homogenized maximal monotone vector field

Given a maximal monotone family  $\mathfrak{B}$  in  $M_{\Omega,p}(\mathbb{R}^N)$  that is  $Q$ -periodic for an open non-degenerate parallelogram  $Q$  in  $\mathbb{R}^n$ , its homogenization  $\mathfrak{B}_{hom}$  is normally given by the non-variational formula (8). In this section, we shall give a variational formulation for the vector field  $\mathfrak{B}_{hom}$  in terms of a suitably homogenized selfdual Lagrangian  $L_{hom}$  derived from the  $\Omega$ -dependent selfdual Lagrangian associated to  $\mathfrak{B}$ .

**Theorem 3.2** *Assume  $\mathfrak{B} \in M_{\Omega,p}(\mathbb{R}^N)$  is  $Q$ -periodic and let  $L$  be an  $\Omega$ -dependent selfdual Lagrangian such that  $\mathfrak{B}(x, \cdot) = \bar{\partial}L(x, \cdot)$  given by Proposition 3.1. If the operator  $\mathfrak{B}_{hom}$  is given by (8), then  $\mathfrak{B}_{hom} = \bar{\partial}L_{hom}$  where  $L_{hom}$  is the selfdual Lagrangian on  $\mathbb{R}^N \times \mathbb{R}^N$  given by*

$$L_{hom}(\xi, \eta) = \min_{\substack{\varphi \in W_{\#}^{1,p}(Q) \\ g \in L_{\#}^q(Q; \mathbb{R}^N)}} \frac{1}{|Q|} \int_Q L(x, \xi + \nabla \varphi(x), \eta + g(x)) dx. \quad (32)$$

The proof will follow from the following propositions. First, we show that the homogenized Lagrangian  $L_{hom}$  inherits many of the properties of the original  $\Omega$ -dependent Lagrangian  $L$  such as convexity, boundedness and coercivity.

**Proposition 3.4** *Assume  $L$  is an  $\Omega$ -dependent Lagrangian on  $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$  satisfying (25) for some  $p, q > 1$ . Then  $L_{hom}$  is convex and lower semi continuous, and for every  $a^*, b^* \in \mathbb{R}^n$ ,*

$$L_{hom}^*(a^*, b^*) = \inf_{\substack{\varphi \in W_{\#}^{1,q'}(Q) \\ g \in L_{\#}^{p'}(Q; \mathbb{R}^N)}} \frac{1}{|Q|} \int_Q L^*(x, a^* + g(x), b^* + \nabla \varphi(x)) dx, \quad (33)$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{q} + \frac{1}{q'} = 1$ . Furthermore,

$$C_0(|a|^p + |b|^q - 1) \leq L_{hom}(a, b) \leq C_1(1 + |a|^p + |b|^q) \quad \text{for all } a, b \in \mathbb{R}^n. \quad (34)$$

The following gives the relation between the subdifferentials of  $L_{hom}$  and of  $L$ .

**Proposition 3.5** *For each  $a, b \in \mathbb{R}^n$ , the subdifferential map  $\partial L_{hom}(a, b)$  is given by*

$$\partial L_{hom}(a, b) = \frac{1}{|Q|} \int_Q \partial L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)) \, dy,$$

where  $\tilde{\varphi} \in W_{\#}^{1,p}(Q)$  and  $\tilde{g} \in L_{\#}^q(Q; \mathbb{R}^N)$  are such that

$$L_{hom}(a, b) = \frac{1}{|Q|} \int_Q L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)) \, dy.$$

We need a few preliminary facts. For each  $1 < r < \infty$ , set

$$E_r := \{f = \nabla u \in L^r(Q; \mathbb{R}^N); \text{ for some } u \in W_{\#}^{1,r}(Q)\}$$

and

$$E_r + \mathbb{R}^n := \{f + \eta : f \in E_r, \eta \in \mathbb{R}^n\}.$$

The Poincaré-Wirtinger inequality which states that for  $D$  bounded open and convex, there exists  $K := K(r, D) > 0$  such that

$$\|u - \frac{1}{|D|} \int_D u\|_{L^r(D)} \leq K \|\nabla u\|_{W^{1,r}(D)} \text{ for every } u \in W^{1,r}(D),$$

implies that  $E_r + \mathbb{R}^n$  is a convex weakly closed subset of  $L^r(Q; \mathbb{R}^N)$ . The indicator function of  $E_r + \mathbb{R}^n$ ,

$$\chi_{E_r + \mathbb{R}^n}(f) = \begin{cases} 0 & f \in E_r + \mathbb{R}^n, \\ +\infty & f \in L^r(Q; \mathbb{R}^N) \setminus (E_r + \mathbb{R}^n), \end{cases}$$

is therefore convex and lower semi-continuous in  $L^r(Q; \mathbb{R}^N)$ . Assuming that  $r'$  is the conjugate of  $r$ , i.e.,  $\frac{1}{r} + \frac{1}{r'} = 1$ , define

$$E_{r'}^{\perp} := \{g \in L^{r'}(Q; \mathbb{R}^N); \int_{\Omega} \langle f(x), g(x) \rangle_{\mathbb{R}^N} \, dx = 0 \text{ for all } f \in E_r + \mathbb{R}^n\}.$$

The Fenchel-Legendre dual  $\chi_{E_r + \mathbb{R}^n}^*$  of  $\chi_{E_r + \mathbb{R}^n}$  is then given by,

$$\begin{aligned} \chi_{E_r + \mathbb{R}^n}^*(g) &= \sup_{f \in L^r(Q; \mathbb{R}^N)} \left\{ \int_Q \langle f(x), g(x) \rangle_{\mathbb{R}^N} \, dx - \chi_{E_r + \mathbb{R}^n}(f) \right\} \\ &= \sup_{f \in E_r + \mathbb{R}^n} \int_Q \langle f(x), g(x) \rangle_{\mathbb{R}^N} \, dx = \chi_{E_{r'}^{\perp}}(g), \end{aligned}$$

for all  $g \in L^{r'}(Q; \mathbb{R}^N)$ . Also due to the convexity and lower semi-continuity of  $\chi_{E_r + \mathbb{R}^n}$  one has  $\chi_{E_{r'}^{\perp}}^* = \chi_{E_r + \mathbb{R}^n}$ . Similarly one can deduce that,

$$\chi_{E_{r'}^{\perp} + \mathbb{R}^n}^*(f) = \chi_{E_r}(f)$$

for all  $f \in L^r(Q; \mathbb{R}^N)$ . Note also that  $E_r$  is the isometric image of  $W_{\#}^{1,r}(Q)$  by  $\nabla$  and  $E_r^{\perp} = L_{\#}^r(Q; \mathbb{R}^N)$ .

**Proof of Proposition 3.4.** We first prove (33). Fix  $(a^*, b^*) \in \mathbb{R}^n \times \mathbb{R}^n$  and write

$$\begin{aligned} L_{hom}^*(a^*, b^*) &= \sup_{(a,b) \in \mathbb{R}^n \times \mathbb{R}^n} \{ \langle a, a^* \rangle_{\mathbb{R}^N} + \langle b, b^* \rangle_{\mathbb{R}^N} - L_{hom}(a, b) \} \\ &= \sup_{\substack{(a,b) \in \mathbb{R}^n \times \mathbb{R}^n \\ (\varphi, g) \in W_{\#}^{1,p}(Q) \times L_{\#}^q(Q; \mathbb{R}^N)}} \frac{1}{|Q|} \int_Q \left[ \langle a, a^* \rangle_{\mathbb{R}^N} + \langle b, b^* \rangle_{\mathbb{R}^N} - L(x, a + \nabla \varphi(x), b + g(x)) \right] \, dx \\ &= \sup_{\substack{(a,b) \in \mathbb{R}^n \times \mathbb{R}^n \\ (f,g) \in E_p \times E_q^{\perp}}} \frac{1}{|Q|} \int_Q \left[ \langle a, a^* \rangle_{\mathbb{R}^N} + \langle b, b^* \rangle_{\mathbb{R}^N} - L(x, a + f(x), b + g(x)) \right] \, dx. \end{aligned}$$

Setting  $A(x) = a + f(x)$ ,  $B(x) = b + g(x)$  and substituting above we have

$$\begin{aligned}
L_{hom}^*(a^*, b^*) &= \sup_{\substack{A \in E_p + \mathbb{R}^n \\ B \in E_q^\perp + \mathbb{R}^n}} \frac{1}{|Q|} \int_Q \left[ \langle A, a^* \rangle_{\mathbb{R}^N} + \langle B, b^* \rangle_{\mathbb{R}^N} - L(x, A(x), B(x)) \right] dx \\
&= \sup_{\substack{A \in L^p(\Omega; \mathbb{R}^n) \\ B \in L^q(\Omega; \mathbb{R}^n)}} \left\{ \frac{1}{|Q|} \int_Q \left[ \langle A, a^* \rangle_{\mathbb{R}^N} + \langle B, b^* \rangle_{\mathbb{R}^N} - L(x, A(x), B(x)) \right] dx \right. \\
&\quad \left. - \chi_{E_p + \mathbb{R}^n}(A) - \chi_{E_q^\perp + \mathbb{R}^n}(B) \right\}.
\end{aligned}$$

Now using the fact that the Fenchel dual of a sum is their *inf-convolution*, we obtain

$$\begin{aligned}
L_{hom}^*(a^*, b^*) &= \inf_{\substack{f \in L^{q'}(Q; \mathbb{R}^n) \\ g \in L^{p'}(Q; \mathbb{R}^n)}} \left\{ \frac{1}{|Q|} \int_Q L^*(x, a^* - g(x), b^* - f(x)) dx + \chi_{E_{p'}^\perp}(g) + \chi_{E_{q'}}(f) \right\} \\
&= \inf_{\substack{f \in E_{q'} \\ g \in E_{p'}^\perp}} \frac{1}{|Q|} \int_Q L^*(x, a^* - g(x), b^* - f(x)) dx \\
&= \inf_{\substack{\varphi \in W_{\#}^{1, q'}(Q) \\ g \in L_{\#}^{p'}(Q; \mathbb{R}^N)}} \frac{1}{|Q|} \int_Q L^*(x, a^* + g(x), b^* + \nabla \varphi(x)) dx.
\end{aligned}$$

This proves (33), which then implies that  $L_{hom}^{**} = L_{hom}$  and therefore  $L_{hom}$  is convex and lower semi-continuous.

We now prove estimate (34). In fact, the upper bound simply follows from

$$L_{hom}(a, b) \leq \frac{1}{|Q|} \int_Q L(x, a, b) dx \leq C_1(|a|^p + |b|^q + 1).$$

For the lower bound, note first that since  $C_0(|a|^p + |b|^q - 1) \leq L(x, a, b)$  for all  $a, b \in \mathbb{R}^N$ , it follows that

$$L^*(x, a, b) \leq \frac{C_0(p-1)}{(C_0p)^{p'}} |a|^{p'} + \frac{C_0(q-1)}{(C_0q)^{q'}} |b|^{q'} + C_0 \quad \text{for all } a, b \in \mathbb{R}^N.$$

On then get from (33) that

$$L_{hom}^*(a, b) \leq \frac{1}{|Q|} \int_Q L^*(x, a, b) dx \leq \frac{C_0(p-1)}{(C_0p)^{p'}} |a|^{p'} + \frac{C_0(q-1)}{(C_0q)^{q'}} |b|^{q'} + C_0 \quad \text{for all } a, b \in \mathbb{R}^N,$$

from which we get that  $L_{hom}(a, b) = L_{hom}^{**}(a, b) \geq C_0(|a|^p + |b|^q - 1)$  for all  $a, b \in \mathbb{R}^N$ .  $\square$

**Proof of Proposition 3.5.** Setting  $A(a, b) := \frac{1}{|Q|} \int_Q \partial L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)) dy$ , we shall first show that  $A \subset \partial L_{hom}$ . For that consider  $(a_1, b_1) \in \mathbb{R}^N \times \mathbb{R}^N$ ,  $\varphi \in W_{\#}^{1, p}(Q)$  and  $g \in L_{\#}^q(Q; \mathbb{R}^N)$ . From the convexity of  $L$ :

$$\begin{aligned}
L(y, a_1 + \nabla \varphi(y), b_1 + g(y)) &\geq L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)) \\
&\quad + \langle \partial_1 L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)), a_1 + \nabla \varphi(y) - a - \nabla \tilde{\varphi}(y) \rangle_{\mathbb{R}^N} \\
&\quad + \langle \partial_2 L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)), b_1 + g(y) - b - \tilde{g}(y) \rangle_{\mathbb{R}^N}.
\end{aligned}$$

Averaging the above on  $Q$  implies that

$$\frac{1}{|Q|} \int_Q L(y, a_1 + \nabla \varphi(y), b_1 + g(y)) dy \geq L_{hom}(a, b) + \langle A(a, b), (a_1 - a, b_1 - b) \rangle_{\mathbb{R}^N \times \mathbb{R}^N},$$

from which we get

$$L_{hom}(a_1, b_1) \geq L_{hom}(a, b) + \langle A(a, b), (a_1 - a, b_1 - b) \rangle_{\mathbb{R}^N \times \mathbb{R}^N}.$$

This implies that  $A \subset \partial L_{hom}$ . To prove the reverse inclusion, let  $(d, c)$  be in  $\partial L_{hom}(a, b)$ . Since  $L_{hom}$  is convex and lower semi-continuous, we have

$$L_{hom}(a, b) + L_{hom}^*(d, c) = \langle a, d \rangle_{\mathbb{R}^N} + \langle b, c \rangle_{\mathbb{R}^N}.$$

It follows from Proposition 3.4 that there exist  $\varphi \in W_{\#}^{1,q'}(Q)$  and  $g \in L_{\#}^{p'}(Q; \mathbb{R}^N)$  such that

$$L_{hom}^*(a^*, b^*) = \frac{1}{|Q|} \int_Q L^*(x, a^* + g(x), b^* + \nabla \varphi(x)) \, dx,$$

and therefore

$$\frac{1}{|Q|} \int_Q L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)) \, dy + \frac{1}{|Q|} \int_Q L^*(x, d + g(x), c + \nabla \varphi(x)) \, dx = \langle a, d \rangle_{\mathbb{R}^N} + \langle b, c \rangle_{\mathbb{R}^N}.$$

On the other hand,

$$\langle a, d \rangle_{\mathbb{R}^N} + \langle b, c \rangle_{\mathbb{R}^N} = \frac{1}{|Q|} \int_Q \langle a + \nabla \tilde{\varphi}(y), d + g(y) \rangle_{\mathbb{R}^N} + \frac{1}{|Q|} \int_Q \langle b + \tilde{g}(y), c + \nabla \varphi(y) \rangle_{\mathbb{R}^N} \, dy,$$

which together with the previous equality yield

$$\int_Q [L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)) + L^*(y, d + g(y), c + \nabla \varphi(y)) - \langle a + \nabla \tilde{\varphi}(y), d + g(y) \rangle_{\mathbb{R}^N} - \langle b + \tilde{g}(y), c + \nabla \varphi(y) \rangle_{\mathbb{R}^N}] \, dy = 0.$$

Taking into account that the integrand is non-negative we obtain

$$L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)) + L^*(y, d + g(y), c + \nabla \varphi(y)) - \langle a + \nabla \tilde{\varphi}(y), d + g(y) \rangle_{\mathbb{R}^N} - \langle b + \tilde{g}(y), c + \nabla \varphi(y) \rangle_{\mathbb{R}^N} = 0$$

for almost all  $y \in Q$ . This implies that

$$(d + g(y), c + \nabla \varphi(y)) \in \partial L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)) \quad \text{a.e. } y \in Q.$$

Integrating the above over  $Q$  implies that

$$(d, c) \in \frac{1}{|Q|} \int_Q \partial L(y, a + \nabla \tilde{\varphi}(y), b + \tilde{g}(y)),$$

which completes the proof.  $\square$

**Proof of Theorem 3.2** Let  $\eta \in \bar{\partial} L_{hom}(\xi)$  in such a way that  $L_{hom}(\xi, \eta) = \langle \xi, \eta \rangle_{\mathbb{R}^N}$ . From the definition of  $L_{hom}$ , we have

$$L_{hom}(\xi, \eta) = \min_{\substack{\varphi \in W_{\#}^{1,p}(Q) \\ g \in L_{\#}^q(Q; \mathbb{R}^N)}} \frac{1}{|Q|} \int_Q L(x, \xi + \nabla \varphi(x), \eta + g(x)) \, dx.$$

From the coercivity assumptions on  $L$ , it follows that there exist  $\varphi \in W_{\#}^{1,p}(Q)$  and  $g \in L_{\#}^q(Q; \mathbb{R}^N)$  such that

$$L_{hom}(\xi, \eta) = \frac{1}{|Q|} \int_Q L(x, \xi + \nabla \varphi(x), \eta + g(x)) \, dx.$$

Hence

$$\begin{aligned} 0 &= L_{hom}(\xi, \eta) - \langle \xi, \eta \rangle_{\mathbb{R}^N} \\ &= \frac{1}{|Q|} \int_Q L(x, \xi + \nabla \varphi(x), \eta + g(x)) \, dx - \langle \xi, \eta \rangle_{\mathbb{R}^N} \\ &= \frac{1}{|Q|} \int_Q [L(x, \xi + \nabla \varphi(x), \eta + g(x)) - \langle \xi + \nabla \varphi(x), \eta + g(x) \rangle_{\mathbb{R}^N}] \, dx, \end{aligned}$$



and since the integrand is non-negative we obtain

$$L(x, \xi + \nabla\varphi(x), \eta + g(x)) - \langle \xi + \nabla\varphi(x), \eta + g(x) \rangle_{\mathbb{R}^N} = 0 \quad \text{for a.e. } x \in Q,$$

from which we have

$$\eta + g(x) \in \bar{\partial}L(x, \xi + \nabla\varphi(x)) = \mathfrak{B}(x, \xi + \nabla\varphi(x))$$

and finally  $\eta = \int_Q (\eta + g(x)) dx$ . This implies that  $\bar{\partial}L_{hom} \subset \mathfrak{B}_{hom}$  and the equality follows since  $\bar{\partial}L_{hom}$  is itself a maximal monotone operator.  $\square$

## 4 A variational approach to homogenization

We start by studying the homogenization of a class of Lagrangians that is more general than the one introduced in Proposition 3.2. We shall then apply this result to deduce Theorem 1.2 announced in the introduction.

### 4.1 The homogenization of general Lagrangians on $W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$

The following homogenization result does not require the  $\Omega$ -dependent Lagrangian  $L$  to be selfdual nor that the exponents  $p$  and  $q$  to be conjugate.

**Theorem 4.1** *Let  $\Omega$  be a regular bounded domain and  $Q$  an open non-degenerate parallelogram in  $\mathbb{R}^n$ . Let  $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  be an  $\Omega$ -dependent Lagrangian such that:*

- (1) *For each  $a, b \in \mathbb{R}^N$  the function  $x \rightarrow L(x, a, b)$  is  $Q$ -periodic.*
- (2) *There exist constants  $C_0, C_1 \geq 0$  and exponents  $p, q > 1$  such that for every  $x \in \Omega$ ,*

$$C_0(|a|^p + |b|^q - 1) \leq L(x, a, b) \leq C_1(|a|^p + |b|^q + 1).$$

Let  $\{G_\varepsilon; \varepsilon > 0\}$  be the family of functionals on  $W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  defined by

$$G_\varepsilon(u, \tau) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L\left(\frac{x}{\varepsilon}, \nabla u(x), \tau(x) + f(x)\right) dx,$$

and set

$$L_{hom}(a, b) := \min_{\substack{\varphi \in W_{\#}^{1,p}(Q) \\ g \in L_{\#}^q(Q; \mathbb{R}^N)}} \frac{1}{|Q|} \int_Q L(x, a + \nabla\varphi(x), b + g(x)) dx. \quad (35)$$

Equip  $L^q(\Omega; \mathbb{R}^N)$  with the following topology denoted by  $\mathcal{T}$ ,

$$\tau_n \rightarrow \tau \text{ for } \mathcal{T} \quad \text{if and only if} \quad \tau_n \rightarrow \tau \text{ weakly in } L^q(\Omega; \mathbb{R}^N) \text{ and } \operatorname{div}(\tau_n) \rightarrow \operatorname{div}(\tau) \text{ strongly in } W^{-1,q}(\Omega),$$

There exists then a Lagrangian  $G_{hom}$  on  $W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  that is a  $\Gamma$ -limit of  $\{G_\varepsilon; \varepsilon > 0\}$  as  $\varepsilon \rightarrow 0$ . Moreover,  $G_{hom}$  is given by the formula

$$G_{hom}(u, \tau) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L_{hom}(\nabla u(x), \tau(x) + f(x)) dx, \quad (36)$$

**Remark 4.1** Note that when the Lagrangian  $L$  is independent of the third variable, i.e.,

$$L(x, a, b) = \varphi(x, a) \quad \text{for all } (x, a, b) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N,$$

for some function  $\varphi : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ , this homogenization problem is completely understood. Also, when the Lagrangian  $L$  is independent of the second variable then this problem can be dealt using the bi-continuity of the Fenchel dual (see for instance [1, 10]). The proof for the general Lagrangians consists of two parallel parts corresponding to each of these variables and should be done simultaneously for both. The part regarding the first variable is rather standard and the same argument can be found for instance in [1].

The proof of Theorem 4.1 will follow from the following two lemmas.

**Lemma 4.1** *For any  $(u, \tau) \in W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$ , there exists a sequence  $(u_\varepsilon, \tau_\varepsilon) \in W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  such that  $u_\varepsilon \rightarrow u$  strongly in  $L^p(\Omega)$ ,  $\tau_\varepsilon \rightarrow \tau$  strongly in  $L^q(\Omega; \mathbb{R}^N)$  and*

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) \leq G_{hom}(u, \tau).$$

**Lemma 4.2** *Let  $f \in L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div}(f) = 0$ . For any  $(u, \tau) \in W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  and any sequence  $(u_\varepsilon, \tau_\varepsilon)$  such that  $u_\varepsilon \rightarrow u$  strongly in  $L^p(\Omega)$  and  $\tau_\varepsilon \rightarrow \tau$  with the  $\mathcal{T}$ -topology in  $L^q(\Omega; \mathbb{R}^N)$ , we have*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x), \tau_\varepsilon(x) + f(x)\right) dx \geq \int_{\Omega} L_{hom}(\nabla u(x), \tau(x) + f(x)) dx.$$

We first show how Theorem 4.1 follows from the two lemmas above.

The limsup property in the definition of  $\Gamma$ -convergence readily follows from Lemma 4.1. For the liminf property we must show that for any  $(u, \tau) \in W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  and any sequence  $\{(u_\varepsilon, \tau_\varepsilon)\} \subset W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  such that

$$u_\varepsilon \rightarrow u \text{ strongly in } L^p(\Omega) \quad \text{and} \quad \tau_\varepsilon \rightarrow \tau \quad \text{in the } \mathcal{T} \text{ - topology,}$$

we have that

$$\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) \geq G_{hom}(u, \tau).$$

By Lemma 4.2 we have

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon, \tau_\varepsilon + f\right) dx \geq \int_{\Omega} L_{hom}(\nabla u, \tau + f) dx,$$

for every  $f \in L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div}(f) = 0$ . Since

$$\inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon, \tau_\varepsilon + f\right) dx = \liminf_{\varepsilon \rightarrow 0} \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon, \tau_\varepsilon + f\right) dx,$$

we obtain that  $\liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) \geq G_{hom}(u, \tau)$ , as desired.  $\square$

**Proof of Lemma 4.1.** Note that without loss of generality we may assume  $L \geq 0$ . Assume first that  $u$  is an affine function and  $\tau$  is constant on  $\Omega$ , that is

$$u(x) = \langle a, x \rangle + \alpha \quad \text{and} \quad \tau(x) = b,$$

for some  $a$  and  $b$  in  $\mathbb{R}^n$  and  $\alpha \in \mathbb{R}$ . Fix  $\eta \in \mathbb{R}^n$  and let  $\tilde{\varphi}$  and  $\tilde{g}$  to be the minimizers on the formula for  $L_{hom}$  given by (35):

$$L_{hom}(a, b + \eta) = \frac{1}{|Q|} \int_Q L(x, a + \nabla \tilde{\varphi}(x), b + \eta + \tilde{g}(x)). \quad (37)$$

Define

$$u_\varepsilon(x) := u(x) + \varepsilon \tilde{\varphi}\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \tau_\varepsilon(x) := \tau.$$

Note that by Lemma 5.4 in the Appendix,  $\tilde{g}$  can be extended by periodicity to an element of  $L^q_{loc}(\mathbb{R}^N; \mathbb{R}^N)$ , still denoted by  $\tilde{g}$  such that  $\operatorname{div}(\tilde{g}) = 0$ . It follows that

$$\begin{aligned} \limsup_{\varepsilon} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) &= \limsup_{\varepsilon} \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L\left(\frac{x}{\varepsilon}, a + \nabla \tilde{\varphi}\left(\frac{x}{\varepsilon}\right), b + f(x)\right) dx \\ &\leq \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \limsup_{\varepsilon} \int_{\Omega} L\left(\frac{x}{\varepsilon}, a + \nabla \tilde{\varphi}\left(\frac{x}{\varepsilon}\right), b + f(x)\right) dx \\ &\leq \limsup_{\varepsilon} \int_{\Omega} L\left(\frac{x}{\varepsilon}, a + \nabla \tilde{\varphi}\left(\frac{x}{\varepsilon}\right), b + \eta + \tilde{g}\left(\frac{x}{\varepsilon}\right)\right) dx. \end{aligned}$$

By Lemma 5.2 of the Appendix we have as  $\varepsilon \rightarrow 0$ ,

$$\int_{\Omega} L\left(\frac{x}{\varepsilon}, a + \nabla \tilde{\varphi}\left(\frac{x}{\varepsilon}\right), b + \eta + \tilde{g}\left(\frac{x}{\varepsilon}\right)\right) dx \rightarrow \frac{|\Omega|}{|Q|} \int_Q L(y, a + \nabla \tilde{\varphi}(y), b + \eta + \tilde{g}(y)) dy.$$

It then follows from (37) that

$$\limsup_{\varepsilon \rightarrow 0} G_{\varepsilon}(u_{\varepsilon}, \tau_{\varepsilon}) \leq |\Omega| L_{hom}(a, b + \eta),$$

and since  $\eta$  is arbitrary, we have that

$$\limsup_{\varepsilon \rightarrow 0} G_{\varepsilon}(u_{\varepsilon}, \tau_{\varepsilon}) \leq \inf_{\eta \in \mathbb{R}^n} |\Omega| L_{hom}(a, b + \eta).$$

By Lemma 5.1 of the Appendix we have

$$\inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L_{hom}(a, b + f(x)) dx \geq \inf_{\eta \in \mathbb{R}^n} |\Omega| L_{hom}(a, b + \eta),$$

and thus we conclude, as desired

$$\limsup_{\varepsilon \rightarrow 0} G_{\varepsilon}(u_{\varepsilon}, \tau_{\varepsilon}) \leq \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L_{hom}(a, b + f(x)) dx = G_{hom}(u, \tau).$$

Assume now that  $u$  is a piecewise affine function and  $\tau$  is a piecewise constant function on  $\Omega$ , that is for  $\{\hat{\Omega}_j\}_{j \in I_1}$  and  $\{\tilde{\Omega}_k\}_{k \in I_2}$ , both finite polyhedral partitions of  $\Omega$ , we have

$$u(x) = \langle a_j, x \rangle + \alpha_j \text{ for } x \in \hat{\Omega}_j \quad \text{and} \quad \tau(x) = b_k \text{ for } x \in \tilde{\Omega}_k,$$

for fixed  $a_j \in \mathbb{R}^n$  and  $b_k \in \mathbb{R}^n$  and constants  $\alpha_j$ . By considering non-empty intersections  $\hat{\Omega}_j \cap \tilde{\Omega}_k$  and re-indexing them, we can consider  $\{\Omega_i\}_{i \in I}$  a polyhedral partition of  $\Omega$  such that

$$u(x) = \langle a_i, x \rangle + \alpha_i \text{ for } x \in \Omega_i \quad \text{and} \quad \tau(x) = b_i \text{ for } x \in \Omega_i.$$

Analogous to what was done previously, fix  $\{\eta_i\} \subset \mathbb{R}^N$  and let  $\tilde{\varphi}_i$  and  $\tilde{g}_i$  be such that

$$L_{hom}(a_i, b_i + \eta_i) = \frac{1}{|Q|} \int_Q L(x, a_i + \nabla \tilde{\varphi}_i(x), b_i + \eta_i + \tilde{g}_i(x)) dx,$$

and set  $u_{\varepsilon}^i(x) := u(x) + \varepsilon \tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right)$ .

Unfortunately, we cannot consider  $u_{\varepsilon}$  as the piecewise construction of the above functions, as the  $\varphi_i$  won't necessarily match along the interface between the  $\Omega_i$  and thus will not in general be a function in  $W^{1,p}(\Omega)$ . This can be remedied by the following standard construction (see for instance [1]): Let  $\Sigma$  be the interface set between the  $\Omega_i$ , and define for  $\delta > 0$ ,  $\Sigma_{\delta} := \{x \in \Omega : d(x, \Sigma) \leq \delta\}$ . Consider a smooth function  $\Psi_{\delta}$  so that

$$\Psi_{\delta}(x) = \begin{cases} 1 & x \in \Sigma_{\delta} \\ 0 & x \in \Omega \setminus \Sigma_{2\delta}, \end{cases}$$

and define

$$u_{\varepsilon}^{\delta}(x) := (1 - \Psi_{\delta}(x)) u_{\varepsilon}^i(x) + \Psi_{\delta}(x) u(x) \text{ for } x \in \Omega_i \quad \text{and} \quad \tau_{\varepsilon} := \tau.$$

It can be checked that the function  $u_{\varepsilon}^{\delta}$  lies in  $W^{1,p}(\Omega)$ . Note that by Lemma 5.4 of the Appendix, each  $\tilde{g}_i$  can be extended by periodicity to an element of  $L_{loc}^q(\mathbb{R}^N; \mathbb{R}^N)$ , still denoted by  $\tilde{g}_i$  such that  $\operatorname{div}(\tilde{g}_i) = 0$ . Thus  $\operatorname{div}(\eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)) = 0$  on  $\mathbb{R}^N$  and in particular on  $\Omega_i \setminus \Sigma_{\delta}$ . Define  $f_{\varepsilon, \delta}(x) = \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)$  on  $\Omega_i \setminus \Sigma_{\delta}$ . One can also extend (using Theorem 2.5 and Corollary 2.8 in [20])  $f_{\varepsilon, \delta}$  to an element in  $L^q(\Omega; \mathbb{R}^N)$ , still denoted

by  $f_{\varepsilon,\delta}$  such that  $\|f_{\varepsilon,\delta}\|_{L^q(\Omega;\mathbb{R}^N)}$  is bounded and  $\operatorname{div}(f_{\varepsilon,\delta}) = 0$ . Take now any  $0 < t < 1$ , then

$$\begin{aligned}
G_\varepsilon(tu_\varepsilon^\delta, \tau_\varepsilon) &= \inf_{\substack{f \in L^q(\Omega;\mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_\Omega L(\nabla tu_\varepsilon^\delta, \tau_\varepsilon + f) \, dx \\
&\leq \int_\Omega L(\nabla tu_\varepsilon^\delta, \tau_\varepsilon + f_{\varepsilon,\delta}) \, dx \\
&= \sum_i \int_{\Omega_i \setminus \Sigma_\delta} L\left(\frac{x}{\varepsilon}, t(1 - \Psi_\delta) \nabla u_\varepsilon^i + t\Psi_\delta \nabla u + (1-t) \frac{t}{(1-t)} (u - u_\varepsilon^i) \nabla \Psi_\delta, b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx \\
&\quad + \int_{\Sigma_\delta} L(\nabla tu_\varepsilon^\delta, \tau_\varepsilon + f_{\varepsilon,\delta}) \, dx
\end{aligned}$$

Since  $L$  is convex in the middle variable and since  $t(1 - \Psi_\delta) + t\Psi_\delta + (1-t) = 1$ , we obtain

$$\begin{aligned}
G_\varepsilon(tu_\varepsilon^\delta, \tau_\varepsilon) &\leq \sum_i \int_{\Omega_i \setminus \Sigma_\delta} t(1 - \Psi_\delta) L\left(\frac{x}{\varepsilon}, a_i + \nabla \tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx \\
&\quad + \sum_i \int_{\Omega_i \setminus \Sigma_\delta} (1-t) L\left(\frac{x}{\varepsilon}, \frac{t}{(1-t)} (u - u_\varepsilon^i) \nabla \Psi_\delta, b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx \\
&\quad + \int_{\Sigma_{2\delta} \setminus \Sigma_\delta} t\Psi_\delta L\left(\frac{x}{\varepsilon}, \nabla u, b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx \\
&\quad + \int_{\Sigma_\delta} L(\nabla tu_\varepsilon^\delta, \tau_\varepsilon + f_{\varepsilon,\delta}) \, dx.
\end{aligned}$$

For the first term on the right hand side of this inequality we have

$$\int_{\Omega_i \setminus \Sigma_\delta} t(1 - \Psi_\delta) L\left(\frac{x}{\varepsilon}, a_i + \nabla \tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx \leq \int_{\Omega_i \setminus \Sigma_\delta} L\left(\frac{x}{\varepsilon}, a_i + \nabla \tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx.$$

Using the boundedness of  $L$  we get the following estimate for the second term,

$$\begin{aligned}
\int_{\Omega_i \setminus \Sigma_\delta} (1-t) L\left(\frac{x}{\varepsilon}, \frac{t}{(1-t)} (u - u_\varepsilon^i) \nabla \Psi_\delta, b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx &\leq C_1(1-t) \int_{\Omega_i \setminus \Sigma_\delta} \left( \left| \frac{t}{(1-t)} (u - u_\varepsilon^i) \nabla \Psi_\delta \right|^p \right. \\
&\quad \left. + |b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)|^q + 1 \right) \, dx,
\end{aligned}$$

and similarly

$$\int_{\Sigma_{2\delta} \setminus \Sigma_\delta} t\Psi_\delta L\left(\frac{x}{\varepsilon}, \nabla u, b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx \leq C_1 \int_{\Sigma_{2\delta} \setminus \Sigma_\delta} \left( 1 + |\nabla u|^p + |b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)|^q \right) \, dx,$$

as well as

$$\int_{\Sigma_\delta} L(\nabla tu_\varepsilon^\delta, \tau_\varepsilon + f_{\varepsilon,\delta}) \, dx \leq C_1 \int_{\Sigma_\delta} \left( 1 + |\nabla tu_\varepsilon^\delta|^p + |\tau_\varepsilon + f_{\varepsilon,\delta}|^q \right) \, dx.$$

It then follows that

$$\begin{aligned}
G_\varepsilon(tu_\varepsilon^\delta, \tau_\varepsilon) &\leq \sum_i \int_{\Omega_i \setminus \Sigma_\delta} L\left(\frac{x}{\varepsilon}, a_i + \nabla \tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \, dx \\
&\quad + C_1(1-t) \sum_i \int_{\Omega_i \setminus \Sigma_\delta} \left( \left| \frac{t}{(1-t)} (u - u_\varepsilon^i) \nabla \Psi_\delta \right|^p + |b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)|^q + 1 \right) \, dx \\
&\quad + C_1 \int_{\Sigma_{2\delta} \setminus \Sigma_\delta} \left( 1 + |\nabla u|^p + |b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)|^q \right) \, dx \\
&\quad + C_1 \int_{\Sigma_\delta} \left( 1 + |\nabla tu_\varepsilon^\delta|^p + |\tau_\varepsilon + f_{\varepsilon,\delta}|^q \right) \, dx.
\end{aligned}$$

By taking  $\limsup_{\varepsilon \rightarrow 0}$  on both sides and considering  $u_\varepsilon^i \rightarrow u$  on  $L^p(\Omega_i)$ , and then letting  $t \rightarrow 1$  and  $\delta \rightarrow 0$  we finally get,

$$\limsup_{\substack{t \rightarrow 1 \\ \delta \rightarrow 0}} \limsup_{\varepsilon \rightarrow 0} G_\varepsilon(tu_\varepsilon^\delta, \tau_\varepsilon) \leq \sum_i \frac{|\Omega_i|}{|Q|} \int_Q L(x, a_i + \nabla \tilde{\varphi}_i(x), b_i + \eta_i + \tilde{g}_i(x)) \, dx. \quad (38)$$

Also note that,

$$\sum_i \frac{|\Omega_i|}{|Q|} \int_Q L(x, a_i + \nabla \tilde{\varphi}_i(x), b_i + \eta_i + \tilde{g}_i(x)) \, dx = \sum_i |\Omega_i| L_{hom}(a_i, b_i + \eta_i).$$

A diagonalization argument yields from limit (38) the existence of some  $t(\varepsilon)$  and  $\delta(\varepsilon)$  such that  $t(\varepsilon) \rightarrow 1$  and  $\delta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Defining  $u_\varepsilon := t(\varepsilon)u_\varepsilon^{\delta(\varepsilon)}$ , we obtain

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) \leq \sum_i |\Omega_i| L_{hom}(a_i, b_i + \eta_i),$$

and since the  $\{\eta_i\}$  is arbitrary one has

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) \leq \sum_i |\Omega_i| \inf_{\eta_i \in \mathbb{R}^n} L_{hom}(a_i, b_i + \eta_i).$$

Now we use Lemma 5.1 of the Appendix to obtain

$$\sum_i |\Omega_i| \inf_{\eta_i \in \mathbb{R}^n} L_{hom}(a_i, b_i + \eta_i) \leq \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_\Omega L_{hom}(\nabla u(x), \tau(x) + f(x)) \, dx,$$

from which we get  $\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) \leq G_{hom}(u, \tau)$ .

Finally, consider any  $(u, \tau) \in W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$ . There exists then a sequence  $\{u_n\}$  of piecewise affine functions and a sequence  $\{\tau_n\}$  of piecewise constant functions such that  $(u_n, \tau_n) \rightarrow (u, \tau)$ . By Proposition 3.4, the function  $G_{hom}$  are continuous, so we also have

$$\lim_n G_{hom}(u_n, \tau_n) = G_{hom}(u, \tau).$$

For each  $n$ , we have shown the existence of  $(u_n^\varepsilon, \tau_n^\varepsilon)$  such that  $u_n^\varepsilon \rightarrow u_n$  and  $\tau_n^\varepsilon \rightarrow \tau_n$  in  $L^p(\Omega)$  and  $L^q(\Omega; \mathbb{R}^N)$  respectively and

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u_n^\varepsilon, \tau_n^\varepsilon) \leq G_{hom}(u_n, \tau_n),$$

so we get

$$\limsup_n \limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u_n^\varepsilon, \tau_n^\varepsilon) \leq G_{hom}(u, \tau).$$

From the same diagonalization argument as before, there exists some  $n(\varepsilon)$  such that  $n(\varepsilon) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$  for which, by defining  $(u_\varepsilon, \tau_\varepsilon) := (u_{n(\varepsilon)}^\varepsilon, \tau_{n(\varepsilon)}^\varepsilon)$  we obtain

$$u_\varepsilon \rightarrow u \text{ strongly in } L^p(\Omega), \tau_\varepsilon \rightarrow \tau \text{ strongly in } L^q(\Omega; \mathbb{R}^N)$$

and

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) \leq G_{hom}(u, \tau).$$

This concludes the proof of Lemma 4.1.  $\square$

**Proof of Lemma 4.2.** Let  $(u, \tau) \in W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  and  $f \in L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div}(f) = 0$ . We assume that  $u_\varepsilon \rightarrow u$  strongly in  $L^p(\Omega)$  and  $\tau_\varepsilon \rightarrow \tau_\varepsilon$  in  $\mathcal{T}$ . For constant vectors  $a_i, b_i, \eta_i \in \mathbb{R}^n$ , consider as before functions  $\tilde{\varphi}_i \in W_{\#}^{1,p}(Q)$  and  $\tilde{g}_i \in L_{\#}^q(Q; \mathbb{R}^N)$  such that

$$L_{hom}(a_i, b_i + \eta_i) = \frac{1}{|Q|} \int_Q L(x, a_i + \nabla \tilde{\varphi}_i(x), b_i + \eta_i + \tilde{g}_i(x)) \, dx.$$

Denote  $\partial_1 L$  the subdifferential of  $L$  with respect to the middle variable and  $\partial_2 L$  the subdifferential of  $L$  with respect to the last variable. From the above we have both

$$\operatorname{div}\left(\partial_1 L(y, a_i + D\tilde{\varphi}_i(y), b_i + \eta_i + \tilde{g}_i(y))\right) = 0 \text{ a.e. } y \in Q, \quad (39)$$

and

$$\int_Q \langle \partial_2 L(y, a_i + D\tilde{\varphi}_i(y), b_i + \eta_i + \tilde{g}_i(y)), g(y) \rangle dy = 0, \quad (40)$$

for any  $g \in L^q_{\#}(Q; \mathbb{R}^N)$ . It follows from (40) that

$$\partial_2 L(y, a_i + \nabla\tilde{\varphi}_i(y), b_i + \eta_i + \tilde{g}_i(y)) = \nabla w(y) \quad \text{a.e. } y \in Q, \quad (41)$$

for some  $w \in W^1_{\#}(Q)$ . It also follows from Lemma 5.3 that  $w$  can be extended by periodicity to an element in  $W^1_{loc}(\mathbb{R}^N)$ . Now, let  $\hat{u} \in W^{1,p}(\Omega)$  be a piecewise affine functions and  $\hat{\tau} \in L^q(\Omega; \mathbb{R}^N)$  be a piecewise constant function such that for some partition  $\{\Omega_i\}$  of  $\Omega$  we have

$$\hat{u}(x) = \langle a_i, x \rangle + \alpha_i \text{ for } x \in \Omega_i \text{ and } \hat{\tau}(x) = b_i \text{ for } x \in \Omega_i.$$

Consider now for  $x \in \Omega_i$ ,

$$\hat{u}_\varepsilon(x) := \hat{u}(x) + \varepsilon\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right) \text{ and } \hat{\tau}_\varepsilon(x) := \hat{\tau}(x).$$

From the convexity of  $L$  we get

$$\begin{aligned} L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x), \tau_\varepsilon(x) + f(x)\right) &\geq L\left(\frac{x}{\varepsilon}, \nabla \hat{u}_\varepsilon(x), \hat{\tau}_\varepsilon(x) + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \\ &\quad + \langle \partial_1 L\left(\frac{x}{\varepsilon}, \nabla \hat{u}_\varepsilon(x), \hat{\tau}_\varepsilon(x) + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), \nabla u_\varepsilon(x) - \nabla \hat{u}_\varepsilon(x) \rangle \\ &\quad + \langle \partial_2 L\left(\frac{x}{\varepsilon}, \nabla \hat{u}_\varepsilon(x), \hat{\tau}_\varepsilon(x) + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), \tau_\varepsilon(x) - \hat{\tau}_\varepsilon(x) \rangle \\ &\quad + \langle \partial_2 L\left(\frac{x}{\varepsilon}, \nabla \hat{u}_\varepsilon(x), \hat{\tau}_\varepsilon(x) + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), f(x) - \eta_i - \tilde{g}_i\left(\frac{x}{\varepsilon}\right) \rangle. \end{aligned}$$

Consider now smooth functions  $\Psi_i : \Omega_i \rightarrow \mathbb{R}$  with compact support such that  $0 < \Psi_i < 1$ . Multiplying the above convexity inequality by  $\Psi_i$ , integrating over  $\Omega_i$  and adding over all  $i$ , we get the following:

$$\begin{aligned} \int_\Omega L\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon, \tau_\varepsilon + f\right) dx &\geq \sum_i \int_{\Omega_i} L\left(\frac{x}{\varepsilon}, a_i + \nabla\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \Psi_i(x) dx \\ &\quad + \sum_i \int_{\Omega_i} \langle \partial_1 L\left(\frac{x}{\varepsilon}, a_i + \nabla\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), \nabla u_\varepsilon(x) - \nabla \hat{u}_\varepsilon(x) \rangle \Psi_i(x) dx \\ &\quad + \sum_i \int_{\Omega_i} \langle \partial_2 L\left(\frac{x}{\varepsilon}, a_i + \nabla\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), \tau_\varepsilon(x) - \hat{\tau}_\varepsilon(x) \rangle \Psi_i(x) dx \\ &\quad + \sum_i \int_{\Omega_i} \langle \partial_2 L\left(\frac{x}{\varepsilon}, a_i + \nabla\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), f(x) - \eta_i \rangle \Psi_i(x) dx \\ &\quad + \sum_i \int_{\Omega_i} \langle \partial_2 L\left(\frac{x}{\varepsilon}, a_i + \nabla\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), -\tilde{g}_i\left(\frac{x}{\varepsilon}\right) \rangle \Psi_i(x) dx. \end{aligned}$$

Now we deal with each term independently. For the first term on the right hand side of the above expression we have

$$\int_{\Omega_i} L\left(\frac{x}{\varepsilon}, a_i + \nabla\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right) \Psi_i(x) dx \rightarrow \int_{\Omega_i} L_{hom}(a_i, b_i + \eta_i) \Psi_i(x) dx,$$

by virtue of Lemma 5.2.

For the second term, by integrating by parts and by then taking into account (39) we obtain

$$\begin{aligned} &\int_{\Omega_i} \langle \partial_1 L\left(\frac{x}{\varepsilon}, a_i + \nabla\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), \nabla u_\varepsilon(x) - \nabla \hat{u}_\varepsilon(x) \rangle \Psi_i(x) dx \\ &= - \int_{\Omega_i} \langle \partial_1 L\left(\frac{x}{\varepsilon}, a_i + \nabla\tilde{\varphi}_i\left(\frac{x}{\varepsilon}\right), b_i + \eta_i + \tilde{g}_i\left(\frac{x}{\varepsilon}\right)\right), (u_\varepsilon - \hat{u}_\varepsilon) \nabla \Psi_i(x) \rangle dx. \end{aligned}$$

It follows from Lemma 5.2 and Proposition 3.5 below, that if  $\varepsilon \rightarrow 0$  then,

$$\int_{\Omega_i} \langle \partial_1 L(\frac{x}{\varepsilon}, a_i + \nabla \tilde{\varphi}_i(\frac{x}{\varepsilon}), b_i + \eta_i + \tilde{g}_i(\frac{x}{\varepsilon})), (u_\varepsilon - \hat{u}_\varepsilon) \nabla \Psi_i(x) \rangle dx \rightarrow \int_{\Omega_i} \langle \partial_1 L_{hom}(a_i, b_i + \eta_i), (u - \hat{u}) \nabla \Psi_i(x) \rangle dx$$

Integrate by parts on more time to get

$$\int_{\Omega_i} \langle \partial_1 L_{hom}(a_i, b_i + \eta_i), (u - \hat{u}) \nabla \Psi_i(x) \rangle dx = - \int_{\Omega_i} \langle \partial_1 L_{hom}(a_i, b_i + \eta_i), \nabla u - \nabla \hat{u} \rangle \Psi_i(x) dx,$$

from which one has

$$\begin{aligned} \int_{\Omega_i} \langle \partial_1 L(\frac{x}{\varepsilon}, a_i + \nabla \tilde{\varphi}_i(\frac{x}{\varepsilon}), b_i + \eta_i + \tilde{g}_i(\frac{x}{\varepsilon})), \nabla u_\varepsilon(x) - \nabla \hat{u}_\varepsilon(x) \rangle \Psi_i(x) dx &\rightarrow \\ \int_{\Omega_i} \langle \partial_1 L_{hom}(a_i, b_i + \eta_i), \nabla u - \nabla \hat{u} \rangle \Psi_i(x) dx. \end{aligned}$$

For the third term, we use (41) to get  $\partial_2 L(\frac{x}{\varepsilon}, a_i + D\tilde{\varphi}_i(\frac{x}{\varepsilon}), b_i + \eta_i + \tilde{g}_i(\frac{x}{\varepsilon})) = \nabla w(\frac{x}{\varepsilon})$  for some  $w \in W_{\#}^{1,p}(Q)$ . Using an integration by parts, we obtain

$$\begin{aligned} \int_{\Omega_i} \langle \nabla w(\frac{x}{\varepsilon}), \tau_\varepsilon(x) - \hat{\tau}_\varepsilon(x) \rangle \Psi_i(x) dx &= - \int_{\Omega_i} \varepsilon w(\frac{x}{\varepsilon}) \operatorname{div}(\tau_\varepsilon(x) - \hat{\tau}_\varepsilon(x)) \Psi_i(x) dx \\ &\quad - \int_{\Omega_i} \varepsilon w(\frac{x}{\varepsilon}) \langle \nabla \Psi_i(x), \tau_\varepsilon(x) - \hat{\tau}_\varepsilon(x) \rangle dx, \end{aligned}$$

which goes to 0 as  $\varepsilon \rightarrow 0$  since  $\tau_\varepsilon \rightarrow \tau$  in the  $\mathcal{T}$ -topology.

Similarly as above, the fourth term can be seen to converge to

$$\int_{\Omega_i} \langle \partial_2 L_{hom}(a_i, b_i + \eta_i), f(x) - \eta_i \rangle \Psi_i(x) dx,$$

while for the fifth term, we first observe that the function

$$m_i(x) := \langle \partial_2 L(x, a_i + \nabla \tilde{\varphi}_i(x), b_i + \eta_i + \tilde{g}_i(x)), \tilde{g}_i(x) \rangle$$

is  $Q$ -periodic, and thus setting  $(m_i)_\varepsilon(x) := m_i(\frac{x}{\varepsilon})$ , it follows from Lemma 5.2 that  $(m_i)_\varepsilon \rightharpoonup \bar{m}_i$  weakly in  $L^1$ , where

$$\bar{m}_i = \frac{1}{|Q|} \int_Q \langle \partial_2 L(y, a_i + \nabla \tilde{\varphi}_i(y), b_i + \eta_i + \tilde{g}_i(y)), -\tilde{g}_i(y) \rangle dy,$$

which is equal to 0 in view of (40). The fifth term therefore disappears as  $\varepsilon \rightarrow 0$ .

Putting now all of the above together we obtain that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} L(\frac{x}{\varepsilon}, \nabla u_\varepsilon, \tau_\varepsilon + f) dx &\geq \sum_i \int_{\Omega_i} L_{hom}(a_i, b_i + \eta_i) \Psi_i(x) dx \\ &\quad + \sum_i \int_{\Omega_i} \langle \partial_1 L_{hom}(a_i, b_i + \eta_i), \nabla u(x) - \nabla \hat{u}(x) \rangle \Psi_i(x) dx \\ &\quad + \sum_i \int_{\Omega_i} \langle \partial_2 L_{hom}(a_i, b_i + \eta_i), f(x) - \eta_i \rangle \Psi_i(x) dx. \end{aligned}$$

By taking into account the estimate

$$|\partial L_{hom}(a, b)| \leq M(1 + |a|^{p-1} + |b|^{q-1}) \quad \text{for all } a, b \in \mathbb{R}^N,$$

which follows from estimate (34) in Proposition 3.4, and letting  $\Psi_i \uparrow 1$  on each  $\Omega_i$ , it follows from the dominated convergence theorem that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} L(\frac{x}{\varepsilon}, \nabla u_\varepsilon, \tau_\varepsilon + f) dx &\geq \int_{\Omega} L_{hom}(\nabla \hat{u}(x), \hat{\tau}(x) + \tilde{f}(x)) dx \\ &\quad + \int_{\Omega} \langle \partial_1 L_{hom}(\nabla \hat{u}(x), \hat{\tau}(x) + \tilde{f}(x)), \nabla u(x) - \nabla \hat{u}(x) \rangle dx \\ &\quad + \int_{\Omega} \langle \partial_2 L_{hom}(\nabla \hat{u}(x), \hat{\tau}(x) + \tilde{f}(x)), f(x) - \tilde{f}(x) \rangle dx. \end{aligned}$$

where  $\tilde{f} \in L^q(\Omega; \mathbb{R}^N)$  is a function defined by  $\tilde{f}(x) = \eta_i$  on  $\Omega_i$ . The above is valid for arbitrary piecewise affine function  $\hat{u}$ , and piecewise constant functions  $\hat{\tau}, \tilde{f}$ . We can then let  $\hat{u} \rightarrow u$  in  $W^{1,p}(\Omega)$  and  $\hat{\tau} \rightarrow \tau$  and  $\tilde{f} \rightarrow f$  in  $L^q(\Omega; \mathbb{R}^N)$  to obtain

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} L\left(\frac{x}{\varepsilon}, \nabla u_{\varepsilon}, \tau_{\varepsilon} + f\right) dx \geq \int_{\Omega} L_{hom}(\nabla u(x), \tau(x) + f(x)) dx.$$

This completes the proof.  $\square$

Before proceeding to the next subsection, we note the following slight extension of Lemma 4.1, which will be needed for Proposition 4.1 below. We note that the proof is known when  $G_{\varepsilon}$  is independent of the second variable, and here we show that the same proof with minor modification works for general Lagrangians just as in Theorem 4.1.

**Lemma 4.3** *Let  $G_{\varepsilon}$  and  $G_{hom}$  be as in Theorem 4.1. Then, for any  $(u, \tau) \in W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$ , there exist a sequence  $(u_{\varepsilon}, \tau_{\varepsilon})$  such that  $u - u_{\varepsilon} \rightarrow 0$  weakly in  $W^{1,p}(\Omega)$  and  $\tau_{\varepsilon} \rightarrow \tau$  in the  $\mathcal{T}$ -topology. Furthermore,  $u - u_{\varepsilon} \in W_0^{1,p}(\Omega)$  and for this sequence:*

$$\limsup_{\varepsilon \rightarrow 0} G_{\varepsilon}(u_{\varepsilon}, \tau_{\varepsilon}) \leq G(u, \tau).$$

**Proof.** From Theorem 4.1, there exist a sequence  $(\tilde{u}_{\varepsilon}, \tau_{\varepsilon})$  with  $\tilde{u}_{\varepsilon} \rightarrow u$  in  $L^p(\Omega)$  and  $\tau_{\varepsilon} \rightarrow \tau$  in the  $\mathcal{T}$ -topology, such that

$$G_{hom}(u, \tau) = \lim_{\varepsilon \rightarrow 0} G_{\varepsilon}(\tilde{u}_{\varepsilon}, \tau_{\varepsilon}).$$

Up to a subsequence one may assume that

$$\tilde{u}_{\varepsilon} \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega).$$

Pick any  $\varphi \in W_0^{1,p}(\Omega)$  with  $\varphi > 0$  in  $\Omega$ . Define:

$$u_{\varepsilon}(x) := \begin{cases} \tilde{u}_{\varepsilon} & u(x) - \varphi(x) \leq \tilde{u}_{\varepsilon} \leq u(x) + \varphi(x) \\ u(x) - \varphi(x) & \tilde{u}_{\varepsilon}(x) < u(x) - \varphi(x) \\ u(x) + \varphi(x) & u(x) + \varphi(x) < \tilde{u}_{\varepsilon}(x) \end{cases}.$$

Note that  $u_{\varepsilon} - u \in W_0^{1,p}(\Omega)$  and since  $\tilde{u}_{\varepsilon} \rightharpoonup u$  weakly in  $W^{1,p}(\Omega)$ , so  $u_{\varepsilon} \rightharpoonup u$  weakly in  $W^{1,p}(\Omega)$ . Note that  $L + C_0 \geq 0$ . For any  $f \in L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div}(f) = 0$  we have

$$\begin{aligned} G_{\varepsilon}(u_{\varepsilon}, \tau_{\varepsilon}) + C_0|\Omega| &\leq \int_{\{u_{\varepsilon} \neq \tilde{u}_{\varepsilon}\}} \left[ L\left(\frac{x}{\varepsilon}, \nabla u_{\varepsilon}(x), f(x) + \tau_{\varepsilon}(x)\right) + C_0 \right] dx \\ &\quad + \int_{\{u_{\varepsilon} = \tilde{u}_{\varepsilon}\}} \left[ L\left(\frac{x}{\varepsilon}, \nabla u_{\varepsilon}(x), f(x) + \tau_{\varepsilon}(x)\right) + C_0 \right] dx. \end{aligned}$$

For  $x$  in the set  $\{u_{\varepsilon} \neq \tilde{u}_{\varepsilon}\}$ , the norm of  $\nabla \tilde{u}_{\varepsilon}(x)$  is controlled by the norm of  $|\nabla u(x)| + |\nabla \varphi(x)|$ . It follows that

$$\begin{aligned} G_{\varepsilon}(u_{\varepsilon}, \tau_{\varepsilon}) + C_0|\Omega| &\leq \int_{\{u_{\varepsilon} \neq \tilde{u}_{\varepsilon}\}} \left[ C_1 \left( (|\nabla u(x)| + |\nabla \varphi(x)|)^p + |\tau_{\varepsilon}(x) + f(x)|^q + 1 \right) + C_0 \right] dx \\ &\quad + \int_{\Omega} \left[ L\left(\frac{x}{\varepsilon}, \nabla \tilde{u}_{\varepsilon}(x), f(x) + \tau_{\varepsilon}(x)\right) + C_0 \right] dx. \end{aligned}$$

Take now the infimum over all  $f \in L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div}(f) = 0$  and subtract the latter by  $C_0|\Omega|$ . Since  $|\{u_{\varepsilon} \neq \tilde{u}_{\varepsilon}\}| \rightarrow 0$  and  $G_{hom}(u, \tau) = \lim_{\varepsilon} G_{\varepsilon}(\tilde{u}_{\varepsilon}, \tau_{\varepsilon})$ , we obtain

$$\limsup_{\varepsilon \rightarrow 0} G_{\varepsilon}(u_{\varepsilon}, \tau_{\varepsilon}) \leq G(u, \tau).$$



## 4.2 Variational homogenization of maximal monotone operators on $W_0^{1,p}(\Omega)$

In this section we establish a homogenization result for selfdual Lagrangians on  $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$  where  $\frac{1}{p} + \frac{1}{q} = 1$  and then proceed to prove Theorem 1.2.

**Theorem 4.2** *Let  $\Omega$  be a regular bounded domain,  $Q$  be an open non-degenerate parallelogram in  $\mathbb{R}^n$ , and  $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  be an  $\Omega$ -dependent selfdual Lagrangian such that:*

- (1) *For each  $a, b \in \mathbb{R}^N$  the function  $x \rightarrow L(x, a, b)$  is  $Q$ -periodic,*
- (2) *For some constants  $C_0, C_1 \geq 0$ , we have for every  $x \in \mathbb{R}^N$ ,*

$$C_0(|a|^p + |b|^q) \leq L(x, a, b) \leq C_1(|a|^p + |b|^q + 1), \quad (42)$$

where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $u_n^* \rightarrow u^*$  strongly in  $W^{-1,q}(\Omega)$  and let  $u_n$  be solutions and  $\tau_n$  be momenta for the Dirichlet boundary value problems

$$\begin{cases} \tau_n(x) \in \bar{\partial}L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x)\right) & \text{a.e. } x \in \Omega \\ -\operatorname{div}(\tau_n(x)) = u_n^*(x) & x \in \Omega \\ u_n \in W_0^{1,p}(\Omega). \end{cases} \quad (43)$$

Then, up to a subsequence,

$$u_n \rightarrow u \text{ weakly in } W_0^{1,p}(\Omega) \text{ and } \tau_n \rightarrow \tau \text{ weakly in } L^q(\Omega; \mathbb{R}^N),$$

where  $u$  is a solution and  $\tau$  is a momentum of the homogenized problem

$$\begin{cases} \tau(x) \in \bar{\partial}L_{\text{hom}}(\nabla u(x)) & \text{a.e. } x \in \Omega \\ -\operatorname{div}(\tau(x)) = u^*(x) & x \in \Omega \\ u \in W_0^{1,p}(\Omega), \end{cases} \quad (44)$$

where  $L_{\text{hom}}$  is the selfdual Lagrangian on  $\mathbb{R}^N \times \mathbb{R}^N$  defined by

$$L_{\text{hom}}(a, b) := \min_{\substack{\varphi \in W_{\#}^{1,p}(Q) \\ g \in L_{\#}^q(Q; \mathbb{R}^N)}} \frac{1}{|Q|} \int_Q L(x, a + D\varphi(x), b + g(x)) \, dx. \quad (45)$$

This will follow from the following proposition.

**Proposition 4.1** *Let  $\Omega, Q$  and  $L$  be as in Theorem 4.2, and let  $\{F_\varepsilon; \varepsilon > 0\}$  be the family of selfdual Lagrangians on  $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$  defined by*

$$F_\varepsilon(u, u^*) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ -\operatorname{div} f = u^*}} \int_\Omega L\left(\frac{x}{\varepsilon}, \nabla u(x), f(x)\right) \, dx.$$

Then, there exists a selfdual Lagrangian  $F_{\text{hom}}$  on  $\mathbb{R}^N \times \mathbb{R}^N$  that is a  $\Gamma$ -limit of  $\{F_\varepsilon; \varepsilon > 0\}$  on  $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ . It is given by the formula

$$F_{\text{hom}}(u, u^*) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ -\operatorname{div} f = u^*}} \int_\Omega L_{\text{hom}}(\nabla u(x), f(x)) \, dx,$$

where  $L_{\text{hom}}$  is the selfdual Lagrangian on  $\mathbb{R}^N \times \mathbb{R}^N$  defined by (45), and which satisfies for all  $(a, b) \in \mathbb{R}^N \times \mathbb{R}^N$

$$C_0(|a|^p + |b|^q - 1) \leq L_{\text{hom}}(a, b) \leq C_1(|a|^p + |b|^q + 1).$$

**Proof.** Note first that the selfduality and uniform bounds of  $L_{\text{hom}}$  follow from Proposition 3.4. It also follows from Proposition 3.2 that both  $F_\varepsilon$  and  $F_{\text{hom}}$  are selfdual Lagrangians on  $W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ . Given

$(u, u^*) \in W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ , we now show the existence of a sequence  $\{(u_\varepsilon, u_\varepsilon^*) \in W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)\}$  with  $u_\varepsilon \rightarrow u$  weakly in  $W_0^{1,p}(\Omega)$  and  $u_\varepsilon^* \rightarrow u^*$  strongly in  $W^{-1,q}(\Omega)$  and such that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, u_\varepsilon^*) \leq F_{hom}(u, u^*). \quad (46)$$

For that we consider  $\{G_\varepsilon; \varepsilon > 0\}$  be a family of functionals on  $W^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  defined by

$$G_\varepsilon(u, \tau) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L\left(\frac{x}{\varepsilon}, \nabla u(x), \tau(x) + f(x)\right) dx,$$

and

$$G_{hom}(u, \tau) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L_{hom}(\nabla u(x), \tau(x) + f(x)) dx,$$

Take  $\tau \in L^q(\Omega; \mathbb{R}^N)$  such that  $\operatorname{div}(\tau) = u^*$ . It follows from Lemma 4.1 and Lemma 4.3 that there exists  $(u_\varepsilon, \tau_\varepsilon) \in W_0^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N)$  such that  $u_\varepsilon \rightarrow u$  strongly in  $L^p(\Omega)$  and  $\tau_\varepsilon \rightarrow \tau$  strongly in  $L^q(\Omega; \mathbb{R}^N)$  and

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon, \tau_\varepsilon) \leq G_{hom}(u, \tau).$$

The sequence  $u_\varepsilon$  is bounded in  $W_0^{1,p}(\Omega)$ , so we may assume  $u_\varepsilon \rightarrow u$  weakly in  $W_0^{1,p}(\Omega)$ . Since  $\tau_\varepsilon \rightarrow \tau$  strongly in  $L^q(\Omega; \mathbb{R}^N)$ , it follows that  $u_\varepsilon^* := \operatorname{div}(\tau_\varepsilon) \rightarrow \operatorname{div}(\tau) = u^*$  strongly in  $W^{-1,q}(\Omega)$ . Thus, the inequality (46) follows by noticing that  $G_\varepsilon(u_\varepsilon, \tau_\varepsilon) = F_\varepsilon(u_\varepsilon, u_\varepsilon^*)$  and  $G_{hom}(u, \tau) = F_{hom}(u, u^*)$ .

We shall now show that if  $(u, u^*) \in W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$  and  $u_\varepsilon \rightarrow u$  weakly in  $W_0^{1,p}(\Omega)$  and  $u_\varepsilon^* \rightarrow u^*$  strongly in  $W^{-1,q}(\Omega)$  then

$$F_{hom}(u, u^*) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, u_\varepsilon^*). \quad (47)$$

Take an arbitrary element in  $(v, v^*) \in W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ . From the above, there exists  $(v_\varepsilon, v_\varepsilon^*) \in W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$  with  $v_\varepsilon \rightarrow v$  weakly in  $W_0^{1,p}(\Omega)$  and  $v_\varepsilon^* \rightarrow v^*$  strongly in  $W^{-1,q}(\Omega)$  and such that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon, v_\varepsilon^*) \leq F_{hom}(v, v^*).$$

By the self duality of  $F_\varepsilon$  we have

$$\begin{aligned} F_\varepsilon(u_\varepsilon, u_\varepsilon^*) = F_\varepsilon^*(u_\varepsilon^*, u_\varepsilon) &= \sup\{\langle u_\varepsilon, w^* \rangle + \langle u_\varepsilon^*, w \rangle - F_\varepsilon(w, w^*); (w, w^*) \in W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)\} \\ &\geq \langle u_\varepsilon, v_\varepsilon^* \rangle + \langle u_\varepsilon^*, v_\varepsilon \rangle - F_\varepsilon(v_\varepsilon, v_\varepsilon^*), \end{aligned}$$

from which we get

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, u_\varepsilon^*) &\geq \liminf_{\varepsilon \rightarrow 0} \{\langle u_\varepsilon, v_\varepsilon^* \rangle + \langle u_\varepsilon^*, v_\varepsilon \rangle - F_\varepsilon(v_\varepsilon, v_\varepsilon^*)\} \\ &= \langle u, v^* \rangle + \langle u^*, v \rangle - \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon, v_\varepsilon^*) \\ &\geq \langle u, v^* \rangle + \langle u^*, v \rangle - F_{hom}(v, v^*). \end{aligned}$$

Since the above holds for an arbitrary  $(v, v^*) \in W_0^{1,p}(\Omega) \times W^{-1,q}(\Omega)$ , we obtain

$$F_{hom}^*(u^*, u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, u_\varepsilon^*).$$

Taking into consideration that  $F_{hom}$  is selfdual we obtain

$$F_{hom}(u, u^*) = F_{hom}^*(u^*, u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, u_\varepsilon^*),$$

as desired.  $\square$

**Proof of Theorem 4.2.** Since  $(u_n, \tau_n)$  are solutions of (43), it follows that

$$\begin{aligned} 0 &= \int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x), \tau_n(x)\right) dx - \int_{\Omega} \langle \nabla u_n(x), \tau_n(x) \rangle_{\mathbb{R}^N} dx \\ &= \int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x), \tau_n(x)\right) dx - \int_{\Omega} u_n(x) u_n^*(x) dx. \end{aligned} \quad (48)$$

Due to the coercivity assumption on  $L$  and the strong convergence of  $u_n^*$ , it follows that the sequence  $u_n$  is bounded in  $W_0^{1,p}(\Omega)$  and  $\tau_n$  is bounded in  $L^q(\Omega; \mathbb{R}^N)$ . Thus, up to a subsequence,  $u_n \rightharpoonup u$  weakly in  $W_0^{1,p}(\Omega)$  and  $\tau_n \rightharpoonup \tau$  weakly in  $L^q(\Omega; \mathbb{R}^N)$ . We also have  $\operatorname{div}(\tau_n) = u_n^* \rightarrow u^* = \operatorname{div}(\tau)$  strongly in  $W^{-1,q}(\Omega)$ , from which we indeed have  $\tau_n \rightarrow \tau$  in the  $\mathcal{T}$ -topology (introduced in Theorem 4.1).

Taking  $f \in L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div} f = 0$ , it follows from (48) that

$$\begin{aligned} \int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x), \tau_n(x)\right) dx &= \int_{\Omega} u_n(x) u_n^*(x) dx \\ &= - \int_{\Omega} u_n(x) \operatorname{div}(\tau_n + f) dx \\ &= \int_{\Omega} \langle \nabla u_n(x), \tau_n + f \rangle_{\mathbb{R}^N} dx \\ &\leq \int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x), \tau_n + f(x)\right) dx. \end{aligned}$$

This indeed shows that

$$\int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x), \tau_n(x)\right) dx = \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x), \tau_n(x) + f(x)\right) dx.$$

Let

$$G_{\varepsilon_n}(v, \hat{\tau}) := \inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla v(x), \hat{\tau}(x) + f(x)\right) dx.$$

It then follows that  $\int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x), \tau_n(x)\right) dx = G_{\varepsilon_n}(u_n, \tau_n)$ . Define  $H : W_0^{1,p}(\Omega) \times L^q(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}$  by  $H(v, \tilde{\tau}) = \int_{\Omega} \langle \nabla v(x), \tilde{\tau}(x) \rangle_{\mathbb{R}^N} dx$ . Note that  $H$  is continuous if we consider the weak topology of  $W_0^{1,p}(\Omega)$  and the  $\mathcal{T}$ -topology for  $L^q(\Omega; \mathbb{R}^N)$ . It then follows from Lemma 4.2 that

$$\begin{aligned} \int_{\Omega} L_{hom}(\nabla u(x), \tau(x)) dx - H(u, \tau) &\leq \liminf_{\varepsilon_n \rightarrow 0} [G_{\varepsilon_n}(u_n, \tau_n) - H(u_n, \tau_n)] \\ &= \liminf_{\varepsilon_n \rightarrow 0} \left[ \int_{\Omega} L\left(\frac{x}{\varepsilon_n}, \nabla u_n(x), \tau_n(x)\right) dx - \int_{\Omega} u_n(x) \operatorname{div}(\tau_n(x)) dx \right] \\ &= 0. \end{aligned}$$

On the other hand, we have that

$$\int_{\Omega} L_{hom}(\nabla u(x), \tau(x)) dx - H(u, \tau) = \int_{\Omega} \left[ L_{hom}(\nabla u(x), \tau(x)) - \langle \nabla u(x), \tau(x) \rangle_{\mathbb{R}^N} \right] dx \geq 0. \quad (49)$$

which means that the latter is indeed zero, i.e.,

$$\int_{\Omega} \left[ L_{hom}(\nabla u(x), \tau(x)) - \langle \nabla u(x), \tau(x) \rangle_{\mathbb{R}^N} \right] dx = 0.$$

Since the integrand is itself non-negative we have

$$L_{hom}(\nabla u(x), \tau(x)) - \langle \nabla u(x), \tau(x) \rangle_{\mathbb{R}^N} = 0 \quad \text{a.e. } x \in \Omega,$$

which together with  $-\operatorname{div}(\tau(x)) = u^*(x)$ , yields

$$\begin{cases} \tau(x) \in \bar{\partial} L_{hom}(\nabla u(x)), & \text{a.e. } x \in \Omega, \\ -\operatorname{div}(\tau(x)) = u^*(x), & x \in \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

## 5 Appendix

We shall here state some of the results used throughout the proof.

**Lemma 5.1** *Assume  $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function such that  $C_0(|a|^p + |b|^q - 1) \leq L(a, b) \leq C_1(|a|^p + |b|^q + 1)$  for all  $a, b \in \mathbb{R}^n$  where  $p, q > 1$  are two constants. Suppose  $\Omega$  is a bounded open domain in  $\mathbb{R}^N$  and  $\tau_1 \in L^p(\Omega; \mathbb{R}^n)$  and  $\tau_2 \in L^q(\Omega; \mathbb{R}^n)$  are two piecewise constant functions such that*

$$\tau_1(x) = a_i, \quad x \in \Omega_i, \quad \text{and} \quad \tau_2(x) = b_i, \quad x \in \Omega_i,$$

where  $\{\Omega_i\}_{i \in I}$  is a finite polyhedral partitions of  $\Omega$ , and  $\{a_i\}_{i \in I}, \{b_i\}_{i \in I}$  are two sequences  $\in \mathbb{R}^n$ . Then

$$\min_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L(\tau_1, \tau_2(x) + f(x)) \, dx \geq \sum_{i \in I} |\Omega_i| \inf_{\eta_i \in \mathbb{R}^n} L(a_i, b_i + \eta_i).$$

**Proof** We first prove a stronger result (actually an equality) when the set index  $I$  is a singleton. For any constant  $\eta \in \mathbb{R}^n$  we have

$$\min_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L(a, b + f(x)) \, dx \leq \int_{\Omega} L(a, b + \eta) \, dx = |\Omega| L(a, b + \eta),$$

from which we obtain

$$\min_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L(a, b + f(x)) \, dx \leq \inf_{\eta \in \mathbb{R}^n} |\Omega| L(a, b + \eta),$$

Let now  $\tilde{f}$  be the element in  $L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div} \tilde{f} = 0$  such that

$$\int_{\Omega} L(a, b + \tilde{f}(x)) \, dx = \min_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L(a, b + f(x)) \, dx.$$

Using Jensen's inequality, we obtain

$$\begin{aligned} \inf_{\eta \in \mathbb{R}^n} |\Omega| L(a, b + \eta) &\leq |\Omega| L\left(a, b + \frac{1}{|\Omega|} \int_{\Omega} \tilde{f}(x) \, dx\right) \\ &= |\Omega| L\left(\frac{1}{|\Omega|} \int_{\Omega} a \, dx, \frac{1}{|\Omega|} \int_{\Omega} b + \tilde{f}(x) \, dx\right) \\ &\leq \int_{\Omega} L(a, b + \tilde{f}(x)) \, dx \\ &= \min_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L(a, b + f(x)) \, dx. \end{aligned}$$

This completes the proof for  $I$  being a singleton. Now we prove it for the general case. Note first that, using the above argument on each  $\Omega_i$  we have

$$\inf_{\substack{g \in L^q(\Omega_i; \mathbb{R}^N) \\ \operatorname{div} g = 0}} \int_{\Omega_i} L(a_i, b_i + g(x)) \, dx = \inf_{\eta_i \in \mathbb{R}^n} |\Omega_i| L(a_i, b_i + \eta_i). \quad (50)$$

One also can easily deduce that

$$\inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L(\tau_1(x), \tau_2(x) + f(x)) \, dx \geq \sum_i \inf_{\substack{f_i \in L^q(\Omega_i; \mathbb{R}^N) \\ \operatorname{div} f_i = 0}} \int_{\Omega_i} L(a_i, b_i + f_i(x)) \, dx. \quad (51)$$

In fact if  $\inf_{\substack{f \in L^q(\Omega; \mathbb{R}^N) \\ \operatorname{div} f = 0}} \int_{\Omega} L(\tau_1(x), \tau_2(x) + f(x)) \, dx = \int_{\Omega} L(\tau_1(x), \tau_2(x) + \bar{f}(x)) \, dx$  for some  $\bar{f} \in L^q(\Omega; \mathbb{R}^N)$  with  $\operatorname{div}(\bar{f}) = 0$ , then

$$\begin{aligned} \int_{\Omega} L(\tau_1(x), \tau_2(x) + \bar{f}(x)) \, dx &= \sum_{i \in I} \int_{\Omega_i} L(a_i, b_i + \bar{f}(x)) \, dx \\ &\geq \sum_{i \in I} \inf_{\substack{f_i \in L^q(\Omega_i; \mathbb{R}^N) \\ \operatorname{div} f_i = 0}} \int_{\Omega_i} L(a_i, b_i + f_i(x)) \, dx. \end{aligned}$$

The proof therefore follows from combining (50) and (51).  $\square$

The following three Lemmas are standard and we refer to [25] for the proof.

**Lemma 5.2** *Let  $r \geq 1$  and  $f \in L^r(Q)$ . Then  $f$  can be extended by periodicity to a function (still denoted by  $f$ ) belonging to  $L^r_{loc}(\mathbb{R}^N)$ . Moreover, if  $(\varepsilon_k)$  is a sequence of positive real numbers converging to 0 and  $g_k(x) = g(\frac{x}{\varepsilon_k})$ .*

*If  $1 \leq r < \infty$ , then  $f_k \rightarrow M(f) = \frac{1}{|Q|} \int_Q f(x) \, dx$  weakly in  $L^r_{loc}(\mathbb{R}^N)$ ,*

*and*

*if  $r = \infty$ , then  $f_k \rightarrow M(f)$  weak\* in  $L^\infty(\mathbb{R}^N)$ .*

**Lemma 5.3** *Let  $r > 1$  and  $u \in W^{1,r}_{\#}(Q)$ , then  $u$  can be extended by periodicity to an element of  $W^{1,r}_{loc}(\mathbb{R}^N)$ .*

**Lemma 5.4** *Let  $r > 1$  and  $r' = \frac{r}{r-1}$ . Let  $g \in L^{r'}(Q; \mathbb{R}^N)$  such that  $\int_Q \langle g(x), \nabla v(x) \rangle \, dx = 0$  for every  $v \in W^{1,r}_{\#}(Q)$ . Then  $g$  can be extended by periodicity to an element of  $L^{r'}_{loc}(\mathbb{R}^N; \mathbb{R}^N)$ , still denoted by  $g$ , such that  $\operatorname{div}(g) = 0$  in  $D'(\mathbb{R}^N)$ .*

## References

- [1] H. Attouch, *Variational convergence for functions and operators*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.
- [2] H. Attouch, G. Beer, *On the convergence of subdifferentials of convex functions*. Arch. Math. (Basel) 60 (1993), no. 4, 389–400.
- [3] A. Braides, *An Introduction to Homogenization and Gamma-convergence*. School on homogenization, Trieste, 1993. Notes.
- [4] A. Braides,  *$\Gamma$ -convergence for beginners*. Oxford Lecture Series in Mathematics and its Applications, 22. Oxford University Press, Oxford, 2002.
- [5] A. Braides, V. Chiadò Piat, *Remarks on the homogenization of connected media*. Nonlinear Anal. 22 (1994), no. 4, 391–407.
- [6] A. Braides, V. Chiadò Piat, A. Defranceschi, *Homogenization of almost periodic monotone operators*. Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1992), no. 4, 399–432.
- [7] J. Byström, *Correctors for some nonlinear monotone operators*. J. Nonlinear Math. Phys. 8 (2001), no. 1, 8–30.
- [8] V. Chiadò Piat, G. Dal Maso, A. Defranceschi, *G-convergence of monotone operators*. Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990), no. 3, 123–160.
- [9] G. Dal Maso, *An introduction to  $\Gamma$ -convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhauser Boston, Inc., Boston, MA, 1993.

- [10] A. Damlamian, N. Meunier, J. Van Schaftingen, *Periodic homogenization for convex functionals using Mosco convergence*. Ric. Mat. 57 (2008), no. 2, 209–249.
- [11] A. Damlamian, N. Meunier, J. Van Schaftingen, *Periodic homogenization of monotone multivalued operators*. Nonlinear Anal. 67 (2007), no. 12, 3217–3239.
- [12] R. De Arcangelis, *A general homogenization result for almost periodic functionals*. J. Math. Anal. Appl. 156 (1991), no. 2, 358–380.
- [13] S.P. Fitzpatrick, *Representing monotone operators by convex functions*, Proc. Centre for Math. Analysis 20 (1989), p. 59-65.
- [14] G. A. Francfort, F. Murat, L. Tartar, *Homogenization of monotone operators in divergence form with  $x$ -dependent multivalued graphs*. Ann. di Matematica. 2009.
- [15] N. Fusco, G. Moscaricello, *On the homogenization of quasilinear divergence structure operators*. Ann. Mat. Pura Appl. (4) 146 (1987), 1-13.
- [16] N. Ghoussoub, *Anti-selfdual Lagrangians: Variational resolutions of non self-adjoint equations and dissipative evolutions*, AIHP-Analyse non linéaire, 24 (2007) p.171-205.
- [17] N. Ghoussoub, *A variational theory for monotone vector fields*. J. Fixed Point Theory Appl. 4 (2008), no. 1, 107–135.
- [18] N. Ghoussoub, *Self-dual Partial Differential Systems and Their Variational Principles*. Springer Monographs in Mathematics, Springer, New York 2008.
- [19] N. Ghoussoub, A. Moameni, *Selfdual variational principles for periodic solutions of Hamiltonian and other dynamical systems*. Comm. Partial Differential Equations 32 (2007), no. 4-6, 771–795.
- [20] V. Girault, P. A. Raviart, *Finite element methods for Navier-Stokes equations. Theory and algorithms*. Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.
- [21] G. Nguetseng, H. Nnang, *Homogenization of nonlinear monotone operators beyond the periodic setting*. Electron. J. Differential Equations 2003, No. 36, 24 pp. (electronic).
- [22] G. Nguetseng, J. L. Woukeng, *Deterministic homogenization of parabolic monotone operators with time dependent coefficients*. Electron. J. Differential Equations 2004, No. 82, 23 pp. (electronic).
- [23] R.R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Math. 1364, Springer Verlag, New York, Berlin, Tokyo, (1998), 2nd edition 1993.
- [24] V. C. Piat, A. Defranceschi, *Homogenization of Monotone Operators*. Nonlinear Anal. 14 (1990), no. 9, 717–732.
- [25] P. Suquet, *Plasticite et Homogeneisation* Thesis Univ. Paris VI, 1982.