Cototality and the skip operator

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joint work with
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Motivation from symbolic dynamics

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2. If we can enumerate L_X, then we can compute a member of X.

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This is an interesting property for a set to have.

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A set A is *cototal* if $A \leq_e \overline{A}$.

Four classes of enumeration degrees

3. A degree $\mathbf{a}$ is *cototal* if it contains a cototal set.

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total $\Rightarrow$ graph-cototal $\Rightarrow$ cototal $\Rightarrow$ Solon cototal.

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The degrees that contain Σ_2^0 sets are called Σ_2^0 *enumeration degrees*.

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Proof.

Fix a Σ^0_2 set A and an approximation $\{A_s\}_{s<\omega}$. Let

$$f(a) = \begin{cases} 0, & \text{if } a \notin A; \\ \text{the least } s \text{ such that } a \in A_t \text{ for all } t \geq s-1, & \text{otherwise.} \end{cases}$$

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This shows that cototal does not imply total.

Complements of maximal independent sets

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Theorem

Every cototal enumeration degree contains the complement of a maximal independent set for the graph $\omega^{<\omega}$.

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More universal examples of cototal degrees

Theorem (McCarthy)

An enumeration degree is cototal if and only if it contains:

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Theorem (McCarthy)

Every cototal enumeration degree contains the language of a minimal subshift.

The skip and the jump

Definition

$X \leq_e Y$ if and only if there is a c.e. set Γ such that
$$X = \Gamma[Y] = \{x\colon (\exists D)\ \langle x, D\rangle \in \Gamma \ \wedge\ D \subseteq Y\}.$$

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1. Both notions induce operations on degrees.
2. Both notions produce a set $\not\leq_e A$.
3. Note that $A' = K_A \oplus A^\Diamond \equiv_e A \oplus A^\Diamond$.
4. In other words, the jump is the “increasing version” of the skip.

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Neither property holds for the enumeration jump.

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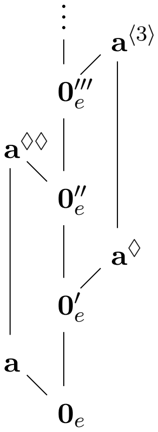
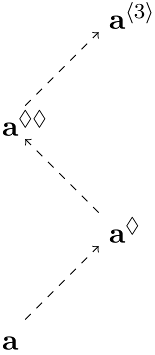
Proof.

Start with S that is not total, but of total degree. Skip-invert to A. Then the degree of A is not cototal, but it is *Solon cototal*, because the complement of K_A is of total degree. $\square$

Iterated skips





Two properties of skips:

1. If $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{a}^{\Diamond} \leq \mathbf{b}^{\Diamond}$;
2. $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$.

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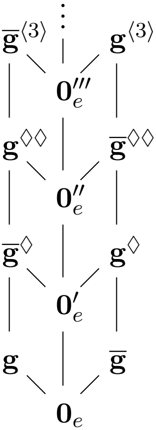
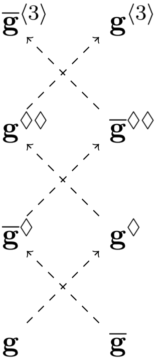
If G is arithmetically generic, then the skips of G and $\overline{G}$ form a double zigzag.





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Such set A and B must be above all hyperarithmetical sets.

Thank you!