Weak König's Lemma for Convex Trees

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Interval Analysis and Constructive Mathematics
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Intermediate Value Theorem (IVT)

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If $f : [0,1] \to \mathbf{R}$ is a uniformly continuous function with $f(0) < 0 < f(1)$, then there exists $x \in [0,1]$ such that $f(x) = 0$.

without Countable Choice, without LEM

$$\mathbf{EL} \vdash \quad \text{WKL} \quad \longrightarrow \quad \text{IVT} \quad \longrightarrow \quad \text{LLPO}$$

with Countable Choice, without LEM

$$\mathbf{BISH} \vdash \quad \text{WKL} \quad \longleftrightarrow \quad \text{IVT} \quad \longleftrightarrow \quad \text{LLPO}$$

without Countable Choice, with LEM

$$\mathbf{RCA}_0 \vdash \quad \text{IVT, LLPO,}$$
$$\mathbf{RCA}_0 \nvdash \quad \text{WKL}$$

WKL and IVT

- From the last observation $\mathbf{RCA}_0 \vdash \mathrm{IVT}, \mathrm{LLPO}$ and $\mathbf{RCA}_0 \nvdash \mathrm{WKL}$, we do not have $\mathrm{IVT} \to \mathrm{WKL}$ in general.
- WKL states "Any infinite binary tree has a path".
- By restricting infinite binary trees to convex ones, get a principle which is equivalent to IVT over some suitable setting (without CC, without LEM).

Real number and function

- A sequence $x = (p_n)_n$ of rationals are *regular* if

$$\forall mn(|p_m - p_n| < 2^{-m} + 2^{-n})$$

- We say x is a *real* ($x \in \mathbf{R}$) if x is regular.
  For $x = (p_n)_n$, x_n denotes p_n.
- The equivalence relation $=_{\mathbf{R}}$ between reals are defined by

$$(p_n)_n =_{\mathbf{R}} (q_n)_n \overset{\text{def}}{\iff} \forall n(|p_n - q_n| \leq 2^{-n+2})$$

The following functions are well-defined

$$\begin{array}{ll} (x \pm_{\mathbf{R}} y)_n = x_{2n+1} \pm y_{2n+1} & |x|_n = |x_n| \\ \max\{x, y\}_n = \max\{x_n, y_n\} & \min\{x, y\}_n = \min\{x_n, y_n\} \\ (x \cdot_{\mathbf{R}} y)_n = x_{2kn+1} \cdot y_{2kn+1}, & \text{where } k = \max\{|x|_0 + 2, |y|_0 + 2\} \end{array}$$

Uniformly continuous function on $[0,1]$

- A uniformly continuous function $f : [0,1] \to \mathbf{R}$ consists of

$$\varphi : \mathbf{Q} \times \mathbf{N} \to \mathbf{Q}, \qquad \nu : \mathbf{N} \to \mathbf{N}$$

s.t.

$$(f(p))_n = \varphi(p,n) \in \mathbf{R}$$
$$\forall n \in \mathbf{N} \forall p,q \in \mathbf{Q}(|p-q| < 2^{-\nu(n)} \to |f(p) - f(q)| < 2^{-n}).$$

For each $x \in [0,1]$, $f(x) \in \mathbf{R}$ is given by

$$(f(x))_n = \varphi(\min\{\max\{x_{\mu(n)}, 0\}, 1\}, n+1),$$

where $\mu(n) = \nu(n+1) + 1$.

Strict order $<_{\mathbf{R}}$

Let x and y are reals.

Order $<_{\mathbf{R}}$

- x is *positive* if $\exists n(x_n > 2^{-n+2})$.
- x is *negative* if $\exists n(x_n < -2^{-n+2})$.
- $x <_{\mathbf{R}} y$ if $y -_{\mathbf{R}} x$ is positive.

Some properties of $<_{\mathbf{R}}$

- $x =_{\mathbf{R}} x' \wedge y =_{\mathbf{R}} y' \wedge x <_{\mathbf{R}} y \to x' <_{\mathbf{R}} y'$
- We have $\forall x, y \in \mathbf{R} \forall n(x_n < y_n \vee x_n = y_n \vee y_n < x_n)$.
- We CANNOT prove $\forall x, y \in \mathbf{R}(x <_{\mathbf{R}} y \vee x =_{\mathbf{R}} y \vee y <_{\mathbf{R}} x)$ constructively. (LPO)

Order $\leq_{\mathbf{R}}$

Let x and y are reals.

Order $\leq_{\mathbf{R}}$

- $x \leq y$ if $y -_{\mathbf{R}} x$ is not positive.

Some properties of $\leq_{\mathbf{R}}$

- $x =_{\mathbf{R}} x' \wedge y =_{\mathbf{R}} y' \wedge x \leq_{\mathbf{R}} y \to x' \leq_{\mathbf{R}} y'$
- We CANNOT prove $\forall x, y \in \mathbf{R}(x \leq_{\mathbf{R}} y \vee_{\mathbf{R}} y \leq_{\mathbf{R}} x)$ constructively. (LLPO)
- We CANNOT prove $\forall x, y \in \mathbf{R}(x \leq_{\mathbf{R}} y \vee_{\mathbf{R}} \neg x \leq_{\mathbf{R}} y)$ constructively. (WLPO)
- We CAN prove that $\forall x, y \in \mathbf{R}(\neg x <_{\mathbf{R}} y \to y \leq_{\mathbf{R}} x)$.

In what follows, we omit $\mathbf{R}$ in $=_{\mathbf{R}}$, $+_{\mathbf{R}}$, $-_{\mathbf{R}}$, $<_{\mathbf{R}}$, $\leq_{\mathbf{R}}$, etc..

IVT in constructive mathematics

Usual proof of IVT

For a uniformly continuous function $f : [0,1] \to \mathbf{R}$, define l_n and r_n as follows:

$$l_0 = 0, r_0 = 1;$$

$$l_{n+1} = \begin{cases} \frac{l_n + r_n}{2} & \text{if } f(\frac{l_n+r_n}{2}) \leq 0 \\ l_n & \text{otherwise} \end{cases};$$

$$r_{n+1} = l_{n+1} + 2^{-(n+1)}.$$

Take $x = \lim_{n\to\infty} l_n$. Then $f(x) = 0$.

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Binary sequence and binary tree

Some notations for binary sequence

- $\{0,1\}^*$: the set of finite sequences of 0 and 1.
- $|s|$: the length of binary sequence.
- $s \preceq t$: s is an initial segment of t, i.e., $s = \langle t(0), ..., t(k) \rangle$ for some $k < |t|$.
- $s * t = \langle s(0), ..., s(|s|-1), t(0), ...t(|t|-1) \rangle$

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- $T \subseteq \{0,1\}^*$ is a *binary tree* if it is closed under initial segments, i.e., $s \preceq t \wedge t \in T$ implies $s \in T$.

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  Note that an infinite tree contains branches with any length.

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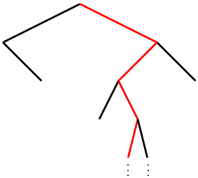
  Note that an infinite tree contains branches with any length.
- A *path* of T is a function $\alpha : \mathbf{N} \to \{0,1\}$ s.t. $\overline{\alpha}n \in T$ for any n, where $\overline{\alpha}n = \langle \alpha(0), ..., \alpha(n-1)\rangle$.

WKL

Weak König's Lemma (WKL)

Any infinite binary tree $T \subseteq \{0,1\}^*$ has a path.



Fact

- WKL is usually proved as follows:
  For an infinite tree T, define α by

$$\alpha(n) = \begin{cases} 0 & \text{if } \{t \in T : \overline{\alpha}n \preceq t\} \text{ is infinite;} \\ 1 & \text{otherwise.} \end{cases} \qquad \leftarrow \mathrm{WLPO}$$

- Constructively, the above construction of α is not allowed.
- Some infinite recursive trees have no recursive path.

WKL for convex trees

Definition

- $s \sqsubset t$ iff $\exists u \preceq s(u * \langle 0 \rangle \preceq s \wedge u * \langle 1 \rangle \preceq t)$.
- For a tree T, let $T_n = \{s \in T : |s| = n\}$.
- A tree T is *convex* if $|u| = n$, $s \sqsubseteq u \sqsubseteq t$, $s \in T_n$ and $t \in T_n$ imply $u \in T$ for each n.

WKL_c

Any infinite binary convex tree has a path.

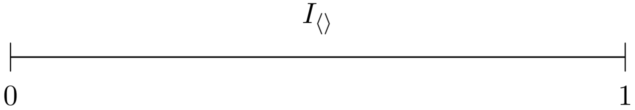
Fact

Trivially WKL implies WKL_c.

Assignment of intervals to binary sequences

For each $s \in \{0,1\}^*$, let $I_s = [l_s, r_s]$ be as follows:

$$l_{\langle\rangle} = 0; \quad l_{s*\langle 0\rangle} = l_a; \quad l_{s*\langle 1\rangle} = l_s + 2^{-(|s|+1)}; \quad r_s = l_s + 2^{-|s|}$$



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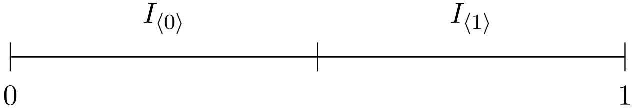

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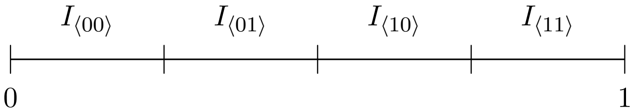

$\mathrm{WKL}_c \to \mathrm{IVT}$

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  1. $|a_n| = |b_n| = n$ and $a_n \sqsubseteq b_n$,
  2. $f(l_{a_n}) < 0 < f(r_{b_n})$,
  3. $\forall c \in \{0,1\}^n (a_n \sqsubset c \sqsubseteq b_n \to |f(l_c)| < 2^{-n})$.

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- Let $T_n = \{u \in \{0,1\}^n \mid a_n \sqsubseteq u \sqsubseteq b_n\}$ for each n, and let $T = \bigcup_{n=0}^{\infty} T_n$.

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- Let $x = \sum_{i=0}^{\infty} \alpha(i) \cdot 2^{-(i+1)}$.

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- Let $x = \sum_{i=0}^{\infty} \alpha(i) \cdot 2^{-(i+1)}$.
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- Thus $f(x) = 0$.

Construction of a_n and b_n

- Let $S = \{u \in \{0,1\}^{n+1} \mid \exists v \in \{0,1\}^n (a_n \sqsubseteq v \sqsubseteq b_n \wedge v \preceq u)\}$
- Divide S into S_-, S_0 and S_+ s.t.

$$c \in S_- \to (f(l_c))_{n+2} < -2^{-(n+2)},$$
$$c \in S_0 \to |(f(l_c))_{n+2}| \leq 2^{-(n+2)},$$
$$c \in S_+ \to 2^{-(n+2)} < (f(l_c))_{n+2}.$$

- If S_- is inhabited, then choose the right-most such $c \in S$ as a_{n+1}. Otherwise set $a_{n+1} = a_n * \langle 0 \rangle$.
- If $\{u \in S_+ \mid a_{n+1} \sqsubset u\}$ is inhabited, then choose the left-most such $c \in S_+$ and choose the right-most d s.t. $a_{n+1} \sqsubseteq d \sqsubset c$ as b_{n+1}. Otherwise set $b_{n+1} = b_n * \langle 1 \rangle$.

Some lemmata for $\mathrm{IVT} \to \mathrm{WKL}_c$

Lemma

Let T be a tree, and let x be a real number such that

$$\forall n \exists a \in T_n (|x - l_a| < 2^{-n}).$$

Then there exists an infinite convex subtree T' of T having at most two nodes at each level, i.e., $\forall n(|T_n| \leq 2)$ and

$$\forall n \forall a' \in T'_n (|x - l_{a'}| < 2^{-n+1}).$$

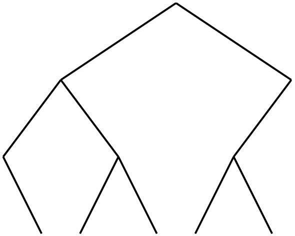
Lemma

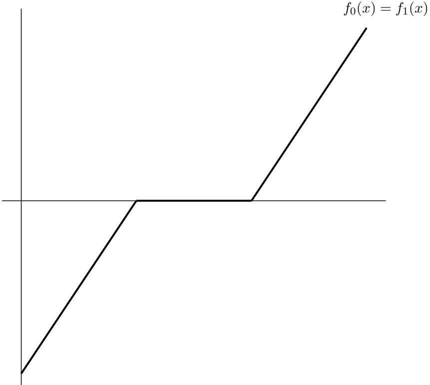
IVT *implies that every infinite convex tree T s.t. $\forall n(|T_n| \leq 2)$ for each n has a path.*

IVT $\to$ WKL$_c$

- Let $(a_n)_n$ and $(b_n)_n$ be sequences of $\{0,1\}^*$ such that $T_n = \{c \in \{0,1\}^n \mid a_n \sqsubseteq c \sqsubseteq b_n\}$ for each n.
- For each n, define a uniformly continuous function $f_n : [0,1] \to \mathbf{R}$ by

$$f_n(x) = \min\{(l_{a_n}+1)^{-1}(3x - l_{a_n} - 1), 0\} + \max\{(2 - r_{b_n})^{-1}(3x - r_{b_n} - 1), 0\}.$$





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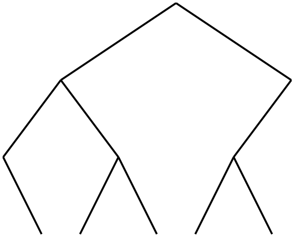

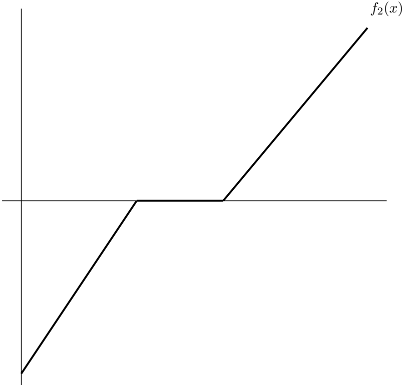

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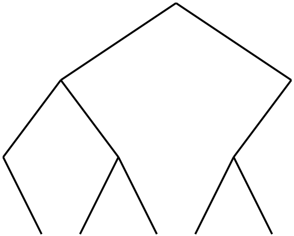

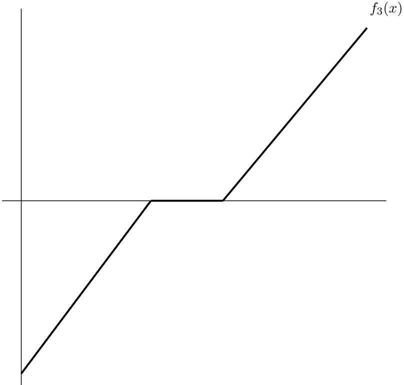

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- Let $(a_n)_n$ and $(b_n)_n$ be sequences of $\{0,1\}^*$ such that $T_n = \{c \in \{0,1\}^n \mid a_n \sqsubseteq c \sqsubseteq b_n\}$ for each n.
- For each n, define a uniformly continuous function $f_n : [0,1] \to \mathbf{R}$ by

$$f_n(x) = \min\{(l_{a_n}+1)^{-1}(3x - l_{a_n} - 1), 0\} + \max\{(2 - r_{b_n})^{-1}(3x - r_{b_n} - 1), 0\}.$$

- Let $f(x) = \sum_{n=0}^{\infty} 2^{-(n+1)} f_n(x)$.
- Then there exists $x \in [0,1]$ such that $f(x) = 0$.
- For each n, $\exists a \in T_n(|(3x-1) - l_a| < 2^{-n})$.
- There is an infinite convex subtree T' of T s.t. $\forall n(|T_n| \leq 2)$.

$\mathrm{IVT} \to \mathrm{WKL}_c$

- Let $(a_n)_n$ and $(b_n)_n$ be sequences of $\{0,1\}^*$ such that $T_n = \{c \in \{0,1\}^n \mid a_n \sqsubseteq c \sqsubseteq b_n\}$ for each n.
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- Let $f(x) = \sum_{n=0}^{\infty} 2^{-(n+1)} f_n(x)$.
- Then there exists $x \in [0,1]$ such that $f(x) = 0$.
- For each n, $\exists a \in T_n(|(3x-1) - l_a| < 2^{-n})$.
- There is an infinite convex subtree T' of T s.t. $\forall n(|T_n| \leq 2)$.
- By the previous lemma, there is a path in T', and hence in T.

Concluding remarks

- These proofs can be formalized over $\mathbf{EL}_0$, which has only Σ^0_1 induction.
- In particular, these proofs do not require countable choice.
- WKL_c can be characterized as a combination of LLPO and a fragment of countable choice.

References

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