

p -adic L -functions for $GL(n+1) \times GL(n)$ II

Fabian Januszewski

KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)

Plan of the lectures:

- Review of the analytic theory of Rankin-Selberg L -functions following Jacquet, Piatetski-Shapiro and Shalika
- The relative modular symbol and algebraicity of special values (Kazhdan-Mazur-Schmidt, Kasten-Schmidt, Raghuram-Shahidi, J., Raghuram)
- Archimedean periods: Non-vanishing and period relations (Kasten-Schmidt, Sun, J., Grobner-Lin)
- p -adic distributions attached to finite slope classes (Kazhdan-Mazur-Schmidt, Schmidt, J.)
- Boundedness in the nearly ordinary case (Schmidt, J.)
- Functional equation (J.)
- Manin congruences and independence of weight (J.)
- Interpolation formulae (Schmidt, J.)

■ Analytic Theory

J. Cogdell, *Analytic theory of L-functions for $GL(n)$* .
Proc. 'School on Automorphic Forms, L-functions, and Number Theory'
Institute for Advanced Studies Hebrew University, Jerusalem, March 12–16, 2001.

■ Algebraicity of L-values

A. Raghuram, *Critical values for Rankin-Selberg L-functions for $GL(n) \times GL(n-1)$ and the symmetric cube L-functions for $GL(2)$* . *Forum Math.* **28** (2016) No. 3, 457-489.

■ (Archimedean) Periods

B. Sun, *The non-vanishing hypothesis at infinity for Rankin-Selberg convolutions*.
Journal of the American Mathematical Society **30** (2017), 1–25.

F. Januszewski, *On Period Relations for Automorphic L-functions I*.
Transactions of the American Mathematical Society, in press.

H. Grobner, J. Lin, *Special values of L-functions and the refined Gan-Gross-Prasad Conjecture*, Preprint.

■ Construction of p -adic L-functions

F. Januszewski, *Non-abelian p -adic Rankin-Selberg L-functions and non-vanishing of central L-values*. <http://arxiv.org/abs/1708.02616>

Table of Contents

Analytic Theory

- Automorphic Representations
- Rankin-Selberg ζ -integrals
- Local L -functions: Non-archimedean case
- Local L -functions: Archimedean case
- Global Rankin-Selberg L -functions

Automorphic cohomology

- Regular algebraic representations
- Automorphic cohomology
- Automorphic cohomology classes
- Evaluation of cohomology classes
- Non-vanishing of archimedean periods
- Rationality of special values
- Towards p -adic L -functions

Some history

Automorphic Representations

F/\mathbf{Q} number field, \mathbf{A} ring of over \mathbf{Q} , $\mathbf{A}_F = \mathbf{A} \otimes_{\mathbf{Q}} F$
 Π irred. cusp. autom. rep. of $GL_n(\mathbf{A}_F)$
 $GL_n \supset B_n = T_n U_n$ standard upper Borel subgroup
 $\mathbf{1} \neq \psi : F \setminus \mathbf{A}_F \rightarrow \mathbf{C}^\times$, extends to $U_n(\mathbf{A}_F)$ via

$$(u_{ij}) \mapsto \prod_{i=1}^{n-1} \psi(u_{i,i+1}).$$

For any cusp for $\varphi \in \Pi \subseteq L^2(GL_n(F) \setminus GL_n(\mathbf{A}_F))$ we have its Fourier transform

$$\left[W_\varphi : g \mapsto \int_{U_n(F) \setminus U_n(\mathbf{A}_F)} \varphi(ug) \psi^{-1}(u) du \right] \in \text{Ind}_{U_n(\mathbf{A}_F)}^{GL_n(\mathbf{A}_F)} \psi.$$

Shalika: $\mathscr{W}(\Pi, \psi) := \{W_\varphi \mid \varphi \in \Pi\}$ is non-zero, hence isomorphic to Π , since

$$\varphi(g) = \sum_{\gamma \in U_{n-1}(F) \setminus GL_{n-1}(F)} W_\varphi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot g \right).$$

Rankin-Selberg ζ -integrals

$n \geq 1$ for the rest of the lectures

Π, Σ irred. cusp. autom. rep. of $\mathrm{GL}_{n+1}(\mathbf{A}_F)$ and $\mathrm{GL}_n(\mathbf{A}_F)$

Jacquet, Piatetski-Shapiro and Shalika consider for $\varphi \in \Pi, \varphi' \in \Sigma$:

$$I(s; \varphi, \varphi') := \int_{\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbf{A}_F)} \varphi \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \varphi'(g) |\det g|^{s-\frac{1}{2}} dg$$

Then:

- $I(s; \varphi, \varphi')$ converges absolutely and uniformly on compact sets for all $s \in \mathbf{C}$
- $I(s; \varphi, \varphi')$ defines an entire function in s , bounded in vertical strips
- With $\tilde{\varphi}(g) := \varphi({}^t g^{-1})$ it satisfies the functional equation

$$I(s; \varphi, \varphi') = I(1-s; \tilde{\varphi}, \tilde{\varphi}').$$

- $I(s; \varphi, \varphi')$ is Eulerian if $W_\varphi \in \mathcal{W}(\Pi, \psi)$ and $W_{\varphi'} \in \mathcal{W}(\Sigma, \psi^{-1})$ are factorizable
- For $n = 1$ this is the ζ -integral of Jacquet-Langlands for $\mathrm{GL}(2)$

Rankin-Selberg ζ -integrals

Assume $W_\varphi = \otimes_v W_v$ and $W_{\varphi'} = \otimes_v W'_v$ with

$$W_v \in \mathscr{W}(\Pi_v, \psi_v), \quad W'_v \in \mathscr{W}(\Sigma_v, \psi_v^{-1})$$

Then for $\Re(s) \gg 0$:

$$\begin{aligned} I(s; \varphi, \varphi') &= \int_{U_n(\mathbf{A}_F) \backslash \mathrm{GL}_n(\mathbf{A}_F)} W_\varphi\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right) W_{\varphi'}(g) |\det g|^{s-\frac{1}{2}} dg \\ &= \prod_v \int_{U_n(F_v) \backslash \mathrm{GL}_n(F_v)} W_v\left(\begin{pmatrix} g_v & \\ & 1 \end{pmatrix}\right) W'_v(g_v) |\det g_v|^{s-\frac{1}{2}} dg_v \\ &= \prod_v \Psi(s; W_v, W'_v) \end{aligned}$$

Local L -functions: Non-archimedean case

Assume $v \nmid \infty$ and write q for the residue field cardinality.

For varying $W_v \in \mathscr{W}(\Pi_v, \psi_v)$ and $W'_v \in \mathscr{W}(\Sigma_v, \psi_v^{-1})$ the ζ -integrals

$$\Psi(s; W_v, W'_v) = \int_{U_n(F_v) \backslash GL_n(F_v)} W_v\left(\begin{pmatrix} g_v & \\ & 1 \end{pmatrix}\right) W'_v(g_v) |\det g_v|^{s-\frac{1}{2}} dg_v$$

span a fractional $\mathbf{C}[q^s, q^{-s}]$ -ideal in $\mathbf{C}(q^{-s})$ containing 1.

Definition (Non-archimedean local L -functions)

Define $L(s, \Pi_v \times \Sigma_v)$ as the unique generator of the form

$$L(s, \Pi_v \times \Sigma_v) = \frac{1}{P(q^{-s})}, \quad P(X) \in \mathbf{C}[X], \quad P(0) = 1.$$

Local L -functions: Non-archimedean case

Spherical case: If Π_v and Σ_v are *spherical*, Shintani's explicit formulae show

$$L(s, \Pi_v \times \Sigma_v) = \Psi(s; W_v^0, W_v^{\prime 0}) = \frac{1}{\det(\mathbf{1} - q^{-s}A \otimes A')}$$

where $W_v^0 \in \mathscr{W}(\Pi_v, \psi_v)^{\mathrm{GL}_{n+1}(\mathcal{O})}$, $W_v^{\prime 0} \in \mathscr{W}(\Sigma_v, \psi_v^{-1})^{\mathrm{GL}_n(\mathcal{O})}$ are the spherical vectors and

$$A \in \mathrm{GL}_{n+1}(\mathbf{C}), \quad A' \in \mathrm{GL}_n(\mathbf{C})$$

are the Satake parameters for Π_v and Σ_v .

General case: $L(s, \Pi_v \times \Sigma_v)$ is always in the image of the map

$$\mathscr{W}(\Pi_v, \psi_v) \otimes \mathscr{W}(\Sigma_v, \psi_v^{-1}) \rightarrow \mathbf{C}(q^{-s}),$$

$$W_v \otimes W_v' \mapsto \Psi(s; W_v, W_v'),$$

i.e. it is a *finite linear combination of local ζ -integrals*.

Caution: The essential vectors do **not** yield $L(s, \Pi_v \times \Sigma_v)$ in general.

Local L -functions: Archimedean case

Assume $v \mid \infty$ and write $K_v \subseteq GL_{n+1}(F_v)$ and $K'_v \subseteq GL_n(F_v)$ for the (standard) maximal compact subgroups.

$$\Pi_v \supset \Pi_v^\infty \supset \Pi_v^{(K_v)}$$

- Π_v : irreducible Hilbert space representation of $GL_{n+1}(F_v)$
- Π_v^∞ : smooth vectors, irreducible Fréchet representation of $GL_{n+1}(F_v)$
- $\Pi_v^{(K_v)}$: K_v -finite vectors, irreducible (\mathfrak{g}, K) -module

Conceptually, consider the pair (Π, Σ) as an automorphic representation $\tilde{\Pi}$ of

$$\tilde{G} := GL(n+1) \times GL(n).$$

$$\rightsquigarrow \tilde{\Pi}_v^\infty = \Pi_v^\infty \hat{\otimes} \Sigma_v^\infty \text{ (completed projective tensor product)}$$

$$\rightsquigarrow \tilde{\Pi}_v^{(K_v \times K'_v)} = \Pi_v^{(K_v)} \otimes \Sigma_v^{(K'_v)} \text{ (algebraic tensor product)}$$

Local L -functions: Archimedean case

Originally, Jacquet and Shalika define $L(s, \Pi_v \times \Sigma_v)$ via LLC.

Results of Jacquet-Shalika, Cogdell-Piatetski-Shapiro and Stade show:

- For $\Re(s) \gg 0$, $\Psi(s; -, -)$ extends to a continuous map

$$\mathscr{W}(\Pi_v^\infty, \psi_v) \hat{\otimes} \mathscr{W}(\Sigma_v^\infty, \psi_v^{-1}) \rightarrow \mathbf{C}$$

- For every $\widetilde{W}_v \in \mathscr{W}(\Pi_v^\infty, \psi_v) \hat{\otimes} \mathscr{W}(\Sigma_v^\infty, \psi_v^{-1})$ there is an *entire* function $H(s)$ with

$$\Psi(s; \widetilde{W}_v) = H(s) \cdot L(s, \Pi_v \times \Sigma_v)$$

- If \widetilde{W} is K -finite, then $H(s)$ is a *polynomial* in s .
- For every $s_0 \in \mathbf{C}$ there is a K -finite $\widetilde{W}_v = W_v \otimes W'_v$ with $H(s_0) \neq 0$.
- Therefore, we find a K -finite $\widetilde{W}_v \in \mathscr{W}(\Pi_v, \psi_v)^{(K_v)} \otimes \mathscr{W}(\Sigma_v, \psi_v^{-1})^{(K'_v)}$ with

$$\Psi(s; \widetilde{W}_v) = L(s, \Pi_v \times \Sigma_v)$$

Caution: This is **not known** for $GL(m) \times GL(n)$!

Global Rankin-Selberg L -functions

Put for $\Re(s) \gg 0$:

$$\Lambda(s, \Pi \times \Sigma) := \prod_v L(s, \Pi_v \times \Sigma_v) = L(s, \Pi_\infty \times \Sigma_\infty) \cdot L(s, \Pi \times \Sigma)$$

Then:

- $\Lambda(s, \Pi \times \Sigma)$ extends *holomorphically* to all $s \in \mathbf{C}$, bounded in vertical strips
- $\Lambda(s, \Pi \times \Sigma)$ satisfies a functional equation

$$\Lambda(s, \Pi \times \Sigma) = \varepsilon(s, \Pi \times \Sigma) \Lambda(1-s, \Pi^\vee \times \Sigma^\vee)$$

- There is are finitely many K -finite cusp forms $\varphi_i \in \Pi$, $\varphi'_i \in \Sigma$, with

$$\Lambda(s, \Pi \times \Sigma) = \sum_i I(s; \varphi_i, \varphi'_i)$$

and we may assume: $W_{\varphi_i} = \otimes_v W_{i,v}$, $W_{\varphi'_i} = \otimes_v W'_{i,v}$, with:

- For almost all $v \nmid \infty$, $W_{\varphi_i, v}$, $W'_{\varphi_i, v}$ are the normalized spherical vectors
- For all $v \mid \infty$, $W_{\varphi_i, v}$, $W'_{\varphi_i, v}$ are K -finite

Analytic Theory

- Automorphic Representations
- Rankin-Selberg ζ -integrals
- Local L -functions: Non-archimedean case
- Local L -functions: Archimedean case
- Global Rankin-Selberg L -functions

Automorphic cohomology

- Regular algebraic representations
- Automorphic cohomology
- Automorphic cohomology classes
- Evaluation of cohomology classes
- Non-vanishing of archimedean periods
- Rationality of special values
- Towards p -adic L -functions

Some history

Regular algebraic representations

Let for $v \mid \infty$:

- $\tilde{K}_v := Z(F_v)^0 \cdot K_v$
- $\mathfrak{g}_v := \mathbf{C} \otimes_{\mathbf{R}} \text{Lie}(\text{GL}_{n+1}(F_v)) = (\mathbf{C} \otimes_{\mathbf{R}} F_v)^{n+1 \times n+1}$
- L_v : an irreducible rational representation of $\text{res}_{F_v/\mathbf{R}} \text{GL}(n+1)$

Theorem (Speh, Enright, Vogan-Zuckerman)

Let Π_v be an irreducible admissible generic representation of $\text{GL}_{n+1}(F_v)$. TFAE:

- $H^\bullet(\mathfrak{g}_v, \tilde{K}_v; \Pi_v \otimes L_v) \neq 0$
- $\Pi_v^{(K_v)} \cong A_{\mathfrak{q}}(\lambda)$ is a cohomologically induced standard module with $\mathfrak{q} \subseteq \mathfrak{g}_v$ a ϑ -stable Borel subalgebra and $\lambda = H^0(\mathfrak{u}_{n+1}; L_v^\vee)$
- $$\Pi_v \cong \begin{cases} J_\lambda = \text{Ind}_{P_{n+1}(\mathbf{R})}^{\text{GL}_{n+1}(\mathbf{R})} \left[D(\ell_1) |\det|^{w/2} \otimes D(\ell_2) |\det|^{w/2} \otimes \dots \right], \\ \ell = 2\lambda + 2\rho_{n+1} - \mathbf{w} : \mathbf{w} = \lambda_i + \lambda_{n+2-i}, & \text{if } F_v \cong \mathbf{R} \\ J_\lambda = \text{Ind}_{B_{n+1}(\mathbf{R})}^{\text{GL}_{n+1}(\mathbf{R})} \left[z^{a_1} \bar{z}^{b_1} \otimes \dots \otimes z^{a_{n+1}} \bar{z}^{b_{n+1}} \right], \lambda = (\lambda^t, \lambda^{\bar{t}}), \\ a = \lambda^t + \rho_{n+1}, b = \lambda^{\bar{t}} - \rho_{n+1} = \mathbf{w} - \lambda^t - \rho_{n+1}, & \text{if } F_v \cong \mathbf{C} \end{cases}$$

Definition (Clozel)

A cuspidal automorphic representation Π of $GL_{n+1}(\mathbf{A}_F)$ is called *regular algebraic* if for all $v \mid \infty$ there is an L_v such that

$$H^*(\mathfrak{g}_v, \tilde{K}_v; \Pi_v \otimes L_v) \neq 0.$$

Define:

- $G_{n+1} := \text{res}_{F/\mathbf{Q}} GL(n+1)$
- S_{n+1} : maximal \mathbf{Q} -split torus in the center $Z_{n+1} \subseteq G_{n+1}$
- $K_{n+1} := \prod_{v \mid \infty} K_v$
- $\tilde{K}_{n+1} := S_{n+1}(\mathbf{R})^0 \cdot K_{n+1} = \mathbf{R}_{>0} \cdot \prod_{v \mid \infty} K_v$
- For $K \subseteq G_{n+1}(\mathbf{A}^{(\infty)})$ compact open:

$$\mathcal{X}_{n+1}(K) := G_{n+1}(\mathbf{Q}) \backslash G_{n+1}(\mathbf{A}) / \tilde{K}_{n+1} \cdot K$$

- L : rational representation of G_{n+1}
- \underline{L} : sheaf on $\mathcal{X}_{n+1}(K)$ attached to L

Π is regular algebraic iff there is a rational representation L of G_{n+1} satisfying:

- There is $\mathbf{w} \in \mathbf{Z}$:

$$L \cong L^{c,v} \otimes (N_{F/\mathbf{Q}} \circ \det)^{\otimes \mathbf{w}}$$

- There is a $G_{n+1}(\mathbf{A}^{(\infty)})$ -module embedding

$$\Pi^{(\infty)} \subseteq \varinjlim_K H^\bullet(\mathcal{X}_{n+1}(K); \underline{L})$$

If this is the case, we have in fact

$$H^\bullet(\mathfrak{g}_{n+1}, \tilde{K}_{n+1}; \Pi \otimes L) \subseteq \varinjlim_K H_i^\bullet(\mathcal{X}_{n+1}(K); \underline{L})$$

where *inner cohomology* is defined as

$$H_i^\bullet(\mathcal{X}_{n+1}(K); \underline{L}) := \text{image} [H_c^\bullet(\mathcal{X}_{n+1}(K); \underline{L}) \rightarrow H^\bullet(\mathcal{X}_{n+1}(K); \underline{L})]$$

We observe:

$$\begin{aligned}
 H^\bullet(\mathfrak{g}_{n+1}, \tilde{K}_{n+1}; \Pi \otimes L) &= H^\bullet(\mathfrak{g}_{n+1}, \tilde{K}_{n+1}; \Pi_\infty \otimes L) \otimes \Pi^{(\infty)} \\
 &= \left[\bigwedge^{\bullet} (\mathfrak{g}_{n+1} / \tilde{\mathfrak{k}}_{n+1})^* \otimes \Pi_\infty \otimes L \right]^{\tilde{K}_{n+1}} \otimes \Pi^{(\infty)} \\
 &= \bigwedge^{\bullet - \dim(\mathfrak{n} \cap \mathfrak{p})} (\mathfrak{l} \cap \tilde{\mathfrak{k}}_{n+1})^* \otimes \Pi^{(\infty)}
 \end{aligned}$$

for $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n} \subseteq \mathfrak{g}_{n+1} = \mathfrak{p} \oplus \tilde{\mathfrak{k}}_{n+1}$ a θ -stable Borel subalgebra.

In particular: The multiplicity of $\Pi^{(\infty)}$ in *bottom degree*

$$b_{n+1} := \dim(\mathfrak{n} \cap \mathfrak{p}) = r_1 \cdot \left[\frac{(n+1)^2}{4} \right] + r_2 \cdot \frac{(n+1)n}{2}$$

is **one**. Furthermore, there's the numerical **miracle** (S., KMS, J.)

$$b_{n+1} + b_n = \dim \mathcal{Y}_n(K'), \quad \mathcal{Y}_n(K') := G_n(\mathbf{Q}) \backslash G_n(\mathbf{A}) / K_n \cdot K'$$

Automorphic cohomology classes

Departing from $W^{(\infty)} \in \mathcal{W}(\Pi^{(\infty)}, \psi^{(\infty)})^K$ we obtain a de Rham cocycle

$$\sum_i \omega_i \otimes W_{\infty, i} \otimes W^{(\infty)} \otimes l_i \in \left[\bigwedge^{b_{n+1}} (\mathfrak{g}_{n+1} / \tilde{\mathfrak{k}}_{n+1})^* \otimes \mathcal{W}(\Pi, \psi)^K \otimes L \right]^{\tilde{K}_{n+1}}$$

By Fourier inversion, writing $W_{\varphi_i} = W_{\infty, i} \otimes W^{(\infty)}$, get a cohomology class

$$\begin{aligned} \sum_i \omega_i \otimes \varphi_i \otimes l_i &\in \left[\bigwedge^{b_{n+1}} (\mathfrak{g}_{n+1} / \tilde{\mathfrak{k}}_{n+1})^* \otimes \Pi^K \otimes L \right]^{\tilde{K}_{n+1}} \\ &\subseteq H_c^{b_{n+1}}(\mathcal{X}_{n+1}(K); \underline{L}) \end{aligned}$$

Likewise, any $W'^{(\infty)} \in \mathcal{W}(\Sigma^{(\infty)}, \psi^{(\infty), -1})^{K'}$ yields a cuspidal cohomology class

$$\sum_j \omega'_j \otimes \varphi'_j \otimes l'_j \in H_c^{b_n}(\mathcal{X}_n(K'); \underline{L}')$$

Evaluation of cohomology classes

Put $G := G_{n+1} \times G_n$.

Observe that for $L \subseteq G_n(\mathbf{A}^{(\infty)})$ compact open and sufficiently small:

$$\begin{array}{ccc}
 \mathcal{Y}_n(L) & \longrightarrow & \mathcal{X}_{n+1}(K) \times \mathcal{X}_n(K') \\
 \parallel & & \parallel \\
 \mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbf{A}_F) / K_n \cdot L & \xrightarrow{g \mapsto (\mathrm{diag}(g,1),g)} & G(\mathbf{Q}) \backslash G(\mathbf{A}) / (\tilde{K}_{n+1}K \times \tilde{K}_nK')
 \end{array}$$

is *proper*. Together with Poincaré duality we obtain a rationality preserving map

$$\begin{aligned}
 H_c^{b_{n+1}}(\mathcal{X}_{n+1}(K); \underline{L}) \otimes H_c^{b_n}(\mathcal{X}_n(K'); \underline{L}') &\rightarrow H_c^{\dim \mathcal{Y}_n}(\mathcal{Y}_n(L); \underline{L} \otimes \underline{L}') \\
 &\cong H^0(\mathcal{Y}_n(L); \underline{L}^\vee \otimes \underline{L}'^\vee) \\
 &= (L^\vee \otimes L'^\vee)^{\Gamma_L} \quad [\Gamma_L = \mathrm{GL}_n(F) \cap L] \\
 &\supseteq \underbrace{\sum_{v \in \mathbf{Z}} \mathrm{Hom}_{G_n}(L \otimes L', (N_{F/\mathbf{Q}} \circ \det)^{\otimes v})}_{\dim \leq 1}
 \end{aligned}$$

Evaluation of cohomology classes

For the constructed automorphic cohomology classes this yields

$$\sum_{i,j} (\omega_i \otimes \varphi_i \otimes l_i) \otimes (\omega'_j \otimes \varphi'_j \otimes l'_j) \mapsto$$
$$\sum_{i,j} \langle \omega^{G_n}, (\omega_i \wedge \omega'_j) |_{\mathcal{Y}_n} \rangle \cdot \left(\eta_\nu(l_i \otimes l'_j) \cdot l\left(\frac{1}{2} + \nu; \varphi_i, \varphi'_j\right) \right)_\nu$$

Here:

- $\omega^{G_n} \in \Lambda^{\dim \mathcal{Y}_n} \mathfrak{g}_n / \mathfrak{k}_n$: a generator (an invariant measure on $G_n(\mathbf{R}) / K_n$)
- $\langle -, - \rangle : \Lambda^{\dim \mathcal{Y}_n} \mathfrak{g}_n / \mathfrak{k}_n \otimes \Lambda^{\dim \mathcal{Y}_n} (\mathfrak{g}_n / \mathfrak{k}_n)^* \rightarrow \mathbf{C}$ the canonical pairing
- ν runs through the integers satisfying

$$\mathrm{Hom}_{G_n}(L \otimes L', (N_{F/\mathbf{Q}} \circ \det)^{\otimes \nu}) \neq 0$$

If existent, those are in bijection with the critical values of $L(s, \Pi \times \Sigma)$
(Kasten-Schmidt, Raghuram)

Recall: $s_0 \in \frac{1}{2} + \mathbf{Z}$ critical for $L(s, \Pi \times \Sigma) \Leftrightarrow$

neither $L(s, \Pi_\infty^\vee \times \Sigma_\infty)$ nor $L(1 - s, \Pi_\infty^\vee \times \Sigma_\infty^\vee)$ have a pole at $s = s_0$

Evaluation of cohomology classes

For each ν the ζ -integral $I(\frac{1}{2} + \nu; \varphi_i, \varphi'_j)$ evaluates to

$$\int_{\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbf{A}_F)} \varphi_i \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \varphi'_j(g) |\det g|^\nu dg = L\left(\frac{1}{2} + \nu, \Pi \times \Sigma\right)$$
$$\cdot \prod_{\nu \nmid \infty} \left[\frac{\Psi(\mathbf{s}; W_\nu, W'_\nu)}{L(\mathbf{s}, \Pi_\nu \times \Sigma_\nu)} \right]_{\mathbf{s} = \frac{1}{2} + \nu} \cdot \underbrace{\prod_{\nu \mid \infty} \Psi\left(\frac{1}{2} + \nu; W_{\nu,i}, W'_{\nu,j}\right)}_{\text{contributes to archimedean period}}$$

- For almost all $\nu \nmid \infty$ and all $\mathbf{s}_0 \in \mathbf{C}$:

$$\left[\frac{\Psi(\mathbf{s}; W_\nu, W'_\nu)}{L(\mathbf{s}, \Pi_\nu \times \Sigma_\nu)} \right]_{\mathbf{s} = \mathbf{s}_0} = 1$$

- For the other $\nu \nmid \infty$ we may hope to choose W_ν, W'_ν 'suitably'
- **Problem:** For $\nu \mid \infty$ the test vectors $W_{\nu,i}$ and $W'_{\nu,j}$ are **not** known explicitly.
(For $n \geq 2$ there is no explicit realization of Π_ν and Σ_ν known)

Non-vanishing of archimedean periods

Theorem (Kasten-Schmidt, Int. J. Number Theory 2013)

If $n = 2$ and F/\mathbf{Q} totally real, then for all $s_0 \in \mathbf{C}$, $\delta \in \{0, 1\}^{[F:\mathbf{Q}]}$:

$$\dim \operatorname{Hom}_{\mathfrak{g}_n, K_n} \left(\Pi_\infty^{(K_{n+1})} \otimes \Sigma_\infty^{(K_n)}, \operatorname{sgn}_\infty^\delta \otimes |\cdot|^{-s_0} \right) \leq 1$$

Theorem (Aizenbud-Gourevich-Sayag, Comp. Math. 2008; Sun-Zhu, Ann. Math. 2012)

For all $n \geq 1$, F/\mathbf{Q} arbitrary, and all $s_0 \in \mathbf{C}$, $\delta \in \{0, 1\}^{r_1}$:

$$\dim \operatorname{Hom}_{G_n(\mathbf{R})} \left(\Pi_\infty^\infty \widehat{\otimes} \Sigma_\infty^\infty, \operatorname{sgn}_\infty^\delta \otimes |\cdot|^{-s_0} \right) \leq 1$$

Theorem (B. Sun, JAMS 2017)

For all $n \geq 1$, F/\mathbf{Q} arbitrary and all $s_0 = \frac{1}{2} + \nu$ critical for $L(s, \Pi \times \Sigma)$:

$$\sum_{i,j} \langle \omega^{G_n}, (\omega_i \wedge \omega'_j) |_{\mathcal{O}_n} \rangle \cdot \eta_\nu(l_i \otimes l'_j) \cdot \prod_{\nu|\infty} \Psi\left(\frac{1}{2} + \nu; W_{\nu,i}, W'_{\nu,j}\right) \neq 0$$

Rationality of special values

Theorem (Schmidt, KMS, Kasten-Schmidt, Raghuram-Shahidi, J., Raghuram, Sun)
Let F/\mathbf{Q} be arbitrary, Π and Σ regular algebraic of balanced cohomological weight. Then there are periods $\Omega_v^\varepsilon \in \mathbf{C}^\times$ with the following property: For each finite order Hecke character $\chi : F^\times \backslash \mathbf{A}_F^\times \rightarrow \mathbf{C}^\times$, and each $s = \frac{1}{2} + \nu$ critical for $L(s, \Pi \times \Sigma)$:

$$\frac{L(\frac{1}{2} + \nu, \Pi \times (\Sigma \otimes \chi))}{G(\chi) \frac{(n+1)n}{2} \Omega_v^{(-1)^\nu \operatorname{sgn} \chi}} \in \mathbf{Q}(\Pi, \Sigma, \chi).$$

Theorem (Hida, Duke 1994; J., TAMS 2018)

Let $1 \leq n \leq 2$ and F totally real if $n = 2$ or F/\mathbf{Q} and n arbitrary and assume the continuity of cohomologically induced functionals. Then for all $s_0 = \frac{1}{2} + \nu_0$ and $s_1 = \frac{1}{2} + \nu_1$ critical for $L(s, \Pi \times \Sigma)$ and all signs ε :

$$\frac{\Omega_{\nu_0}^{(-1)^{\nu_0} \varepsilon}}{\Omega_{\nu_1}^{(-1)^{\nu_1} \varepsilon}} \in \frac{(2\pi i)^{[F:\mathbf{Q}]_{\nu_0}}}{(2\pi i)^{[F:\mathbf{Q}]_{\nu_1}}} \cdot \mathbf{Q}(\Pi, \Sigma).$$

Remark: If F/\mathbf{Q} is an imaginary CM field Grobner-Lin obtained a similar relation under different hypotheses.

Rationality of special values

Theorem (J., Math. Ann. 2017, TAMS 2018)

Let F/\mathbf{Q} be arbitrary, Π and Σ cuspidal regular algebraic. Then:

- The $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbf{A}^{(\infty)})$ -module $\Pi^{(K_\infty)} \otimes \Sigma^{(K'_\infty)}$ admits a $\mathbf{Q}(\Pi, \Sigma)$ -rational structure, unique up to scalars
- As $(\mathfrak{g}_\infty, K_\infty^0) \times G(\mathbf{A}^{(\infty)})$ -module: $\Pi^{(K_\infty)} \otimes \Sigma^{(K'_\infty)} = \bigoplus_\varepsilon \tilde{\Pi}_\infty^\varepsilon$
and each $\tilde{\Pi}_\infty^\varepsilon$ admits a $\mathbf{Q}(\Pi, \Sigma, i^{\frac{(n+1)n}{2}})$ -rational structure, unique up to scalars

Put $E := \mathbf{Q}(\Pi, \Sigma, i^{\frac{(n+1)n}{2}})$ and define two E -rational structures:

- $\tilde{\Pi}_B^+$: via $H^b(\mathcal{X}(K); \underline{L}_E \otimes \underline{L}'_E)$
- $\tilde{\Pi}_{\text{dR}}^+$: via $H^b(\mathfrak{g}_\infty; \tilde{K}_\infty; \tilde{\Pi}_E^+ \otimes L_E \otimes L'_E)$ using E -rational test vectors

We have the analogy:

$$\begin{array}{ccc} \tilde{\Pi}_B^+ \otimes_E \mathbf{C} & \xrightarrow{\sim} & \tilde{\Pi}_{\text{dR}}^+ \otimes_E \mathbf{C} \\ H_B(M(\Pi) \otimes M(\Sigma))^+ \otimes_E \mathbf{C} & \xrightarrow{\sim} & H_{\text{dR}}(M(\Pi) \otimes M(\Sigma))/F^0 \otimes_E \mathbf{C} \end{array}$$

Towards p -adic L -functions

Naively, given an L -series $\sum_{n=1}^{\infty} a_n n^{-s}$, one may chop it into pieces

$$\mu_\nu(x + (\mathfrak{p}^\beta)) := \sum_{n \equiv x (\mathfrak{p}^\beta)} \frac{a_n}{n^\nu}$$

and obtain for given s a \mathbf{C} -valued distribution μ . Integration against characters yields

$$\int \chi(x) d\mu_\nu(x) = \sum_{n=1}^{\infty} \frac{\chi(n) a_n}{n^\nu}$$

Automorphically this means rewriting:

$$I(s, \varphi_i, \varphi'_i) = \sum_{x \in \mathcal{C}(\mathfrak{p}^\beta)} \int_{\det^{-1}(x)} \varphi_i \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \varphi'_i(g) |\det g|^\nu dg$$

Therefore, one may define

$$\mu(x + (\mathfrak{p}^\beta)) := \int_{\det^{-1}(x)} \varphi_i \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \varphi'_i(g) |\det g|^\nu dg$$

Towards p -adic L -functions

The cohomological interpretation of this intuition is:

$$\begin{array}{ccc}
 \mathcal{Y}_n(L_\beta) & \xlongequal{\quad} & \bigsqcup_{x \in C(p^\beta)} \overbrace{\Gamma_x \backslash \mathrm{GL}_n(F \otimes_{\mathbf{Q}} \mathbf{R}) / K_n}^{= \det^{-1}(x)} \\
 \det \downarrow & & \downarrow \\
 C(p^\beta) & \xlongequal{\quad} & C(p^\beta)
 \end{array}$$

where $L_\beta \subseteq \mathrm{GL}_n(\mathbf{A}_F^{(\infty)})$ is a suitable compact open such that

$$C(p^\beta) = F^\times \backslash \mathbf{A}_F^\times / F_{\mathbf{R}}^+ \det(L_\beta)$$

Then we may define a distribution by integrating cohomology classes separately over the fibers of the determinant.

Analytic Theory

- Automorphic Representations
- Rankin-Selberg ζ -integrals
- Local L -functions: Non-archimedean case
- Local L -functions: Archimedean case
- Global Rankin-Selberg L -functions

Automorphic cohomology

- Regular algebraic representations
- Automorphic cohomology
- Automorphic cohomology classes
- Evaluation of cohomology classes
- Non-vanishing of archimedean periods
- Rationality of special values
- Towards p -adic L -functions

Some history

Some history

- **1993:** Claus Schmidt constructs p -adic **measures** for $GL(3) \times GL(2)$ over \mathbf{Q} for trivial cohomological weights, spherical and ordinary at p , **non-vanishing** of archimedean period due to Barry Mazur
- **2000:** Kazhdan-Mazur-Schmidt construct a p -adic **distribution** for $GL(n+1) \times GL(n)$ over \mathbf{Q} for representations spherical and ordinary at p , **not** bounded, archimedean non-vanishing hypothesis explicitly formulated
- **2001:** Schmidt improves the KMS construction to obtain a p -adic **measure** in the ordinary case but has to exclude $p \leq n+1$
- **2005:** Utz extends Schmidt's construction to all p for $n = 2, 3, 4$
- **2009:** **all** ordinary p for all n , arbitrary number fields (J.)
- **2013:** Kasten-Schmidt prove **non-vanishing** of periods for $GL(3) \times GL(2)/\mathbf{Q}$
- **2014:** arbitrary coh. weights over totally real fields, **functional equation** (J.)
- **2015:** arbitrary coh. weights over arbitrary number fields (J.)
- **2017:** Sun proves **non-vanishing** of archimedean periods in **all** cases
- **2017:** allow for near ordinarity at p , **Manin congruences**, extend to Hida's universal nearly ordinary cohomology for $GL(n+1) \times GL(n)$ (J.)

To be continued.

Thank you for your attention.