September 26, 2018

Quantum Symmetries: Hopf actions on noncom. algebras

/$\Bbbk$ field

(Thank organizers to inviting me to participate —
my work so far is on border of the topics presented here
but my current research plans involve some themes here.
So my goal here is to introduce some work that I've done
in Quantum Symmetry & to present some questions/themes
that I'm pursuing in future work)

My training is in Ring Theory & Representation Theory, and
I started to become interested in Hopf algebras during my
postdoc, which naturally involved actions of Hopf algebras
on (noncommutative) algebras.

I noticed that there were lots of results like
"If one has an action of a Hopf algebra H on an algebra A,
then [something with invariants A^H, $A\#H$, ...]"
But there weren't a ton of results on when such actions exist.
Here, by action, I mean
A is an algebra in the rep'n category $\mathrm{Rep}(H)$.

0. Why care? Classical symmetry vs. Quantum Symmetry
1. No Quantum Symmetry (results & sketch of one proof)
2. Genuine Quantum Symmetry (some results)
3. Questions that I have for the audience
which already be answered by now

0. Why care? Classical Symmetry vs. Quantum Symmetry.

$$\text{actions of} \begin{cases}\text{groups} \\ \text{Lie algebras}\end{cases} \text{on [manifolds / varieties / ... or their function algebras]}$$

$$\text{by} \begin{cases}\text{automorphisms} \\ \text{derivations.}\end{cases}$$

$$\text{This boils down to} \begin{cases}\text{actions} \\ \text{coactions}\end{cases} \text{of} \begin{cases}\text{cocommutative} \\ \text{commutative}\end{cases} \text{Hopf algebras on com. algs}$$

Ex. actions of $\underbrace{\Bbbk G}_{\text{group algebra}}$, $\underbrace{U(\mathfrak{g})}_{\text{univ. enveloping alg}}$ & coactions of $\underbrace{\mathcal{O}(G)}_{\text{coordinate alg of an algebraic/Lie group}}$

In fact, by a result of Cartier-Kostant-Milnor-Moore:

Theorem: $\Bbbk = \overline{\Bbbk}$, char $\Bbbk = 0$. If H is a cocommutative Hopf alg, then

$$H \cong U(\underbrace{P(H)}_{\substack{\text{Lie alg of primitive elts} \\ x \in H \text{ so that} \\ \Delta(x) = x \otimes 1 + 1 \otimes x}}) \rtimes \Bbbk \underbrace{G(H)}_{\substack{\text{group of grouplike elts} \\ x \in H \text{ so that} \\ \Delta(x) = x \otimes x}}$$

Assume $\Bbbk = \overline{\Bbbk}$, char $\Bbbk = 0$ from now on

Corollary $\Bbbk = \overline{\Bbbk}$, char $\Bbbk = 0$. If H is a finite-dim'l cocom. Hopf algebra, then $H \cong \Bbbk G$; a group algebra.

By Quantum Symmetry, we mean that we have an algebra that admits an

$$\begin{cases}\text{action} \\ \text{coaction}\end{cases} \text{of a} \begin{cases}\text{noncocommutative} \\ \text{noncommutative}\end{cases} \text{Hopf algebra.}$$

One mechanism of getting Quantum symmetry: —

Take a classical symmetry: $\underset{\substack{\text{cocom.}\\ \text{Hopf algebra}}}{H} \curvearrowright \underset{\substack{\text{commutative}\\ \text{algebra}}}{A}$

& simultaneously deform H & A to get

$\underset{\substack{\text{noncocom.}\\ \text{Hopf alg}}}{H} \curvearrowright \underset{\substack{\text{noncom.}\\ \text{algebra.}}}{A}$

Example: $X = \mathbb{A}^2$. $A = \mathcal{O}(X) = \Bbbk[x,y]$ — 1-parameter deformation $q \in \Bbbk^\times$ — $\frac{\Bbbk\langle x,y\rangle}{(xy-qyx)}$

$\mathcal{O}(SL_2(\Bbbk))$ coacts on $\mathcal{O}(X) \rightsquigarrow \mathcal{O}_q(SL_2(\Bbbk))$ coacts on $\Bbbk_q[x,y]$

$U(\mathfrak{sl}_2)$ acts on $\Bbbk[x,y] \rightsquigarrow U_q(\mathfrak{sl}_2)$ acts on $\Bbbk_q[x,y]$

But there are loads of instances of quantum symmetry that don't arise as above.

Natural Question: Given an action of a Hopf algebra H on an alg A, does the action factor through the action of a cocommutative Hopf algebra?

Given a Hopf algebra H, and an algebra $A \in \mathrm{Rep}(H)$, we have

(a) No Quantum Symmetry if $\exists$ Hopf ideal $I^{\neq 0}$ of H so that H/I is cocommutative & get induced action of $H/I \curvearrowright A$.

(b) Genuine Quantum Symmetry if $\nexists$ Hopf ideal $I^{\neq 0}$ of H so that get induced action of $H/I \curvearrowright A$.

(c) Universal Quantum Symmetry study Hopf algebra $\mathcal{O}_A$

that coaction on A universally:

won't have time to discuss further

$\forall$ coactions ρ of a Hopf alg H on A

$\exists!$ Hopf algebra map $\pi: \mathcal{O}_A \to H$

so that the diagram commutes

$$\begin{tikzcd} & A\otimes \mathcal{O}_A \arrow[d, dashed, "id\otimes\pi"] \\ A \arrow[ur, "\rho_{univ}"] \arrow[r, "\forall\rho"'] & A\otimes H \end{tikzcd}$$

1. No Quantum Symmetry

↳ Restrict our attn to actions of finite dim'l Hopf algs H

Two important classes of fin dim'l H

semisimple	pointed
(semisimple as an algebra)	(all simple H-comodules are 1-dimensional)
Folk use group-theoretic techniques to study these	& Lie-theoretic techniques to study these

arXiv/1301.4161 **Theorem** [Etingof-W] If H a semisimple Hopf algebra that acts a commutative domain A, then $H \curvearrowright A$ factors through the action of a group.

Pf/ Assume $\nexists\, I \neq 0$ Hopf ideal of $H \ni \exists$ induced action $H/I \curvearrowright A$.

STS: H is cocommutative by CCKM theorem.

Get H^* coacts on A so that $\nexists$ induced coaction of a proper Hopf subalgebra H' on A

↳ STS H^* is commutative, when $\rho: A \to A\otimes H^*$

Theorem [EW] Semisimple Hopf algebras K only have finitely many right coideal subalgs [subalgs $B \ni \Delta(B) \subseteq B\otimes K$]

Pf/ We establish a bijection between:

{set of right coideal subalgs B of K}

&

{set of quadruples $(\mathcal{M}, F_{\mathcal{M}}, G, \sigma)$ for $\mathcal{C} = Rep(K)$ fusion categ.}

$\mathcal{M} = Rep(B)$ indecomp. $\mathcal{C}$-module category; $F_{\mathcal{M}}$ forget $\mathcal{M} \to Vec$; G: $\mathcal{C} \to \mathcal{M}$ given by restriction; σ: for forget $F_{\mathcal{C}} : \mathcal{C} \to Vec$, $\sigma : F_{\mathcal{M}} G \xrightarrow{\sim} F_{\mathcal{C}}$

up to equivalence, and we show that there are finitely many 4-tuples up to equiv.

Now take $X = maxSpec(A) = \{$characters $\chi : A \to \Bbbk\}$

Form $\gamma : X \longrightarrow \coprod$ the collection of d-dim'l coideal subalgs of H^*

$\chi \longmapsto A_\chi := \rho_\chi(A)$ for $\rho_\chi = (\chi \otimes id_{H^*})\rho$

Define $X_0 := \{\chi \in X \mid A_\chi$ has max dim'n$\}$

Consider $\gamma|_{X_0}$ since X_0 irred., $\gamma|_{X_0}$ reg., $im\,\gamma$ is finite by Thm↑

is a constant map say $\gamma|_{X_0}(\chi) = B \quad \forall \chi \in X_0$

Argue that H^* coaction on A restricts to coaction of Hopf subalg $\langle B \rangle$ gen by B on A.

By assumption, $H^* = \langle B \rangle$, $B = \rho_\chi(A)$ for some χ, commutative

$\therefore H^*$ is commutative, as desired. ///.

arXiv/

1409.1644 * There are also many no quantum symmetry results for

1509.01165 semisimple Hopf actions on many quantizations of commutative domains

1602.00532 (work with Juan Cuadra & Pavel Etingof).

1605.00560

* also by Skryabin in Char $\Bbbk > 0$

* Goswami & others in the C^* algebra context "No go theorems"

2. Genuine Quantum Symmetry

There's also classification results on actions of

1403.4673 • finite dim'l nonsemisimple pointed Hopf algs of "finite Cartan type" as classified by Andruskiewitsch–Schneider on commutative domains

1511.09320 w/ Etingof

1303.7203 w/ Chan, Kirkman, Zhang • semisimple Hopf algs on (noncom). "Artin-Schelter" regular algs — graded algs that behave like $\Bbbk[x,y]$ homologically (gldim = 2, Gorenstein condition) — subject to a "trivial determinant" on grade preserving Hopf algs

Why care about AS-regular algebras?

One has that a connected graded algebra A is AS-regular if and only if its E(A)= Ext-algebra Ext_A^*(k,k) is Frobenius.

Moreover, E(A) is symmetric iff A is Calabi-Yau.

Ex $G \leq SL_2(\Bbbk)$ finite, $H = \Bbbk G$, $A = \Bbbk[x,y]$, action given by matrix multiplication

1007.06977, 1610.01220, 1809.06524 w/ Conner, Kirkman, Moore — * There's also a "noncommutative McKay correspondence" & other results in noncommutative invariant theory in this context.

1410.7696 w/ Kinser • finite-dim'l nonsemisimple pointed Hopf algs (Taft alg, $u_q(sl_2)$, D(Taft) Drinfeld double) on path algs $\Bbbk Q$ of finite, loopless, no parallel arrows, preserving path length filtration

working preparation w/ Etingof & Kinser • finite-dim'l (esp. semisimple) Hopf algs on $\Bbbk Q$ & on tensor algs in general preserving (path length) grading — in the context of studying tensor algs in finite tensor categorys.