CIRCULAR (YET SOUND) PROOFS

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Joint work with
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What is this talk about?

Tree Resolution

Regular Resolution

General Resolution

Circular Resolution

NEW!

exponentially stronger!

Inference rules

Standard rules:

$$\frac{C \vee X \qquad D \vee \overline{X}}{C \vee D} \qquad \frac{C}{C \vee D} \qquad \frac{}{X \vee \overline{X}}$$

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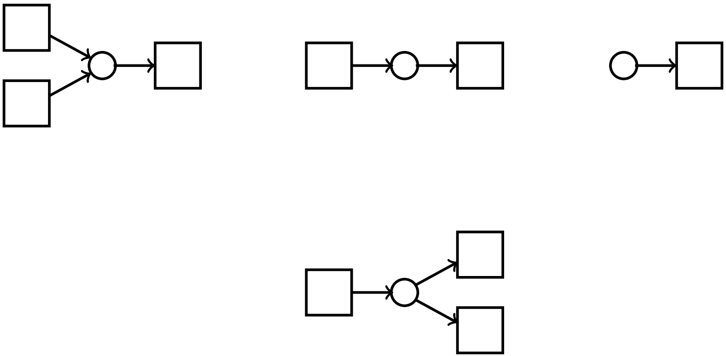
$$\frac{C \vee X \qquad D \vee \overline{X}}{C \vee D} \qquad\qquad \frac{C}{C \vee D} \qquad\qquad \frac{}{X \vee \overline{X}}$$

Symmetric rules:

$$\frac{C \vee X \qquad C \vee \overline{X}}{C} \qquad\qquad \frac{C}{C \vee X \qquad C \vee \overline{X}} \qquad\qquad \frac{}{X \vee \overline{X}}$$

Graphical representation of proofs



Circular arguments

Want: $E, F \vdash A$

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$\boxed{A}$

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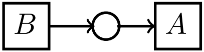

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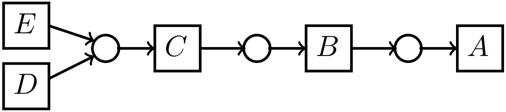

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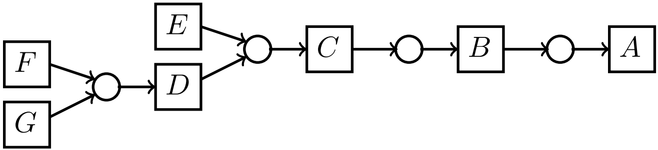
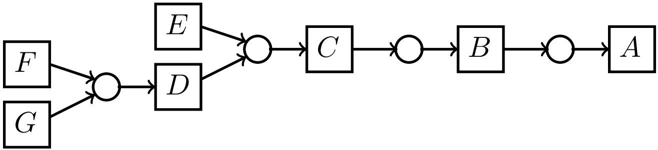


Circular arguments

Want: $E, F \vdash A$

Subgoal: $E, F \vdash G$



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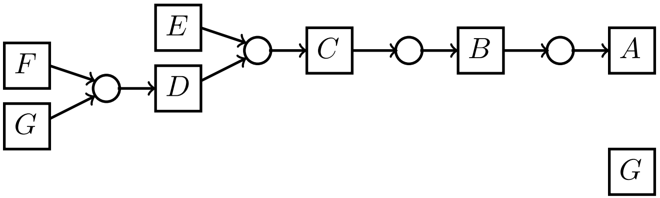

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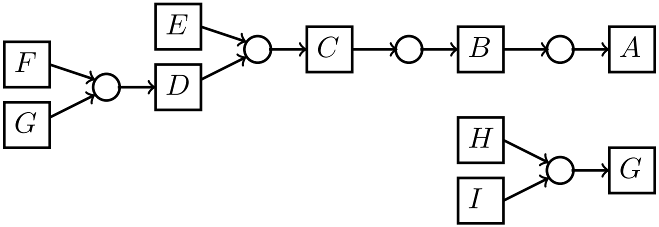

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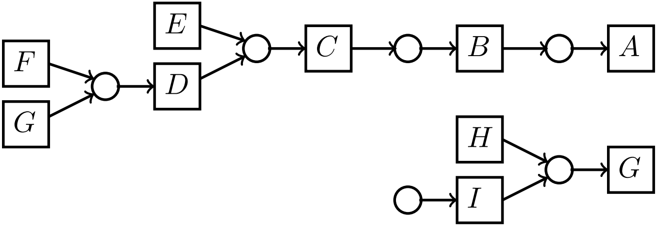

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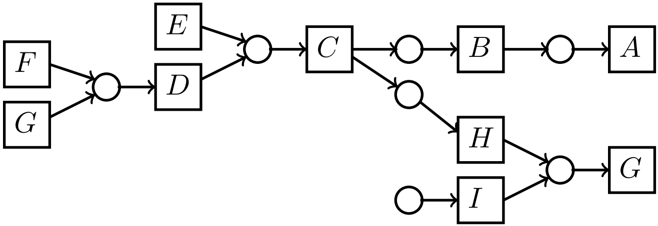

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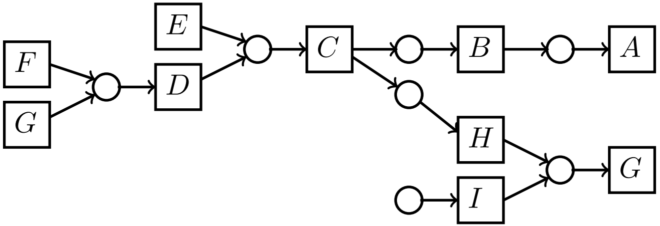

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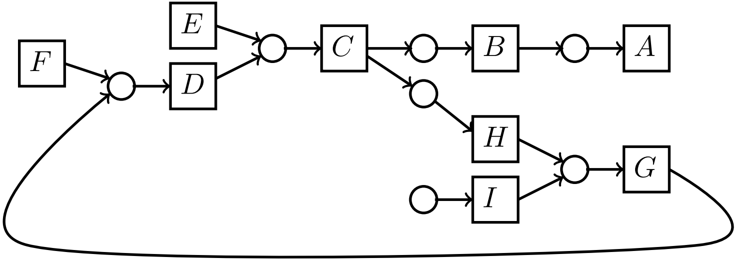

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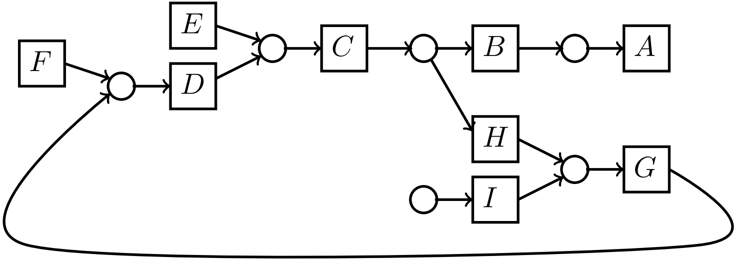

...???...

Circular Pre-proofs

Definition: A pre-proof is a pair (Π, B) where:

- Π is an ordinary proof $C_1, C_2, \ldots, C_m$,
- B is a set of backedges; i.e. pairs (i, j) s.t. $j < i$ and $C_j = C_i$.

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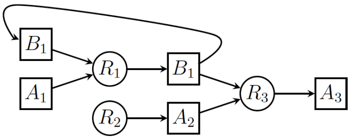
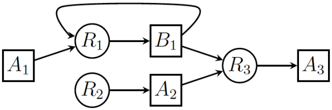
Example:

$$\Pi' : (\Pi = (B_1, A_1, B_1, A_2, A_3), B = \{(3, 1)\})$$





Some terminology and notation

$$\Pi' : ((C_1, C_2, \ldots, C_m), B)$$

Terminology and notation:

- $G(\Pi)$: the graph representation of Π.
- $N^+(u)$: the set of out-neighbours of u.
- $N^-(u)$: the set of in-neighbours of u.
- F: the set of formula vertices (the squares) of $G(\Pi)$.
- I: the set of inference vertices (the circles) of $G(\Pi)$.

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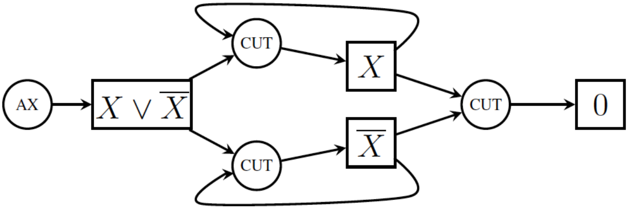
Observe:

- $u \in F$ implies $N^-(u) \subseteq I$ and $N^+(u) \subseteq I$.
- $u \in I$ implies $N^-(u) \subseteq F$ and $N^+(u) \subseteq F$.

Severe unsoundness of pre-proofs



Flow assignments and balance

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- $W^+(u) := \sum_{v \in N^+(u)} W(u)$ for $u \in F$; the out-flow of u.
- $B(u) := W^-(u) - W^+(u)$ for $u \in F$; the balance of $u \in F$.

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- $B(u) := W^-(u) - W^+(u)$ for $u \in F$; the balance of $u \in F$.

- if $B(u) < 0$, then C_u is called a hypothesis.
- if $B(u) > 0$, then C_u is called a conclusion.

Circular Proofs

Definition: A circular proof of A from $A_1, \ldots, A_m$ is a pre-proof for which there exists a flow-assignment such that, for each formula vertex $u \in F$, the following hold:

1. $B(u) < 0$ if $C_u \in \{A_1, \ldots, A_m\}$,
2. $B(u) \geq 0$ if $C_u \notin \{A_1, \ldots, A_m\}$,
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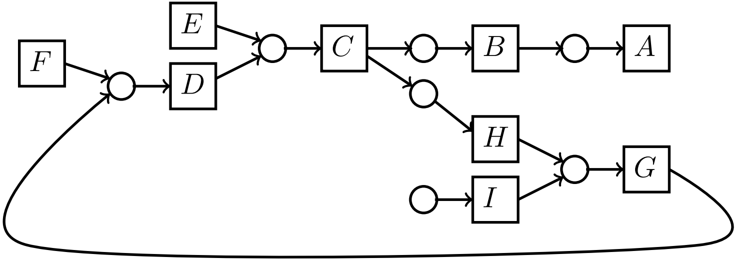
Notes:

- efficient verification: linear programming techniques,
- weights may be assumed small rationals: by LP techniques,
- and even small integers: by flow techniques,

The examples again

Want: $E, F \vdash A$



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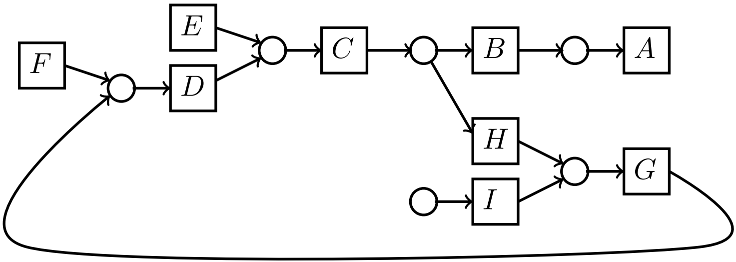

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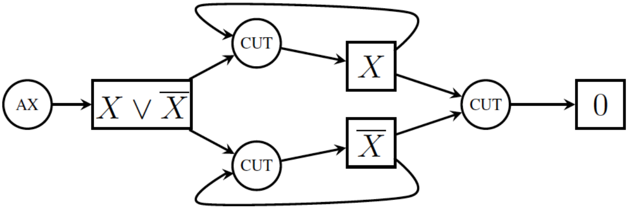
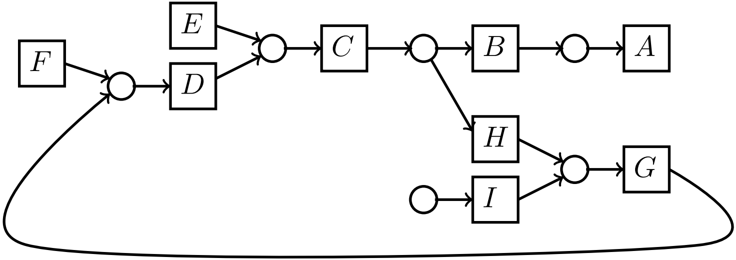


Soundness

Theorem:
If there is a circular proof of A from $A_1, \ldots, A_m$,
then every assignment that satisfies $A_1, \ldots, A_m$ also satisfies A.

1st proof of soundness: by example

$$E, F \vdash A \quad \Longrightarrow \quad E, F \models A$$

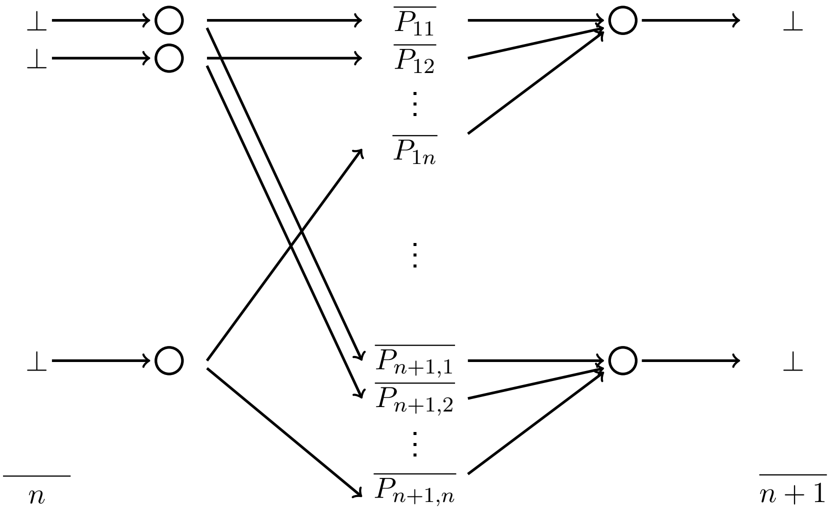


Poly-size circular resolution proof of PHP

Theorem:

PHP_n^{n+1} has poly-size circular resolution refutations.

Proof of PHP



$\perp$
$\perp$
$\perp$
$\overline{P_{11}}$
$\overline{P_{12}}$
$\overline{P_{1n}}$
$\overline{P_{n+1,1}}$
$\overline{P_{n+1,2}}$
$\overline{P_{n+1,n}}$
$\perp$
$\perp$
$\overline{n}$
$\overline{n+1}$

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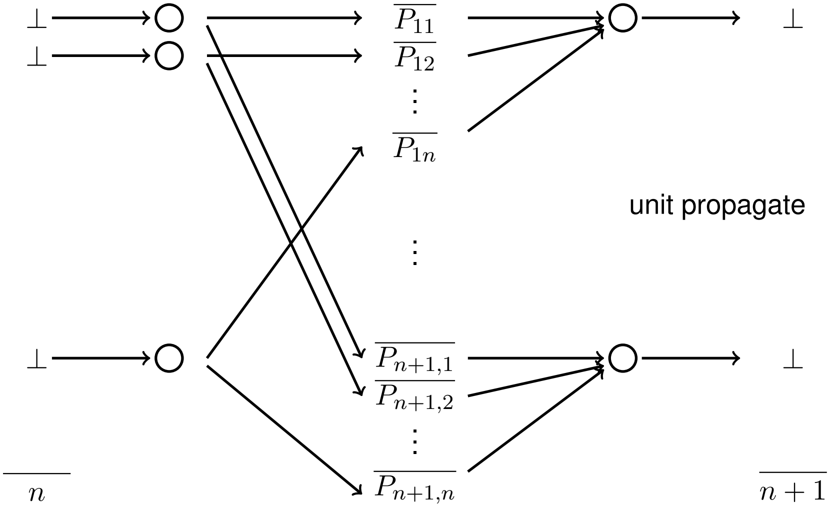

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unit propagate
$\overline{n}$
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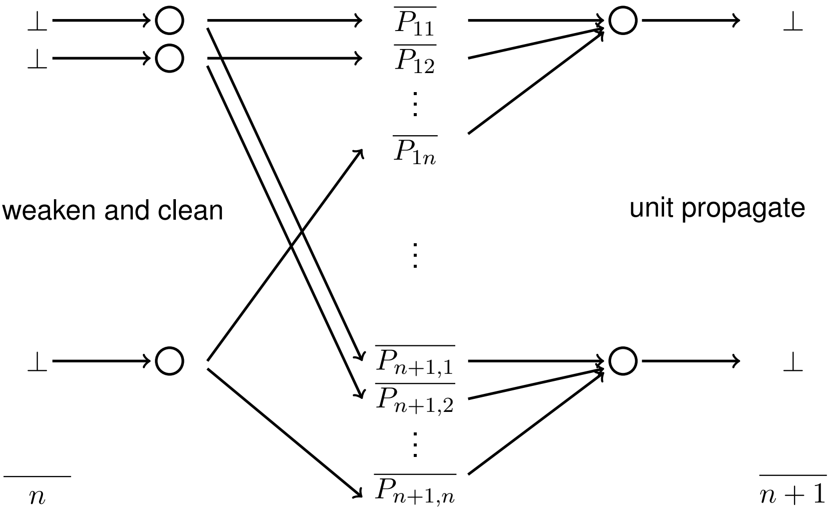
Proof of PHP



⊥
⊥
⊥
$\overline{P_{11}}$
$\overline{P_{12}}$
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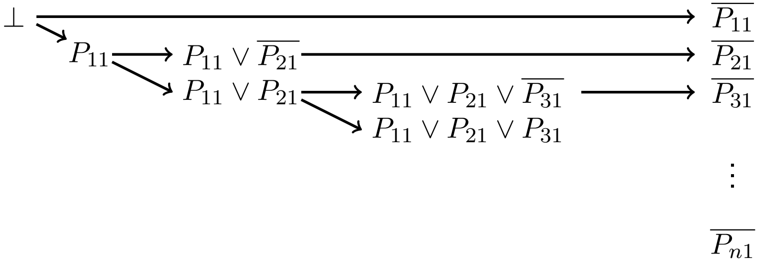
weaken and clean
unit propagate
⊥
⊥
$\overline{n}$
$\overline{n+1}$

Proof of PHP: weaken and clean for hole 1



Next question

WHAT IS CIRCULAR RESOLUTION?

Sherali-Adams proofs on Boolean variables

Variables:

$$X_1, \ldots, X_n \text{ and } \overline{X_1}, \ldots, \overline{X_n}$$

Axioms:

$$\begin{array}{lll} X_i \geq 0 & X_i^2 - X_i \geq 0 & X_i + \overline{X_i} - 1 \geq 0 \\ 1 - X_i \geq 0 & -X_i + X_i^2 \geq 0 & 1 - X_i - \overline{X_i} \geq 0 \end{array}$$

SA Proofs: A refutation of $P_1 \geq 0, \ldots, P_m \geq 0$ (including the axioms) is a polynomial identity of the form

$$\sum_{j=1}^{m} P_j Q_j + Q_0 = -1$$

where each Q_i has the form

$$\sum_{j \in K} c_j^2 \prod_{i \in I_j} X_i \prod_{i \in J_j} \overline{X_i}.$$

Monomial size: max number monomials in $P_i Q_i$ and Q_0.

Equivalence: Circular Resolution $\equiv$ Sherali-Adams

Multiplicative encoding of clauses:

$$\bigvee_{i \in I} X_i \vee \bigvee_{i \in J} \overline{X_i} \quad \mapsto \quad -\prod_{i \in I} \overline{X_i} \prod_{j \in J} X_i \geq 0$$

Additive encoding of clauses:

$$\bigvee_{i \in I} X_i \vee \bigvee_{i \in J} \overline{X_i} \quad \mapsto \quad \sum_{i \in I} X_i + \sum_{j \in J} \overline{X_i} - 1 \geq 0$$

Theorem:

Circular Resolution $\equiv_p$ Sherali-Adams.

(for both encodings)

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$\leq_p$: essentially [Dantchev 2007] (reused in [ALN16]).

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Proof:

$\leq_p$: essentially [Dantchev 2007] (reused in [ALN16]).

$\geq_p$: a normal form result for Sherali-Adams proofs.

2nd proof of soundness: via LP

Assume: α satisfies all the hypotheses.
Define: $Z_u = 1 - \alpha(C_u)$ for each $u \in F$.
Note:

$-Z_u \geq 0$	for each axiom vertex
$Z_u + Z_v - Z_w \geq 0$	for each cut vertex
$Z_u - Z_v - Z_w \geq 0$	for each weakening vertex

Therefore:

$$\sum_{v \in I} W(v) \left(\sum_{u \in N^-(v)} Z_u - \sum_{u \in N^+(v)} Z_u \right) \geq 0.$$

Equivalently:

$$-\sum_{u \in F} B(u) Z_u \geq 0$$

Proof of Circular Resolution $\leq_p$ Sherali-Adams

Define: $M_u =$ "multiplicative encoding of C_u" for each $u \in F$.
Note:

$$\begin{array}{rcll} M_u & = & -X\overline{X} & \text{for axiom } \vdash u \\ -M_u - M_v + M_w & = & (-X - \overline{X} + 1)M_w & \text{for cut } u, v \vdash w \\ -M_u + M_v + M_w & = & (-1 + X + \overline{X})M_u & \text{for weakening } u \vdash v, w \end{array}$$

Therefore:

$$\sum_{v \in I} W(v) \left(\sum_{u \in N^-(v)} M_u - \sum_{u \in N^+(v)} M_u \right) = -\sum_{u \in F} B(u) M_u$$

Now: Add positive multiples of

$$\prod_i X_i \prod_j \overline{X_j} = -M_u \qquad \text{for each } u \text{ s.t. } C_u \neq 0.$$

Get: $M_0 = -1$.

Take-home messages

1- Circular proofs are not always meaningless.

2- PHP has poly-size proofs in Circular Resolution.

3- Indeed Circular Resolution $\equiv_p$ Sherali-Adams.

Acknowledgments

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