Towards MaxSAT-Based Proof Systems

A Practical Perspective

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Workshop on Theory and Practice of Satisfiability Solving

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The SAT disruption

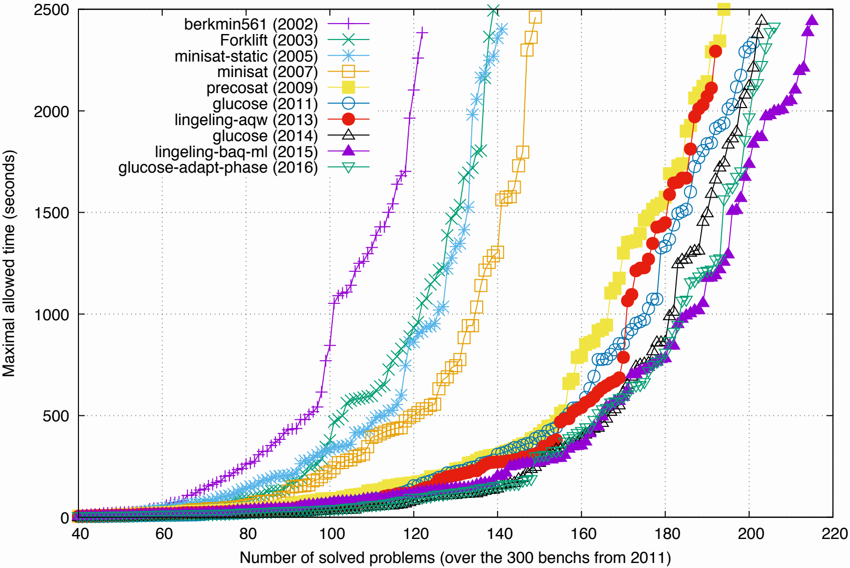
- Key breakthroughs in mid 90s and early 00s

The SAT disruption

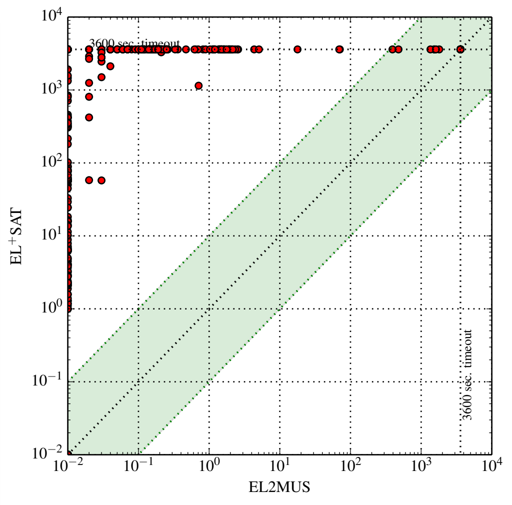
- Key breakthroughs in mid 90s and early 00s
  - **CDCL** SAT solving enabled many successes over the years
  - Hundreds (thousands?) of practical applications

SAT solver evolution

[Source: Simon 2015]



SAT can make the difference – axiom pinpointing



- Instances: $\mathcal{EL}^+$ medical ontologies

How significant is SAT solving?

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Comm. ACM 2010

DOI:10.1145/1839676.1839677

Moshe Y. Vardi

On P, NP, and Computational Complexity

Today's SAT solvers, which enjoy wide industrial usage, routinely solve SAT instances with over one *million* variables. How can a scary **NP**-complete problem be so easy? What is going on?

The answer is that one must read complexity-theoretic claims carefully. Classical **NP**-completeness theory is about *worst-case* complexity.

My point here is not to criticize complexity theory. It is a beautiful theory that has yielded deep insights over the last 50 years, as well as posed fundamental, tantalizing problems, such as the **P** vs. **NP** problem. But an important role of theory is to shed light on practice, and there we have large gaps. We need, I believe, a richer and broader complexity theory, a theory that would explain both the difficulty and the easiness of problems like SAT. More theory, please!

Moshe Y. Vardi, EDITOR-IN-CHIEF

How significant is SAT solving? And SAT oracles?

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When you have a big hammer, look for nails!

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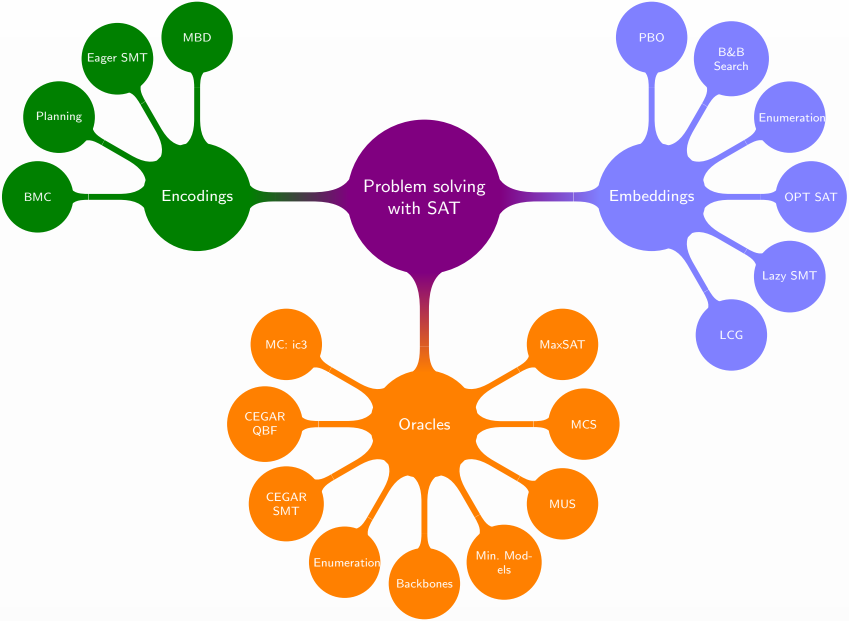
SAT is ubiquitous in problem solving



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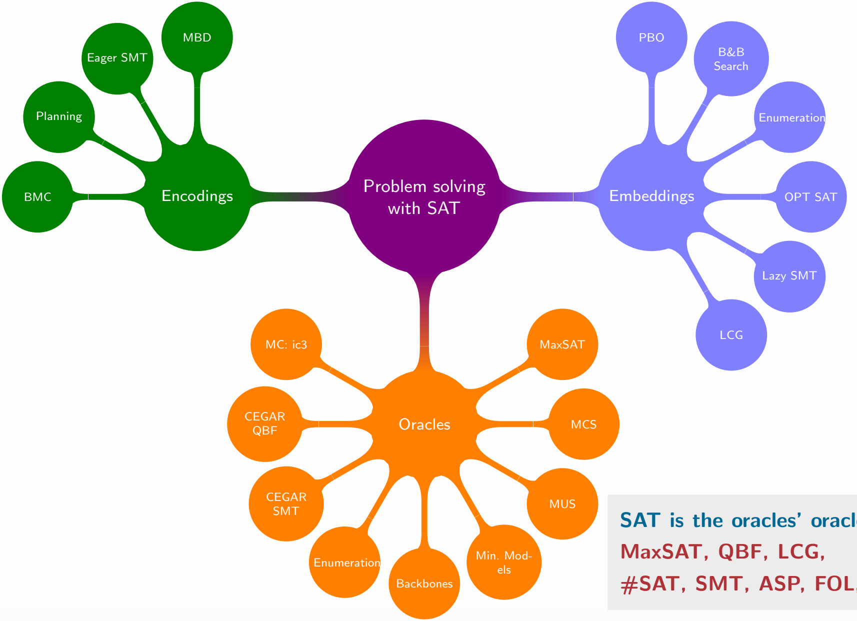

SAT is the oracles' oracle:
MaxSAT, QBF, LCG, #SAT, SMT, ASP, FOL, ...

What is Maximum Satisfiability (MaxSAT)?

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

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- Given unsatisfiable formula find largest satisfiable subset of clauses

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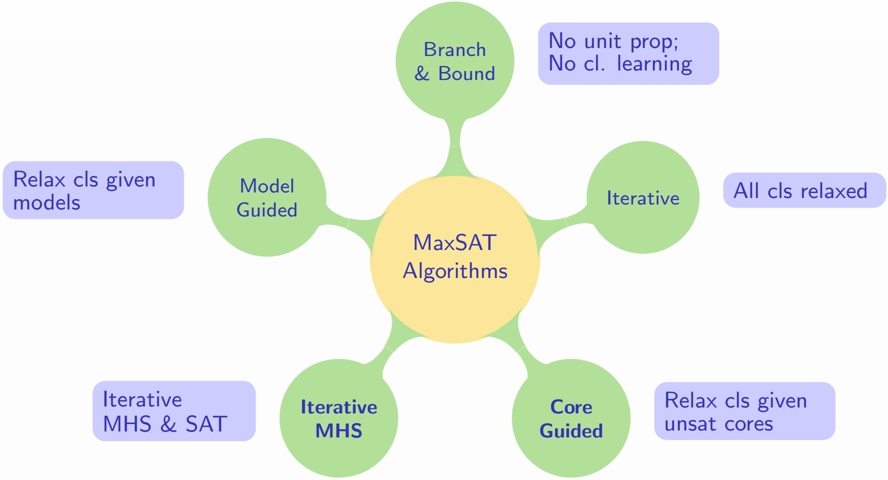
- Given unsatisfiable formula, find largest satisfiable subset of clauses

MaxSAT Variants		Hard Clauses? No	Hard Clauses? Yes
Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

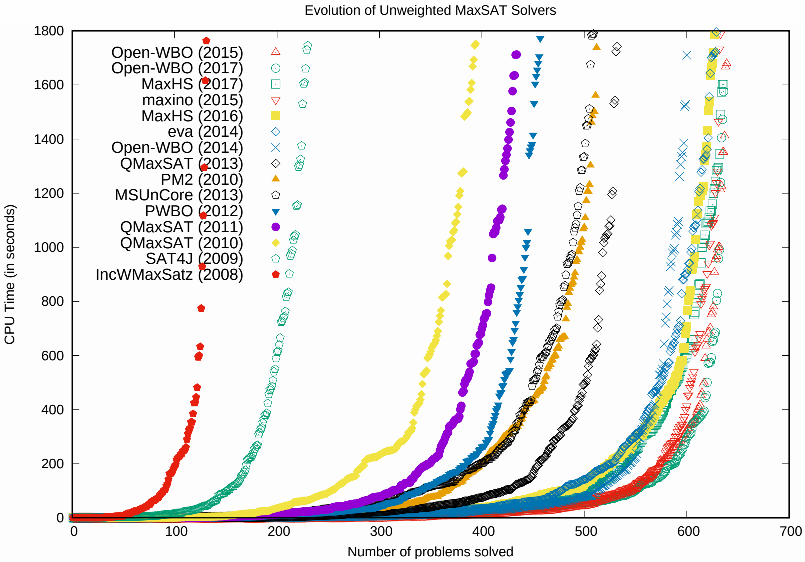
- **Many** practical applications [e.g. SZGN17]

Many MaxSAT approaches



- For practical (**industrial**) instances: **core-guided & MHS** approaches are the most effective [MaxSAT17]

MaxSAT (r)evolution – unweighted instances 2008-2017



Source: [MaxSAT 2017 organizers]

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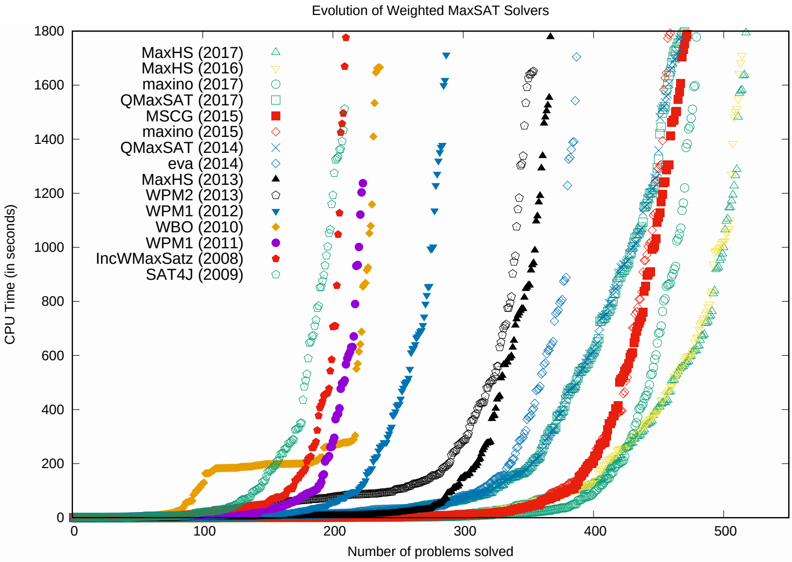

Source: [MaxSAT 2017 organizers]

What about in 2018?

What about in 2018? – complete tracks

Source: [MaxSAT 2017 organizers]

Unweighted			Weighted		
Solver	#Solved	Time (Avg)	Solver	#Solved	Time (Avg)
RC2-B	421	126.32	**RC2-B**	421	256.02
RC2-A	416	138.98	RC2-A	416	267.55
maxino	405	137.50	MaxHS	390	274.87
MaxHS	386	178.06	Pacose	390	348.98
Open-WBO-Gluc	382	171.54	QMaxSAT	381	320.78

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- **Note:** RC2 is a variant of a **2014** algorithm, with some practical optimizations
  - **Core-guided**, based on lower-bound refinement [FM06,MSP07]
  - Exploits **soft** cardinality constraints [MDMS14]
  - Inspired by OLL algorithm, first used in ASP optimization [AKMS12]

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Outline

MaxSAT Solving

Horn MaxSAT

PHP Refutations in Polynomial Time

Outline

MSU3 core-guided algorithm

(M.-S.&Planes,CoRR'07)

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
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$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Example CNF formula

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Formula is UNSAT; $\mathrm{OPT} \leq |\varphi| - 1$; Get unsat core

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$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
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$\sum_{i=1}^{6} r_i \leq 1$

Add relaxation variables and AtMost$k$, $k = 1$, constraint

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Add new relaxation variables and update AtMost$k$, k=2, constraint

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MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

MSU3 core-guided algorithm

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Builds on FM06 seminal work ...

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AtMost$k$/PB constraints used

Relaxed soft clauses become **hard**

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$x_7 \vee$ ... $\vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Note: # of SAT oracle calls grows linear with solution cost!

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

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Outline

MaxSAT Solving

- Core Guided with MSU3 – Example
- Core Guided with RC2 – Example
- MaxHS – Example
- MaxHS – Algorithm

Horn MaxSAT

PHP Refutations in Polynomial Time

Soft cardinality constraints

(Morgado,Dodaro&M.-S.,CP'14)

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$S_1 \leq 1$			

Aux sums: $S_1 = \sum_{i=1}^{6} r_i$;

Add relaxation variables and AtMost1 constraint

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$S_1 \leq 2 \qquad S_2' + \neg(S_1 \leq 1) \leq 1$

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$\boxed{S_1 \leq 2}$ $\boxed{S_2' + \neg(S_1 \leq 1) \leq 1}$

$S_1 \geq 2 \rightarrow S_2' = 0$
$S_1 \leq 1 \rightarrow S_2' \leq 1$

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Instance is now SAT

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MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

Soft cardinality constraints

(Morgado,Dodaro&M.-S.,CP'14)

Builds on other algorithms: FM06, MSP07, ...

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MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

Only AtMost$k$ constraints used

Sums reused with $\neq$ RHSs

Relaxed soft clauses become **hard**

Outline

MaxSAT with Minimum Hitting Sets (MHS)

(Davies&Bacchus,CP'11)

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

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- Find MHS of $\mathcal{K}$:

MaxSAT with Minimum Hitting Sets (MHS)

(Davies&Bacchus,CP'11)

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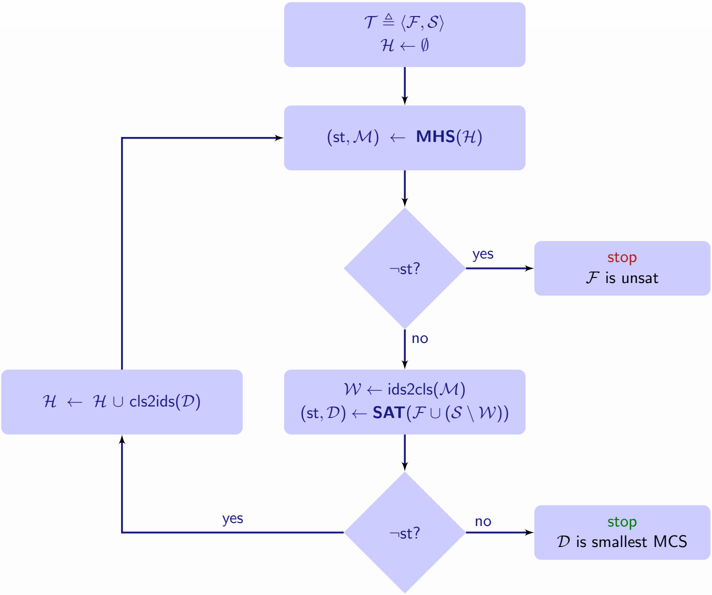
Possibly many MHSes, with one SAT oracle call for each MHS!

Outline

The MaxHS algorithm

(Davies&Bacchus,CP'11)

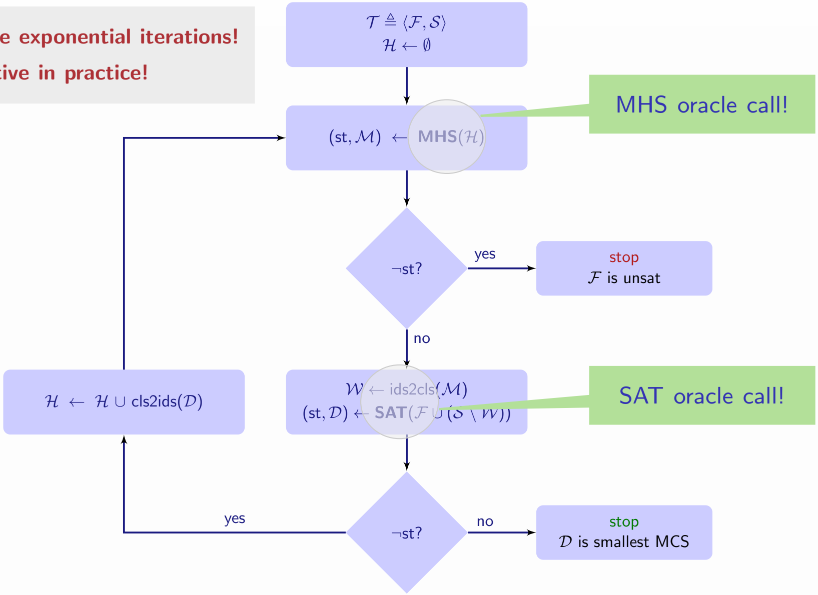


The MaxHS algorithm

(Davies&Bacchus,CP'11)

Worst-case exponential iterations!

But effective in practice!



Outline

MaxSAT Solving

Horn MaxSAT

PHP Refutations in Polynomial Time

Recap Horn MaxSAT

- What is Horn MaxSAT?
  - All soft clauses are Horn
    - Most often, unit soft clauses
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- How hard is Horn MaxSAT?
  - Horn MaxSAT is NP-hard [JS87]
  - Decision K-HornSAT is NP-complete [JS87]
    - **By definition, any problem in NP is reducible to K-HornSAT**
    - But ...

Why use Horn MaxSAT?

- **Practical perspective:**
  - MaxSAT with MHSes is **very efficient** in practice
  - For Horn MaxSAT, we can replace SAT call (worst-case exponential) with LTUR call (worst-case **linear**)

Why use Horn MaxSAT?

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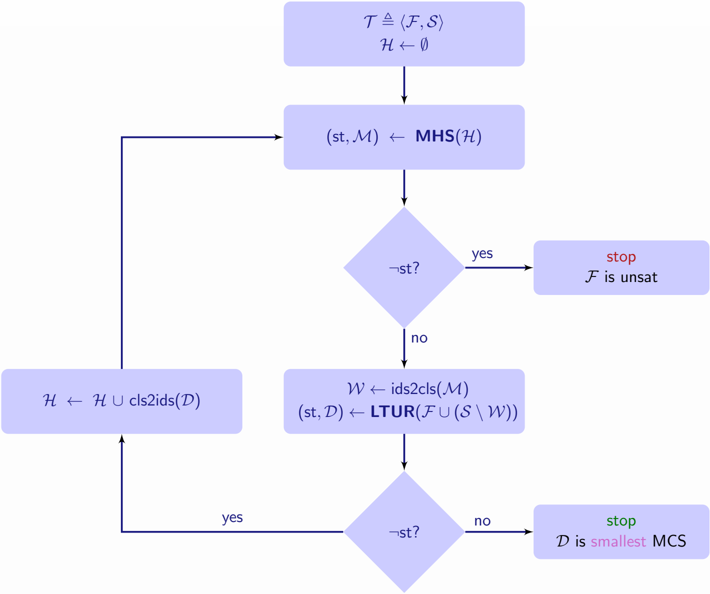
- **Theoretical perspective:**
  - Reducing SAT to Horn MaxSAT & applying a MaxSAT algorithm yields new proof system(s)
    - MaxSAT resolution
    - Core-guided algorithm(s)
    - MaxHS-like algorithms
    - ...
  - Reducing PHP to SAT and then to Horn MaxSAT admits polynomial time refutations for **some** MaxSAT algorithms

A Horn MaxHS algorithm



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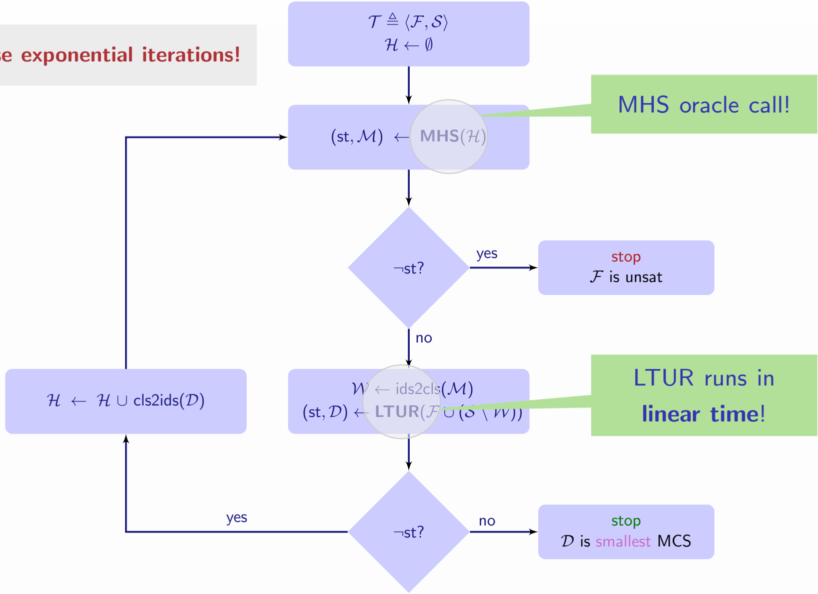
Worst-case exponential iterations!



What can we solve with Horn MaxSAT?

[IMMS17,MSIM17]

SAT $\leq_P$ Horn MaxSAT	and so CSP, ASP, SMT*, ...
CSP $\leq_P$ Horn MaxSAT	direct, besides CSP $\leq_P$ SAT
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  - Sequential counters; totalizers; sorting networks; (pairwise) (cardinality networks); bitwise (for AtMost1) [S05,ES06,ANORC11,...]

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Knapsack $\leq_P$ Horn MaxSAT

- **Horn MaxSAT: enables general-purpose problem solving**

Outline

SAT reduces to Horn MaxSAT

$$\mathcal{F} \triangleq (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

SAT reduces to Horn MaxSAT

$$\mathcal{F} \triangleq (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

- For each x_i, create new variables p_i (for $x_i = 1$) and n_i (for $x_i = 0$)
- p_i and n_i cannot both be assigned 1:
  - Add hard clause $(\neg p_i \vee \neg n_i)$

SAT reduces to Horn MaxSAT

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- p_i and n_i cannot both be assigned 1:
  - Add hard clause $(\neg p_i \lor \neg n_i)$
- Reencode original clauses (as hard clauses):
  - Literal x_i replaced by $\neg n_i$
  - Literal $\neg x_i$ replaced by $\neg p_i$
- Goal is to assign value 1 to each variable, if possible:
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- Original formula is satisfiable iff Horn MaxSAT formula can satisfy n soft clauses (and the hard clauses)
  - I.e., satisfying n soft clauses represents assignment to the n variables consistent with the original clauses **!**

SAT reduces to Horn MaxSAT (Cont.)

$$\mathcal{F} \triangleq (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

- Example:
  - New variables: $p_1, p_2, p_3, n_1, n_2, n_3$
  - Filter impossible assignments: $\{(\neg p_1 \vee \neg n_1), (\neg p_2 \vee \neg n_2), (\neg p_3 \vee \neg n_3)\}$
  - Original clauses reencoded: $(\neg n_1 \vee \neg p_2 \vee \neg n_3) \wedge (\neg n_2 \vee \neg n_3) \wedge (\neg p_1 \vee \neg p_3)$
  - Soft clauses: $\{(p_1), (p_2), (p_3), (n_1), (n_2), (n_3)\}$

SAT reduces to Horn MaxSAT (Cont.)

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  - Soft clauses: $\{(p_1), (p_2), (p_3), (n_1), (n_2), (n_3)\}$
- Encoding is a variant of the dual-rail encoding, used since the mid 80s [BBBCS87]

Pigeonhole formulas – propositional encoding PHP_m^{m+1}

- Variables:
  - $x_{ij} = 1$ iff the i^{th} pigeon is placed in the j^{th} hole, $1 \leq i \leq m+1$, $1 \leq j \leq m$

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- Example encoding, with pairwise encoding for AtMost1 constraint:

Constraint	Clause(s)
$\wedge_{i=1}^{m+1}\mathrm{AtLeast1}(x_{i1}, \ldots, x_{im})$	$(x_{i1} \vee \ldots \vee x_{im})$
$\wedge_{j=1}^{m}\mathrm{AtMost1}(x_{1j}, \ldots, x_{m+1\,j})$	$\wedge_{r=2}^{m+1} \wedge_{s=1}^{r-1} (\neg x_{rj} \vee \neg x_{sj})$

PHP as Horn MaxSAT

- New variables n_{ij} and p_{ij}, for each x_{ij}, $1 \leq i \leq m+1, 1 \leq j \leq m$
- The soft clauses $\mathcal{S}$, with $|\mathcal{S}| = 2m(m+1)$, are given by

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$$\langle \mathcal{H}, \mathcal{S} \rangle = \left\langle \wedge_{i=1}^{m+1} \mathcal{L}_i \wedge \wedge_{j=1}^{m} \mathcal{M}_j \wedge \mathcal{P}, \mathcal{S} \right\rangle$$

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- **No** more than $m(m+1)$ clauses can be satisfied, due to $\mathcal{P}$
- PHP_m^{m+1} is satisfiable iff there exists an assignment that satisfies the hard clauses $\mathcal{H}$ and $m(m+1)$ soft clauses from $\mathcal{S}$

PHP as Horn MaxSAT II

- Clauses in each $\mathcal{L}_i$ and in each $\mathcal{M}_j$, with pairwise encoding

Original Constraint	Encoded To	Clauses
$\wedge_{i=1}^{m+1}\text{AtLeast1}(x_{i1},\ldots,x_{im})$	$\mathcal{L}_i$	$(\neg n_{i1} \vee \ldots \vee \neg n_{im})$
$\wedge_{j=1}^{m}\text{AtMost1}(x_{1j},\ldots,x_{m+1,j})$	$\mathcal{M}_j$	$\wedge_{r=2}^{m+1} \wedge_{s=1}^{r-1} (\neg p_{rj} \vee \neg p_{sj})$

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- **Note:** constraints with key structural properties:

Constraint	Variables
$\mathcal{L}_i$	$(\neg n_{i1} \vee \ldots \vee \neg n_{im})$
$\mathcal{L}_k$	$(\neg n_{k1} \vee \ldots \vee \neg n_{km})$
$\mathcal{M}_j$	$\wedge_{r=2}^{m+1} \wedge_{s=1}^{r-1} (\neg p_{rj} \vee \neg p_{sj})$
$\mathcal{M}_l$	$\wedge_{r=2}^{m+1} \wedge_{s=1}^{r-1} (\neg p_{rl} \vee \neg p_{sl})$

  - Variables in each $\mathcal{L}_i$ **disjoint** from **any** other $\mathcal{L}_k$ and $\mathcal{M}_j$, $k \neq i$
  - Variables in each $\mathcal{M}_j$ **disjoint** from **any** other $\mathcal{M}_l$, $l \neq j$

Outline

Some results from our SAT'17 paper

Claim 1

Core-guided MaxSAT (e.g. MSU3) produces a lower bound on the number of falsified clauses $\geq m(m+1)+1$ in polynomial time

Claim 2

MaxSAT resolution produces a lower bound on the number of falsified clauses $\geq m(m+1)+1$ in polynomial time

Remark

Horn MaxSAT encoding enables polynomial time refutations of the unsatisfiability of PHP instances, using CDCL SAT solvers

Proof of claim 1 – outline

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3. Derive large enough lower bound on # of falsified clauses:

Constr. type	# falsified cls	# constr	In total
$\mathcal{L}_i$	1	$i = 1, \ldots, m+1$	$m+1$
$\mathcal{M}_j$	m	$j = 1, \ldots, m$	$m \cdot m$
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4. Each increase in the value of the lower bound obtained by unit propagation (UP)
   - In total: polynomial number of (linear time) UP runs

Proof of claim 1 – unit propagation steps I

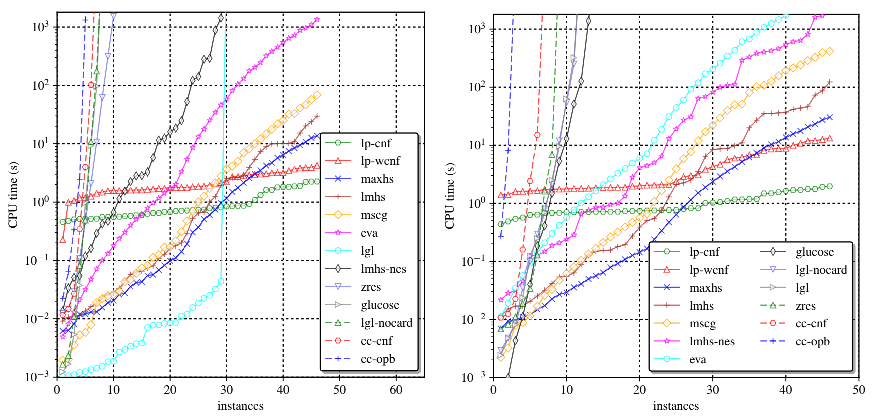
Constr	Hard cls	Soft cls	Relaxed clauses	Updated AtMost$k$ constr	LB incr
$\mathcal{L}_i$	$(\neg n_{i1} \vee \ldots \vee \neg n_{im})$	$(n_{i1}), \ldots, (n_{im})$	$(s_{il} \vee n_{i1})$, $1 \leq l \leq m$	$\sum_{l=1}^{m} s_{il} \leq 1$	1
$\mathcal{M}_j$	$(\neg p_{1j} \vee \neg p_{2j})$	$(p_{1j}), (p_{2j})$	$(r_{1j} \vee p_{1j})$, $(r_{2j} \vee p_{2j})$	$\sum_{l=1}^{2} r_{lj} \leq 1$	1
$\mathcal{M}_j$	$(\neg p_{1j} \vee \neg p_{3j})$, $(\neg p_{2j} \vee \neg p_{3j})$, $(r_{1j} \vee p_{1j})$, $(r_{2j} \vee p_{2j})$, $\sum_{l=1}^{2} r_{lj} \leq 1$	(p_{3j})	$(r_{3j} \vee p_{3j})$	$\sum_{l=1}^{3} r_{lj} \leq 2$	1
		$\ldots$			
$\mathcal{M}_j$	$(\neg p_{1j} \vee \neg p_{m+1j}), \ldots,$ $(\neg p_{mj} \vee \neg p_{m+1j})$, $(r_{1j} \vee p_{1j}), \ldots,$ $(r_{mj} \vee p_{mj})$, $\sum_{l=1}^{m} r_{lj} \leq m-1$	(p_{m+1j})	$(r_{m+1j} \vee p_{m+1j})$	$\sum_{l=1}^{m+1} r_{lj} \leq m$	1

Proof of claim 1 – unit propagation steps II

Clauses	Unit Propagation
$(p_{k+1\,j})$	$p_{k+1\,j} = 1$
$(\neg p_{1j} \vee \neg p_{k+1\,j}), \ldots, (\neg p_{kj} \vee \neg p_{k+1\,j})$	$p_{1j} = \ldots = p_{kj} = 0$
$(r_{1j} \vee p_{1j}), \ldots, (r_{kj} \vee p_{kj})$	$r_{1j} = \ldots = r_{kj} = 1$
$\sum_{l=1}^{k} r_{lj} \leq k-1$	$\left(\sum_{l=1}^{k} r_{lj} \leq k-1\right) \vdash_1 \perp$

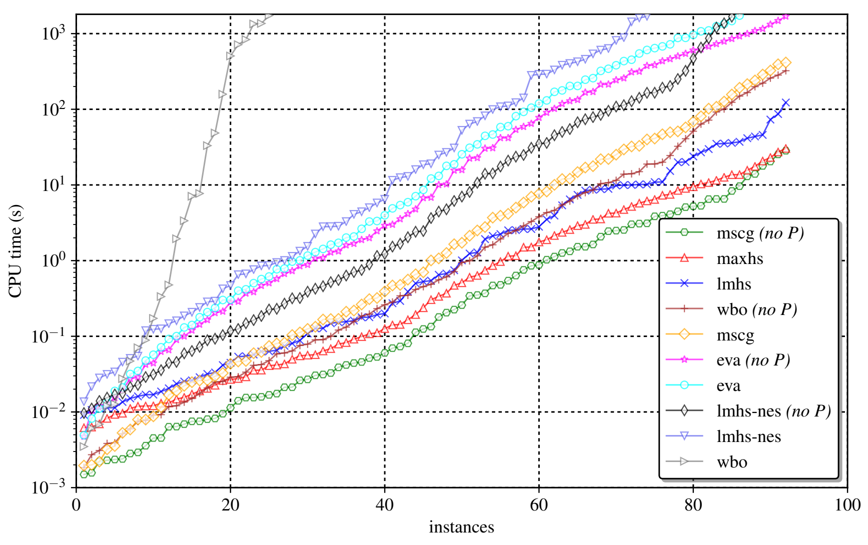
- Key points:
  - For each $\mathcal{L}_i$, UP raises LB by 1
  - For each $\mathcal{M}_j$, UP raises LB by m
  - In total, UP raises LB by $m(m+1)+1$
  - Thus, PHP_m^{m+1} is **unsatisfiable**

Results on PHP instances: pw vs. sc



SAT		**SAT+**		**IHS MaxSAT**		**CG MaxSAT**			**MRes**	**MIP**	**OPB**		**BDD**
minisat	*glucose*	*lgl*	crypto	*maxhs*	*lmhs*	*mscg*	*wbo*	wpm3	*eva*	*lp*	*cc*	sat4j*	*zres*

Effect of $\mathcal{P}$ clauses



Some results from our AAAI'18 paper – see MLB's talk

Remark

Formalize DrMaxSAT proof system, using MaxSAT resolution

Result 1

DrMaxSAT p-simulates RES/CL

$\therefore$ DrMaxSAT stronger proof system than RES/CL

Result 2

MaxSAT refutations of the dual-rail encoded Parity Principle require exponential size 2^{n^ϵ} for some $\epsilon > 0$

$\therefore$ DrMaxSAT does not p-simulate CP

But, several open questions ...

Conclusions & research directions

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- Where to go with Horn MaxSAT?
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- Still many open questions?
  - E.g. MaxHS unreasonably efficient. **Why?**

Questions?

Some references

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