Mapping class group representations via Heisenberg, Schrödinger and Stone-von Neumann

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Aims

Applications of braid group representations:

- Linearity: B_n embeds into $GL_N(\mathbb{R})$

  [Bigelow, Krammer]

  using *Lawrence representations*
- Applications to knot theory (Alexander and Markov theorems)
- Applications to algebraic geometry (invariants of curves in $\mathbb{CP}^2$)

  [Moishezon, Libgober]

Aim:

- Construct analogues of the Lawrence representations for

$$\mathrm{Map}(\Sigma_{g,1}) = \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1}))$$

- ($\rightsquigarrow$ linearity?)
- ($\rightsquigarrow$ extensions to 3-dim. TQFT?)

Representations of braid groups – Burau

[Burau] representation (1935):

$$\sigma_i \quad \longmapsto \quad \mathrm{I}_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus \mathrm{I}_{n-i-1}$$

- This defines $B_n \longrightarrow GL_n(\mathbb{Z}[t^{\pm 1}]) \subset GL_n(\mathbb{R})$
- Q([Birman'74]): *Is this representation injective?* ($\equiv$'faithful')
- A($n \leqslant 3$): Yes [Magnus-Peluso'69]
- A($n \geqslant 5$): No [Moody'91,Long-Paton'93,Bigelow'99]
- A($n = 4$): ??
- Q: *Are the braid groups linear?*
  — Does B_n embed into some $GL_N(\mathbb{F})$?

Representations of braid groups – Lawrence

[Lawrence] representation (1990) — geometric definition.

- $\mathrm{Diff}_\partial(D_n)$ acts on $C_k(D_n)$ (unordered configuration space)
  (D_n = closed 2-disc minus n punctures)
- $B_n = \mathrm{Map}(D_n) = \pi_0(\mathrm{Diff}_\partial(D_n))$ acts on $H_*(C_k(D_n);\mathbb{Z})$
- Two modifications:
  - Choose $\pi_1(C_k(D_n)) \twoheadrightarrow Q$ invariant under the action.
    Then B_n acts on $H_*(C_k(D_n);\mathbb{Z}[Q])$
  - Replace H_* with H_*^{BM} (*Borel-Moore homology*)
    Then $H_*^{BM}(C_k(D_n);\mathbb{Z}[Q])$ is a free $\mathbb{Z}[Q]$-module
    concentrated in degree $* = k$

$$\mathrm{Lawrence}_k \colon B_n \longrightarrow GL_N(\mathbb{Z}[Q]) = \mathrm{Aut}_{\mathbb{Z}[Q]}\left(H_*^{BM}(C_k(D_n);\mathbb{Z}[Q])\right)$$

Representations of braid groups – Lawrence

How is the quotient Q defined?

- $\pi_1(D_n) = F_n \longrightarrow \mathbb{Z} = Q$ "total winding number"
- $\pi_1(C_k(D_n)) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$
  ("total winding number", "self-winding number")

Lemma

This quotient is $\mathrm{Map}(D_n)$-invariant, and hence

$$\mathrm{Lawrence}_k \colon B_n \longrightarrow GL_N(\mathbb{Z}[Q])$$

is well-defined. Moreover, we have $\mathrm{Lawrence}_1 = \mathrm{Burau}$.

Theorem [Bigelow'00,Krammer'00]

$\mathrm{Lawrence}_2$ is faithful (injective). Hence B_n embeds into $GL_N(\mathbb{R})$.

Representations of mapping class groups

- Q: Does $\mathrm{Map}(S)$ embed into $GL_N(\mathbb{F})$ for other surfaces S?
- $\mathrm{Map}(\text{torus}) \cong SL_2(\mathbb{Z}) \subset GL_2(\mathbb{R})$
- $\mathrm{Map}(\Sigma_2) \subset GL_{64}(\mathbb{C})$ [Bigelow-Budney'01]
- In general, wide open.
    - Kontsevich (2006): proposal of a sketch of a construction of a faithful finite-dimensional representation of $\mathrm{Map}(\Sigma_g)$
    - Dunfield (cf. [Margalit'18]): computational evidence suggesting that this will *not* actually be faithful
- From now on, focus on $\Sigma = \Sigma_{g,1}$
  (orientable, genus g, one boundary component)

Main result [Blanchet-P.-Shaukat'21]

A new family of representations of $\mathrm{Map}(\Sigma)$.
("Genuine" analogues of the Lawrence representations)

Representations of MCGs – Moriyama

Simplest analogue of the Lawrence representations:

$$\mathrm{Map}(\Sigma) \quad \circlearrowleft \quad H_k^{BM}(F_k(\Sigma');\mathbb{Z})$$

- $F_k(\quad) =$ *ordered* configuration space
- $\Sigma' = \Sigma \smallsetminus (\text{interval in } \partial\Sigma)$
- *untwisted* $\mathbb{Z}$ coefficients
- $H_k^{BM}(F_k(\Sigma');\mathbb{Z})$ is a free abelian group of finite rank

Theorem [Moriyama'07]

The kernel of this representation is $\mathfrak{J}(k) \subset \mathrm{Map}(\Sigma)$.

- $\mathfrak{J}(k)$ is the k-th term of the *Johnson filtration* of $\mathrm{Map}(\Sigma)$

The Johnson filtration

- Lower central series: $\pi_1(\Sigma) = \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \Gamma_4 \supseteq \cdots$
- $\Gamma_i = [\pi_1(\Sigma), \Gamma_{i-1}]$ (commutators of length i)

Definition [Johnson'81]

$\mathfrak{J}(k)$ = kernel of the action of $\mathrm{Map}(\Sigma)$ on $\pi_1(\Sigma)/\Gamma_{k+1}$.

- $\mathrm{Map}(\Sigma) = \mathfrak{J}(0) \supset \mathfrak{J}(1) \supset \mathfrak{J}(2) \supset \mathfrak{J}(3) \supset \cdots$
- $\mathfrak{J}(1) = \mathrm{Tor}(\Sigma) = \ker\left(\mathrm{Map}(\Sigma) \circlearrowleft H_1(\Sigma;\mathbb{Z})\right)$ *Torelli group*

Theorem [Johnson'81]

$$\bigcap_{k=1}^{\infty} \mathfrak{J}(k) = \{1\}$$

Corollary [Moriyama'07]

$\bigoplus_{k=1}^{\infty} H_k^{BM}(F_k(\Sigma');\mathbb{Z})$ is a faithful (∞-rank) $\mathrm{Map}(\Sigma)$-representation.

Rep. of MCGs – abelian twisted coefficients

- Idea: Enrich the representation by taking homology with *twisted coefficients* $\mathbb{Z}[Q]$, where $\pi_1(C_k(\Sigma')) = B_k(\Sigma) \twoheadrightarrow Q$.
- $Q = \mathfrak{S}_k$ corresponds to the Moriyama representations:
  $$H_k^{BM}(F_k(\Sigma'); \mathbb{Z}) = H_k^{BM}(C_k(\Sigma'); \mathbb{Z}[\mathfrak{S}_k]).$$
- First try abelian quotients Q.

Fact ($k \geqslant 2$)

$$B_k(S)^{ab} \cong \pi_1(S)^{ab} \oplus \left\{ \begin{array}{cc} \mathbb{Z} & S \text{ planar} \\ \mathbb{Z}/(2k-2) & S = S^2 \\ \mathbb{Z}/2 & \text{otherwise.} \end{array} \right\}$$

- If S is non-planar, we can only count the *self-winding number* ("writhe") of S-braids **mod 2**. (or mod $2k-2$ if $S = S^2$)
- In $\mathbb{Z}[B_k(S)^{ab}]$, the corresp. variable t has order two: $t^2 = 1$.
  $\rightsquigarrow$ we get a much "weaker" representation...

Rep. of MCGs – Heisenberg twisted coefficients

Theorem [Bellingeri'04]

$$B_k(\Sigma_{g,1}) \cong \left\langle \sigma_1, \ldots, \sigma_{k-1}, \begin{matrix} a_1, \ldots, a_g \\ b_1, \ldots, b_g \end{matrix} \;\middle|\; \cdots \text{ some relations } \cdots \right\rangle$$

Adding the relations saying that σ_1 is *central* (commutes with every element), we obtain:

$$B_k(\Sigma_{g,1})/\langle\!\langle\, [\sigma_1, x]\,\rangle\!\rangle \cong \left\langle \sigma, \begin{matrix} a_1, \ldots, a_g \\ b_1, \ldots, b_g \end{matrix} \;\middle|\; \begin{matrix} \text{all pairs commute except} \\ a_i b_i = \sigma^2 b_i a_i \end{matrix} \right\rangle$$

Definition

$$\mathcal{H}_g = B_k(\Sigma_{g,1})/\langle\!\langle\, [\sigma_1, x]\,\rangle\!\rangle$$

This is the *genus-g discrete Heisenberg group*. Note that:

$$\mathcal{H}_1 \cong \left\{ \begin{pmatrix} 1 & \mathbb{Z} & \frac{\mathbb{Z}}{2} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3(\mathbb{Q})$$

Rep. of MCGs – Heisenberg twisted coefficients

Lemma

The action $\mathrm{Map}(\Sigma) \circlearrowleft B_k(\Sigma)$ descends to a well-defined action on the quotient $\mathcal{H}_g$.

Proof

- Aim: $\ker(B_k(\Sigma) \twoheadrightarrow \mathcal{H}_g)$ is preserved by the $\mathrm{Map}(\Sigma)$-action.
- This is $\langle\!\langle\, [\sigma_1, x] \,\rangle\!\rangle$, so it is enough to show that σ_1 is fixed by the $\mathrm{Map}(\Sigma)$-action.
- Let $[\varphi] \in \mathrm{Map}(\Sigma) = \mathrm{Diff}(\Sigma)/\!\sim$ be represented by a diffeo. φ that fixes *pointwise* a collar neighbourhood of $\partial\Sigma$.
- The loop of configurations $\sigma_1 \in B_k(\Sigma) = \pi_1(C_k(\Sigma))$ can be homotoped to stay inside this collar neighbourhood.

The Heisenberg group fits into a central extension:

$$1 \to \mathbb{Z} \longrightarrow \mathcal{H}_g \longrightarrow H_1(\Sigma;\mathbb{Z}) \to 1$$

and the $\mathrm{Map}(\Sigma)$-action on $\mathcal{H}_g$ lifts the natural action on $H_1(\Sigma;\mathbb{Z})$.

Rep. of MCGs – Heisenberg twisted coefficients

Corollary

We obtain a *twisted* representation, defined over $\mathbb{Z}[\mathcal{H}_g]$:

$$\mathrm{Map}(\Sigma) \quad \circlearrowleft \quad H_k^{BM}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g]) = \mathcal{V}$$

"Twisted representation" really means we have:

- a $\mathbb{Z}[\mathcal{H}_g]$-module ${}_\tau\mathcal{V}$ for each $\tau \in \mathrm{Aut}^+(\mathcal{H}_g)$
- isomorphisms ${}_{\tau\circ\varphi_*}\mathcal{V} \to {}_\tau\mathcal{V}$ for each $\varphi \in \mathrm{Map}(\Sigma)$, $\tau \in \mathrm{Aut}^+(\mathcal{H}_g)$

(where $\varphi_* \in \mathrm{Aut}^+(\mathcal{H}_g)$ is the action of φ on $\mathcal{H}_g$)

In other words a functor $\mathrm{Ac}(\mathrm{Map}(\Sigma) \circlearrowleft \mathcal{H}_g) \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathcal{H}_g]}$.

(where $\mathrm{Ac}(\mathrm{Map}(\Sigma) \circlearrowleft \mathcal{H}_g)$ is the *action groupoid*)

Note

Replace the coefficients $\mathbb{Z}[\mathcal{H}_g]$ with any $\mathcal{H}_g$-representation W over R to get a twisted $\mathrm{Map}(\Sigma)$-representation $\mathcal{V}(W)$ over R.

Rep. of MCGs – Heisenberg twisted coefficients

Problem

How to untwist this representation?

Three methods:

(1) On the Torelli group $\mathrm{Tor}(\Sigma) \subset \mathrm{Map}(\Sigma)$
for any $\mathcal{H}_g$-representation W.

(2) On the (stably) universal central extension of $\mathrm{Map}(\Sigma)$
for $W = W_{\mathsf{Sch}}$ the Schrödinger representation of $\mathcal{H}_g$.

(3) Directly on the mapping class group $\mathrm{Map}(\Sigma)$
for $W = W_{\mathsf{lin}}$ the linearised tautological representation of $\mathcal{H}_g$.

Rep. of MCGs – restricting to the Torelli group

Proposition

The pre-image of $\mathrm{Inn}(\mathcal{H}_g) \subset \mathrm{Aut}(\mathcal{H}_g)$ under $\alpha\colon \mathrm{Map}(\Sigma) \to \mathrm{Aut}(\mathcal{H}_g)$ is the Torelli group.

We obtain a central extension of the Torelli group:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Tor}}(\Sigma) & \xrightarrow{\widetilde{\alpha}} & \mathcal{H}_g \\ \downarrow & & \downarrow \\ \mathrm{Tor}(\Sigma) & \xrightarrow{\alpha} & \mathrm{Inn}(\mathcal{H}_g) \end{array}$$

The twisted representation provides ${}_{\tau\circ\varphi_*}\mathcal{V} \to {}_{\tau}\mathcal{V}$ for $\varphi \in \mathrm{Tor}(\Sigma)$. Given a lift $\widetilde{\varphi}$ to $\widetilde{\mathrm{Tor}}(\Sigma)$, the element $\widetilde{\alpha}(\widetilde{\varphi})$ provides ${}_{\tau}\mathcal{V} \to {}_{\tau\circ\varphi_*}\mathcal{V}$.

Lemma

The central extension $\widetilde{\mathrm{Tor}}(\Sigma)$ of $\mathrm{Tor}(\Sigma)$ turns out to be trivial.

Rep. of MCGs – untwisting via Schrödinger $\mathcal{H}_g$-rep

Definition (Schrödinger representation)

$\mathcal{H}_g \cong \mathbb{Z}\langle \sigma, a_1, \cdots, a_g\rangle \rtimes \mathbb{Z}\langle b_1, \ldots, b_g\rangle$ $\quad (b_i \cdot a_i = a_i + 2\sigma)$
$\mathcal{H}_g^{\mathrm{Re}} \cong \mathbb{R}\langle \sigma, a_1, \cdots, a_g\rangle \rtimes \mathbb{R}\langle b_1, \ldots, b_g\rangle$
One-dimensional repr. $\mathbb{R}\langle \sigma, a_1, \cdots, a_g\rangle \twoheadrightarrow \mathbb{R}\langle \sigma\rangle \to S^1 = U(1)$
given by $t \mapsto \exp(\hbar t i/2)$ for fixed $\hbar > 0$
Induction $\rightsquigarrow$ unitary representation

$$\mathcal{H}_g \subset \mathcal{H}_g^{\mathrm{Re}} \longrightarrow U(W_{\mathrm{Sch}}) \qquad W_{\mathrm{Sch}} = L^2(\mathbb{R}^g)$$

Theorem ((corollary of) Stone-von Neumann)

For $\varphi \in \mathrm{Aut}(\mathcal{H}_g^{\mathrm{Re}})$ there is a unique inner automorphism $T(\varphi)$ of $U(W_{\mathrm{Sch}})$ such that the following square commutes:

$$\begin{array}{ccc} \mathcal{H}_g^{\mathrm{Re}} & \longrightarrow & U(W_{\mathrm{Sch}}) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle T(\varphi)} \\ \mathcal{H}_g^{\mathrm{Re}} & \longrightarrow & U(W_{\mathrm{Sch}}) \end{array}$$

Rep. of MCGs – untwisting via Schrödinger $\mathcal{H}_g$-rep

Theorem ((corollary of) Stone-von Neumann)

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Since $\mathrm{Inn}(U(W_{\mathrm{Sch}})) = PU(W_{\mathrm{Sch}})$, we obtain

$$\begin{array}{ccc} \widetilde{\mathrm{Map}}(\Sigma)^{\mathrm{univ}} & \xrightarrow{\qquad\qquad\qquad \widetilde{\alpha} \qquad\qquad\qquad} & U(W_{\mathrm{Sch}}) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\Sigma) \xrightarrow{\ \alpha\ } & \mathrm{Aut}^+(\mathcal{H}_g) \subset \mathrm{Aut}^+(\mathcal{H}_g^{\mathrm{Re}}) & \xrightarrow{\ T\ } PU(W_{\mathrm{Sch}}) \end{array}$$

and $\widetilde{\alpha}$ allows us to untwist the representation:

$$\widetilde{\mathrm{Map}}(\Sigma)^{\mathrm{univ}} \longrightarrow U\left(H_k^{BM}(C_k(\Sigma'); W_{\mathrm{Sch}})\right) = U(\mathcal{V}(W_{\mathrm{Sch}}))$$

Rep. of MCGs – untwisting via tautological $\mathcal{H}_g$-rep

$\mathcal{H}_g = \mathbb{Z}^{2g+1} = \mathbb{Z} \times H_1(\Sigma)$ with $(k,x)(l,y) = (k+l+x.y, x+y)$
Give $\mathbb{Z}^{2g+1}$ its usual affine structure (a torsor over itself)
Obs₁: Left multiplication $\mathcal{H}_g \circlearrowleft \mathcal{H}_g$ preserves the affine structure.

$$\mathrm{Taut}_{\mathrm{lin}} \colon \mathcal{H}_g \longrightarrow \mathrm{Aff}(\mathbb{Z}^{2g+1}) \subset GL(W_{\mathrm{lin}}) \qquad W_{\mathrm{lin}} = \mathbb{Z}^{2g+1} \oplus \mathbb{Z}$$

Obs₂: Every (orientation-preserving) automorphism of $\mathcal{H}_g$ preserves the structure of $\mathbb{Z}^{2g+1}$ as a free $\mathbb{Z}$-module.

$$\mathrm{Map}(\Sigma) \xrightarrow{\alpha} \mathrm{Aut}^+(\mathcal{H}_g) \xrightarrow{\mathrm{id}} GL(\mathbb{Z}^{2g+1}) \xhookrightarrow{-\oplus \mathrm{id}_{\mathbb{Z}}} GL(W_{\mathrm{lin}})$$

Lemma

This untwists the representation with coefficients in W_{lin}:

$$\mathrm{Map}(\Sigma) \longrightarrow GL\left(H_k^{BM}(C_k(\Sigma'); W_{\mathrm{lin}})\right) = GL(\mathcal{V}(W_{\mathrm{lin}}))$$

Upper bound on the kernel

Summary

For $k \geq 2$ and $W = \mathcal{H}_g$-representation over R, we obtain:

- a $\mathrm{Tor}(\Sigma)$-representation $\mathcal{V}(W)$ over R;
- a unitary $\widetilde{\mathrm{Map}}(\Sigma)^{\mathsf{univ}}$-representation $\mathcal{V}(W)$ if $W = W_{\mathsf{Sch}}$;
- a $\mathrm{Map}(\Sigma)$-representation $\mathcal{V}(W)$ over $\mathbb{Z}$ if $W = W_{\mathsf{lin}}$.

Lemma

As an R-module, $\mathcal{V}(W) \cong \bigoplus_{\binom{k+2g-1}{k}} W$.

For example, $\mathcal{V}(W_{\mathsf{lin}})$ is a free $\mathbb{Z}$-module of rank $(2g+2)\binom{k+2g-1}{k}$.

Q: What is the kernel of this representation?
Q′: Is it smaller than $\mathfrak{J}(k) = \ker(\mathrm{Moriyama}_k)$?

Upper bound on the kernel

$(k = 2)$
Quotient of (twisted) $\mathrm{Map}(\Sigma)$-representations:

$$H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g]) \twoheadrightarrow H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathfrak{S}_2])$$

induced by $\mathcal{H}_g \twoheadrightarrow (\mathcal{H}_g)^{ab} = \mathbb{Z}/2 \oplus H_1(\Sigma) \twoheadrightarrow \mathbb{Z}/2 = \mathfrak{S}_2$.

- The right-hand side is $H_2^{BM}(F_2(\Sigma'); \mathbb{Z}) = \mathrm{Moriyama}_2$, hence

$$\ker\left(H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g])\right) \quad \subseteq \quad \mathfrak{J}(2)$$

- Let $\gamma \subset \Sigma$ be a simple closed curve that separates off a genus-1 subsurface. Then the Dehn twist T_γ lies in $\mathfrak{J}(2)$.
- Calculations $\Rightarrow$ T_γ acts *non-trivially* in our representation.

Corollary

The kernel of $H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$ is **strictly smaller** than $\mathfrak{J}(2)$.

Example calculation

Set $k = 2$ and $g = 1$. In this case the representation

$$H_2^{BM}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_1])$$

is free of rank 3 over $\mathbb{Z}[\mathcal{H}_1] = \mathbb{Z}[\sigma^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle/(ab = \sigma^2 ba)$

Let γ be a curve isotopic to $\partial\Sigma = \partial\Sigma_{1,1}$. Then T_γ acts via:

$$\begin{bmatrix}
\begin{array}{c}\sigma^{-8}b^2+\sigma^{-4}a^{-2}-\sigma a^{-2}b^2+(\sigma^{-1}-\sigma^{-2})a^{-2}b+\\(\sigma^{-3}-\sigma^{-4})a^{-1}b^2+(\sigma^{-4}-\sigma^{-5})a^{-1}b\end{array} &
\begin{array}{c}(\sigma^2+1-2\sigma^{-1}+\sigma^{-2}+\sigma^{-4})a^{-2}b^2-\sigma a^{-2}b^4+\\(-\sigma^2+\sigma+\sigma^{-1}-\sigma^{-2})a^{-2}b^3-\sigma^{-3}a^{-2}+\\(-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})a^{-2}b\end{array} &
\begin{array}{c}(-1+2\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+\sigma^{-5})a^{-2}b+(\sigma-1)a^{-2}b^3+\\(\sigma^2-\sigma-\sigma^{-1}+2\sigma^{-2}-\sigma^{-3})a^{-2}b^2+(-\sigma^{-3}+\sigma^{-4})a^{-1}b+\\(\sigma^{-4}-\sigma^{-5})a^{-1}b^3+(-\sigma^{-2}+\sigma^{-3}+\sigma^{-5}-\sigma^{-6})a^{-1}b^2+(-\sigma^{-3}+\sigma^{-4})a^{-2}\end{array} \\
\begin{array}{c}-\sigma^{-1}-\sigma^{-3}+2\sigma^{-4}-\sigma^{-5}-\sigma^{-7}+\sigma^{-2}a^2+\\(\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+\sigma^{-5})a+\sigma^{-6}a^{-2}+\\(\sigma^{-3}-\sigma^{-4}-\sigma^{-6}+\sigma^{-7})a^{-1}\end{array} &
\begin{array}{c}1+\sigma^{-2}-\sigma^{-3}+\sigma^{-6}+\sigma^{-6}a^{-2}b^2-\sigma^{-1}b^2+\\(\sigma^{-3}-\sigma^{-4})a^{-1}b^2+(-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})b+\\(\sigma^{-2}-2\sigma^{-3}+\sigma^{-4}+\sigma^{-6}-\sigma^{-7})a^{-1}b-\sigma^{-5}a^{-2}+\\(-\sigma^{-2}+\sigma^{-3}+\sigma^{-5}-\sigma^{-6})a^{-1}+(\sigma^{-5}-\sigma^{-6})a^{-2}b\end{array} &
\begin{array}{c}(-\sigma^{-6}+\sigma^{-7})a^{-2}b+(\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+2\sigma^{-5}-\sigma^{-6})b+\\(-\sigma^{-3}+2\sigma^{-4}-\sigma^{-5}-\sigma^{-7}+\sigma^{-8})a^{-1}b+1-\sigma^{-1}+\sigma^{-2}-\\3\sigma^{-3}+2\sigma^{-4}+\sigma^{-6}-\sigma^{-7}+(-\sigma^{-2}+2\sigma^{-3}-\sigma^{-4}+\sigma^{-5}-2\sigma^{-6}+\sigma^{-7})a^{-1}\\+(\sigma^{-2}-\sigma^{-3})ab+(-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})a+(-\sigma^{-5}+\sigma^{-6})a^{-2}\end{array} \\
\begin{array}{c}-\sigma^{-6}ab+(-\sigma^{-3}+\sigma^{-4}-\sigma^{-7})b-\sigma^{-4}+\\(\sigma^{-1}-\sigma^{-4}+\sigma^{-5})a^{-1}b+\sigma^{-2}a^{-2}b+\\(-\sigma^{-3}+\sigma^{-6})a^{-1}+\sigma^{-5}a^{-2}\end{array} &
\begin{array}{c}(-1-\sigma^{-2}+2\sigma^{-3}-\sigma^{-6})a^{-1}b+\sigma^{-1}a^{-1}b^3+\\\sigma^{-2}a^{-2}b^3+(1-\sigma^{-1}-\sigma^{-3}+\sigma^{-4})a^{-1}b^2+\\(\sigma^{-1}-\sigma^{-2}+\sigma^{-5})a^{-2}b^2+(-\sigma^{-1}+\sigma^{-4}-\sigma^{-5})a^{-2}b+\\(\sigma^{-2}-\sigma^{-5})a^{-1}-\sigma^{-4}a^{-2}\end{array} &
\begin{array}{c}\sigma^{-3}+(\sigma^{-2}-\sigma^{-3}-\sigma^{-5}+\sigma^{-6})a^{-1}+\\(-\sigma^{-1}+\sigma^{-2}-\sigma^{-5}+\sigma^{-6})a^{-1}b^2+(-\sigma^{-2}+\sigma^{-3})a^{-2}b^2+\\(-1+\sigma^{-1}+2\sigma^{-3}-3\sigma^{-4}+\sigma^{-7})a^{-1}b+\\(-\sigma^{-1}+\sigma^{-2}-\sigma^{-5}+\sigma^{-6})a^{-2}b+(-\sigma^{-4}+\sigma^{-5})b^2+\\(\sigma^{-2}-\sigma^{-3}-\sigma^{-5}+\sigma^{-6})b+(-\sigma^{-4}+\sigma^{-5})a^{-2}\end{array}
\end{bmatrix}$$

Exercise: this reduces to the identity if we set $a = b = \sigma^2 = 1$.

Summary and outlook

B_n: Lawrence$_k \cong \bigoplus_{\text{fin}} \mathbb{Z}[\mathbb{Z}^2] \rightsquigarrow$ linearity

$\mathrm{Map}(\Sigma_{g,1})$: Moriyama$_k \rightsquigarrow$ kernel $= \mathfrak{J}(k)$

twisted representations $\mathcal{V}(W) \cong \bigoplus_{\text{fin}} W$

u n t w i s t i n g

$\mathrm{Tor}(\Sigma_{g,1})$ (any W)

$\widetilde{\mathrm{Map}}(\Sigma_{g,1})^{\text{univ}}$ ($W =$ Schrödinger)

$\mathrm{Map}(\Sigma_{g,1})$ ($W = \text{Taut}_{\text{lin}}$)

kernel $\subseteq \mathfrak{J}(k)$ (when $W = \mathbb{Z}[\mathcal{H}_g]$)

kernel $\subsetneq \mathfrak{J}(2)$ (for $k = 2$)

Q: linearity? $\rightsquigarrow$ study $\mathcal{V}(W)$ for well-chosen W

extensions to TQFTs?

Thank you for your attention!