Tropical totally positive cluster varieties

Lara Bossinger

Universidad Nacional Autónoma de México, Unidad Oaxaca

BIRS-CMO "Bases for cluster algebras"
September 26-30 2022

Motivation

In tropical algebraic geometry we have the *Fundamental Theorem* that tells us that three different notions of tropicalization agree.

There is an analogue in Total Positivity stating how two different notions of the *positive part of the tropicalization* agree.

For cluster algebras or cluster varieties we have another notion of (positive) tropicalization: *Fock–Goncharov tropicalization* which does not appear in the Fundamental Theorems.

Question: How is the Fock–Goncharov tropicalization related to the tropicalizations considered in the fundamental theorem?

Tropicalization of an ideal

Given $f \in \mathbb{C}[x_1, \dots, x_n]$ of the form $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathrm{x}^\alpha$ and $w \in \mathbb{R}^n$ we define its *initial form with respect to w*

$$\mathrm{in}_w(f) = \sum_{\substack{\beta : \langle \beta, w \rangle \geq \langle \alpha, w \rangle \\ \forall \alpha, c_\alpha \neq 0}} c_\beta \mathrm{x}^\beta$$

For an ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ we define its *initial ideal with respect to w* as $\mathrm{in}_w(I) = (\mathrm{in}_w(f) : f \in I)$. By the fundamental theorem the *tropicalization of I* is

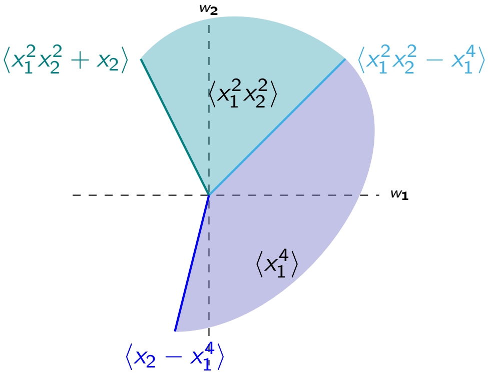
$$\mathrm{Trop}(I) = \overline{\{w \in \mathbb{R}^n : \mathrm{in}_w(I) \not\ni \text{ monomials}\}}$$

$\mathrm{Trop}(I) \subset \mathbb{R}^n$ is a closed subfan of the *Gröbner fan of I*, so $v, w \in \mathbb{R}^n$ lie in the relative interior of the same cone if

$$\mathrm{in}_v(I) = \mathrm{in}_w(I)$$

Example

Let $J = \langle x_1^2 x_2^2 - x_1^4 + x_2 \rangle \subset \mathbb{C}[x_1, x_2]$. The Gröbner fan is



The tropicalization of J consists of the three rays.

The totally positive part

An ideal $I \subset \mathbb{R}[x_1, \dots, x_n]$ is called *totally positive* if it does not contain any non-zero element of $\mathbb{R}_{>0}[x_1, \dots, x_n]$. We define the *totally positive part of the tropicalization of I* as

$$\text{Trop}^+(I) := \{w \in \text{Trop}(I) : \text{in}_w(I) \text{ totally positive}\}$$

$\text{Trop}^+(I) \subset \text{Trop}(I)$ is a closed subfan.

Let $A(\Gamma)$ be a finitely generated cluster algebra and $\mathcal{B} \subset A(\Gamma)$ a set of cluster variables that are algebra generators. Then the kernel of

$$\pi_{\mathcal{B}} : \mathbb{C}[x_1, \dots, x_{|\mathcal{B}|}] \to A(\Gamma)$$

is a prime ideal $I_{\mathcal{B}} \subset \mathbb{C}[x_1, \dots, x_{|\mathcal{B}|}]$ and $A(\Gamma) \cong \mathbb{C}[x_1, \dots, x_{|\mathcal{B}|}]/I_{\mathcal{B}}$ [Fomin–Williams–Zelevinsky].

Question: What does $\text{Trop}^+(I_{\mathcal{B}})$ know about the cluster structure?

Fundamental Theorem of Totally Positive Tropical Geometry

Consider $\mathcal{C} = \bigcup_{n\geq 1} \mathbb{C}((t^{1/n}))$ the field of Puiseux series with valuation

$$\mathrm{val} : \mathcal{C} \setminus \{0\} \to \mathbb{Q}, \quad \mathrm{val}(x(t)) = \min\{u : a_u t^u \text{ term in } x(t)\}.$$

Let $\mathcal{R}^+ = \{x(t) \in \mathcal{C} : \mathrm{val}(x(t)) = u \text{ then the coefficient } a_u \in \mathbb{R}_{>0}\}$.

Theorem 1 (Speyer–Williams)

Let $I \subset \mathcal{C}[x_1, \dots, x_n]$ be an ideal. Then the following sets in $\mathbb{R}^n$ coincide:

1. *The positive part of the tropicalization of I*

$$\mathrm{Trop}^+(I) = \{w \in \mathrm{Trop}(I) : \mathrm{in}_w(I) \textit{ totally positive}\}$$

2. *The closure of the pointwise valuation of $V(I) \cap (\mathcal{R}^+)^n$:*

$$\mathrm{val}(X) = \left\{(\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)) : (x_1, \dots, x_n) \in V(I) \cap (\mathcal{R}^+)^n\right\}.$$

Notation for cluster varieties

Given fixed data $\Gamma = (N \supset N^\circ, M \subset M^\circ, \{\ ,\ \} : N \times N \to \mathbb{Q})$ and its Langlands dual $\Gamma^\vee = (N^\circ \supset dN, M^\circ \subset d^{-1}M, d^{-1}\{\ ,\ \} : N^\circ \times N^\circ \to \mathbb{Q})$ we have cluster varieties

$$\mathcal{A}_\Gamma = \bigcup_{s_0 \sim s} T_{N_s^\circ} \quad \longleftrightarrow \quad \mathcal{X}_{\Gamma^\vee} = \bigcup_{s \sim s_0} T_{M_s^\circ}$$

$$\mathcal{X}_\Gamma = \bigcup_{s_0 \sim s} T_{M_s} \quad \longleftrightarrow \quad \mathcal{A}_{\Gamma^\vee} = \bigcup_{s \sim s_0} T_{N_s}$$

where s_0 is an initial seed, s a seed obtained from s_0 by mutations and $T_{N^\circ} \cong (\mathbb{C}^*)^n$ is a seed torus.

$A(\Gamma)$ is a cluster algebra of rank n with m frozen directions whose cluster variables $A_{i;s}$ are the local coordinates of $\mathcal{A}_\Gamma$:

$$T_{N_s^\circ} = N_s^\circ \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[M_s^\circ]) = \mathrm{Spec}(\mathbb{C}[A_{i;s}^{\pm 1}, \dots, A_{n+m;s}^{\pm 1}])$$

$X_{i;s}$ are the local coordinates of $T_{M_s} \subset \mathcal{X}_\Gamma$.

Fock–Goncharov tropicalization

The transition functions gluing $T_{N_s^\circ}$ to $T_{N_{s'}^\circ}$ are *substraction free* rational maps $\mu_{k;\mathcal{A}} : T_{N_s^\circ} T_{N_{s'}^\circ}$ given by

$$(t_1, \dots, t_n) \mapsto \left(t_1, \dots, \frac{1}{t_k} \left(\prod_{i \in I} t_i^{[b_{ik}]_+} + \prod_{i \in I} t_i^{[-b_{ik}]_+} \right), \dots, t_n \right)$$

$\Rightarrow$ well-defined over *semifields* such as $\mathbb{Z}^T := (\mathbb{Z}, +, \max)$ (or $\mathbb{Q}^T$,$\mathbb{R}^T$)

$$T_{N^\circ}(\mathbb{Z}^T) \equiv N^\circ \otimes_{\mathbb{Z}} \mathbb{Z}^T.$$

Then $\mu_{k;\mathcal{A}}^T : T_{N_s^\circ}(\mathbb{Z}^T) \to T_{N_{s'}^\circ}(\mathbb{Z}^T)$ is a bijection given by

$$(a_1, \dots, a_n) \mapsto \left(a_1, \dots, -a_k + \max \left(\sum_{i \in I} [b_{ik}]_+ a_i, \sum_{i \in I} [-b_{ik}]_+ a_i \right), \dots, a_n \right)$$

The *Fock–Goncharov tropicalization of $\mathcal{A}_\Gamma$* is

$$\mathcal{A}_\Gamma(\mathbb{Z}^T) = \bigcup_{s_0 \sim s} T_{N_s^\circ}(\mathbb{Z}^T) \overset{s}{\equiv} N_s^\circ$$

Duality and $\mathbf{g}$-vectors

The tropical space $\mathcal{A}_\Gamma(\mathbb{Z}^T)$ has a natural dual $\mathcal{X}_{\Gamma^\vee}(\mathbb{Z}^T) \overset{s}{\equiv} M_s^\circ$ and a duality pairing

$$\langle -, - \rangle : \mathcal{A}_\Gamma(\mathbb{Z}^T) \times \mathcal{X}_{\Gamma^\vee}(\mathbb{Z}^T) \to \mathbb{Z}$$

In particular, given $\mathbf{g} \in \mathcal{X}_{\Gamma^\vee}(\mathbb{Z}^T)$ we have $\langle -, \mathbf{g} \rangle : \mathcal{A}_\Gamma(\mathbb{Z}^T) \to \mathbb{Z}$.

Assumption 1

The cluster algebra is $\mathbb{Z}_{\geq 0}$-graded and has a ϑ-basis (respecting the grading)

$$A(\Gamma) = \bigoplus_{q \in \Theta \subseteq \mathcal{X}_{\Gamma^\vee}(\mathbb{Z}^T)} \mathbb{C}\vartheta_q.$$

If $A(\Gamma)$ is of finite type, ϑ_q is the cluster monomial with $\mathbf{g}$-vector q.

$\mathbf{g}$-vectors as a valuation

Proposition 1 (Fujita–Oya, B.-Cheung-Nájera Chávez-Magee)

Let $A(\Gamma)$ be a $\mathbb{Z}_{\geq 0}$-graded cluster algebra with ϑ-basis and s a seed. Then the map $\mathbf{g}_s : A(\Gamma) \setminus \{0\} \to M_s^\circ$ induced by

$$\begin{array}{ccc} A(\Gamma)\setminus\{0\} & \xrightarrow{\quad \mathbf{g}_s \quad} & M_s^\circ \\ & {\scriptstyle \vartheta_q \mapsto q} \searrow \qquad \nearrow {\scriptstyle \overset{s}{\equiv}} & \\ & \mathcal{X}_{\Gamma^\vee}(\mathbb{Z}^T) & \end{array}$$

is a full rank valuation with finitely generated value semigroup $\mathrm{im}(\mathbf{g}_s)$.

A generating set $\mathcal{B} \subset A(\Gamma)$ is called a *Khovanskii basis for $\mathbf{g}_s$* if $\{\mathbf{g}_s(b) : b \in \mathcal{B}\}$ generates the value semigroup.

Example: If $A(\Gamma)$ is of finite type, then for every seed s the set of all cluster variables is a Khovanskii basis for $\mathbf{g}_s$.

Full rank assumption

Assumption 2

The extended exchange matrix $\tilde{B}_s = \begin{bmatrix} B_s \\ B_{f;s} \end{bmatrix} \in \mathbb{Z}^{(n+m)\times n}$ *of* $A(\Gamma)$ *is of full rank and can complete it to an invertible square matrix*

$$\widetilde{B_s} = \begin{bmatrix} B_s & B'_s \\ B_{f;s} & * \end{bmatrix} \in GL_{n+m}(\mathbb{Z}).$$

where $B'_s = -diag(d_1, \dots, d_n)\ B^T_{f;s}\ diag(d_{n+1}, \dots, d_{n+m})^{-1}$, *and* $*$ *is an integral* $(m \times m)$*-matrix.*

We obtain induced lattice isomorphisms (*cluster ensemble maps*)

$$N \xrightarrow{\widetilde{B_s}} M^\circ, \quad N^\circ \xrightarrow{\widetilde{B_s}^T} M, \quad N^\circ \xrightarrow{-\widetilde{B_s}^T} d^{-1}M, \quad dN \xrightarrow{-\widetilde{B_s}} M^\circ$$

In particular, for $m \in M^\circ$ then $(-\widetilde{B_s}^{-1} m) \in dN$ defines a map $d^{-1}M \to \mathbb{Z}$.

A piecewise linear map on the cluster complex

Fix a generating set $\mathcal{B}$ of $A(\Gamma)$, a seed s and the *cluster complex*

$$\Delta_s = \bigcup_{s \sim s'} \sigma_{s';s} \subset \mathcal{X}_{\Gamma^\vee}(\mathbb{R}^T) \stackrel{s}{=} M^\circ_{\mathbb{R};s} \cong M_{\mathbb{R};s}$$

We define a piecewise linear map $\varphi_{\mathcal{B}} : \mathrm{supp}(\Delta_s) \to \mathbb{R}^{\mathcal{B}}$ on $\sigma_{s;s}$ by

$$u \mapsto \left(\left\langle u, -\widetilde{B_s}^{-1} \mathbf{g}_s(A) \right\rangle \right)_{A \in \mathcal{B}}$$

For $s' \sim s$ consider $s \stackrel{\mu_{i_1}}{\to} s_1 \stackrel{\mu_{i_2}}{\to} \dots \stackrel{\mu_{i_{\ell-1}}}{\to} s_{\ell-1} \stackrel{\mu_{i_\ell}}{\to} s'$ and define $\varphi_{\mathcal{B}}$ on $\sigma_{s';s}$

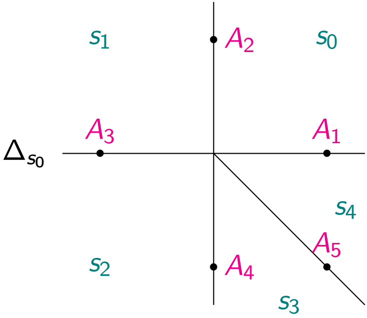
$$u \mapsto \left(\left\langle \mu^T_{i_\ell;\mathcal{X}} \circ \dots \circ \mu^T_{i_1;\mathcal{X}}(u), -\widetilde{B_{s'}}^{-1} \mathbf{g}_{s'}(A) \right\rangle \right)_{A \in \mathcal{B}}$$

We write $\varphi_{s';\mathcal{B}} : \mathrm{supp}(\sigma_{s';s}) \to \mathbb{R}^{\mathcal{B}}$ for the linear pieces of $\varphi_{\mathcal{B}}$.

[Kaveh–Manon] tropical geometry over the semifield of PL functions on a fan in a lattice

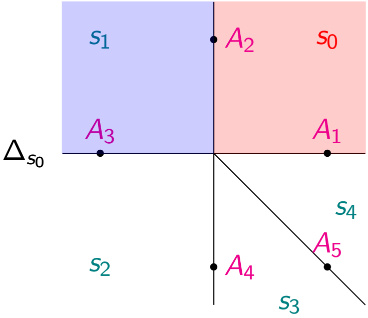
Example type A_2

Consider the exchange matrix $B_{s_0} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then the cluster complex Δ_{s_0} is the following simplicial fan generated by the **g**-vectors of all cluster variables $\mathcal{B} := \{A_1, \dots, A_5\}$:



Example type A_2

Consider the exchange matrix $B_{s_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then the cluster complex Δ_{s_0} is the following simplicial fan generated by the **g**-vectors of all cluster variables $\mathcal{B} := \{A_1, \dots, A_5\}$:



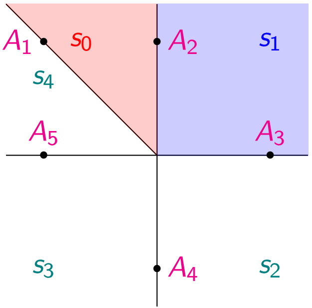
$$\varphi_{s_0;\mathcal{B}} = -B_{s_0}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\varphi_{s_1;\mathcal{B}}$

Example type A_2

Consider the exchange matrix $B_{s_0} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then the cluster complex Δ_{s_0} is the following simplicial fan generated by the **g**-vectors of all cluster variables $\mathcal{B} := \{A_1, \ldots, A_5\}$:

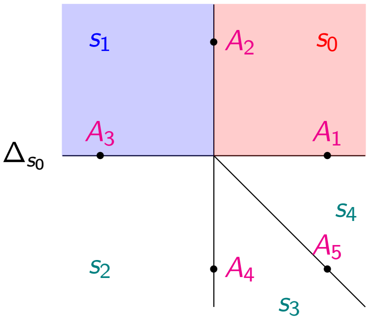
$$\mu^T_{1;\mathcal{X}}(\Delta_{s_0}) = \Delta_{s_1}$$



$$\varphi_{s_1;\mathcal{B}} = -B_{s_1}^{-1} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example type A_2

Consider the exchange matrix $B_{s_0} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then the cluster complex Δ_{s_0} is the following simplicial fan generated by the **g**-vectors of all cluster variables $\mathcal{B} := \{A_1, \dots, A_5\}$:

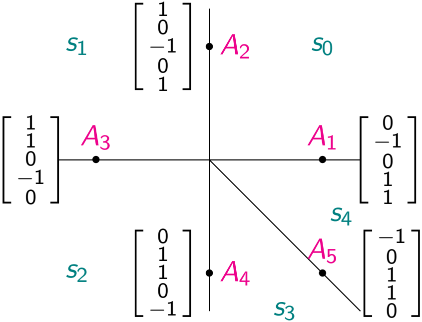


$$\varphi_{s_0;\mathcal{B}} = -B_{s_0}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\varphi_{s_1;\mathcal{B}} = -B_{s_1}^{-1} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example type A_2

Consider the exchange matrix $B_{s_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then the cluster complex Δ_{s_0} is the following simplicial fan generated by the **g**-vectors of all cluster variables $\mathcal{B} := \{A_1, \dots, A_5\}$:



The image $\varphi_{\mathcal{B}}(\mathrm{supp}(\Delta_{s_0}))$ is a simplicial fan in $\mathbb{R}^5$ with five maximal cones whose ray generators are as indicated above.

The positive part of the tropicalization and the cluster complex

Recall, given a seed s the valuations $\mathbf{g}_s : A(\Gamma) \to M_s^\circ$ and given a set of algebra generators $\mathcal{B}$ of $A(\Gamma)$ the presentation $A(\Gamma) \cong \mathbb{C}[x_1, \dots, x_{|\mathcal{B}|}]/I_{\mathcal{B}}$

Theorem 2 (B., arxiv:2208.01723)

Let s be a seed and fix a Khovanskii basis $\mathcal{B}$ simultaneously for $\mathbf{g}_s$ and $\mathbf{g}_{\mu_k(s)}$ containing all cluster variables $A_{i;s}$ and $A_{i;\mu_k(s)}$. Then for all $s' \in \{\mu_k(s) : k\} \cup \{s\}$ the images of $\sigma_{s';s} \in \Delta_s$ under $\varphi_{\mathcal{B}}$ satisfy

$$\varphi_{\mathcal{B}}(\sigma_{s';s}) \subset \mathrm{Trop}^+(I_{\mathcal{B}})$$

are adjacent maximal prime cone whose associated initial ideals $\mathrm{in}_{\varphi_{\mathcal{B}}(\sigma_{s';s})}(I_{\mathcal{B}})$ satisfy

$$\mathbb{C}[x_1, \dots, x_{|\mathcal{B}|}]/\,\mathrm{in}_{\varphi_{\mathcal{B}}(\sigma_{s';s})}(I_{\mathcal{B}}) \cong \mathbb{C}[im(\mathbf{g}_{s'})].$$

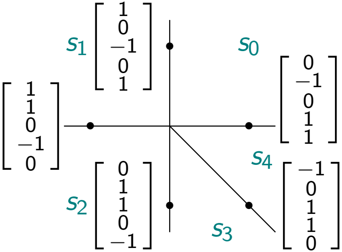
Example type A_2

We have $\mathcal{B} = \{A_1, \dots, A_5\}$ and $I_{\mathcal{B}}$ is generated by

$$A_1A_3 - A_2 - 1, \quad A_2A_4 - A_3 - 1, \quad A_3A_5 - A_4 - 1,$$
$$A_4A_1 - A_5 - 1, \quad A_5A_2 - A_1 - 1$$

We choose a point in the interior of $\varphi_{\mathcal{B}}(\sigma_{s_0;s_0})$: $w = [1, -1, -1, 1, 2]^T$ which gives the initial ideal $\mathrm{in}_w(I_{\mathcal{B}})$ generated by

$$\overset{1}{A_1}\overset{-1}{A_3} - \cancel{\overset{-1}{A_2}} - \overset{0}{1}, \quad \overset{-1}{A_2}\overset{1}{A_4} - \cancel{\overset{-1}{A_3}} - \overset{0}{1},$$
$$\overset{-1}{A_3}\overset{2}{A_5} - \cancel{\overset{1}{A_4}} - \overset{1}{A_4}, \quad \overset{1}{A_4}\overset{1}{A_1} - \overset{2}{A_5} - \cancel{\overset{0}{1}},$$
$$\overset{2}{A_5}\overset{-1}{A_2} - \overset{1}{A_1} - \cancel{\overset{0}{1}}$$



$\rightsquigarrow$ totally positive binomial prime ideal, so $\varphi_{\mathcal{B}}(\sigma_{s_0;s_0}) \subset \mathrm{Trop}^+(I_{\mathcal{B}})$ maximal cone.

Tropicalizing a positive parametrization

Recall, by the Laurent phenomenen $A(\Gamma) \subset \mathbb{C}[M_s^\circ]$. Given the lattice isomorphism $\widetilde{B_s}: N_s \to M_s^\circ$ due to positivity of the Laurent Phenomenon we get a *positive parametrization*

$$\begin{array}{rcl} \Psi_s : \mathbb{C}[A_{1;s}^{\pm 1}, \dots, A_{n+m;s}^{\pm 1}] \cong \mathbb{C}[M_s^\circ] & \longrightarrow & \mathbb{C}[N_s] \cong \mathbb{C}[X_{1;s}^{\pm 1}, \dots, X_{n+m;s}^{\pm 1}] \\ \text{cluster variable } \ A & \mapsto & \Psi_s(A) \in \mathbb{Z}_{\geq 0}[X_{1;s}^{\pm 1}, \dots, X_{n+m;s}^{\pm 1}] \end{array}$$

A set of cluster variables $\mathcal{B}$ gives a map of tori

$$T_{M_s} \to (\mathbb{C}^*)^{\mathcal{B}}, \quad t \mapsto (\Psi_s(A)(t))_{A \in \mathcal{B}}$$

Tropicalizing yields the piecewise linear function

$$\psi_s : \mathcal{X}_\Gamma(\mathbb{R}^T) \stackrel{s}{\equiv} M_{\mathbb{R};s} \to \mathbb{R}^{\mathcal{B}}, \quad u \mapsto \Big(\Psi_s(A)^T(u)\Big)_{A \in \mathcal{B}}$$

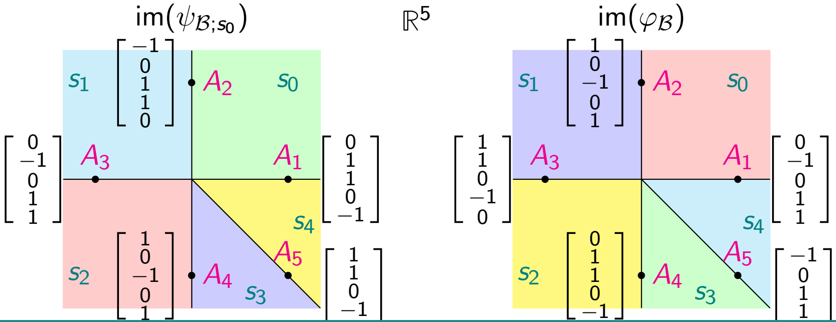
where $\Psi_s(A)^T(u) = \max_{n \in \mathrm{supp}(\Psi_s(A))} \{\langle u, n \rangle\}$.

Example type A_2

The piecewise linear map $\psi_{\mathcal{B};s_0}$ is given by

$$\begin{pmatrix} (p^*)^{-1}(A_1)\\ (p^*)^{-1}(A_2)\\ (p^*)^{-1}(\frac{A_2+1}{A_1})\\ (p^*)^{-1}(\frac{A_1+1+A_2}{A_1A_2})\\ (p^*)^{-1}(\frac{A_1+1}{A_2})\end{pmatrix} = \begin{pmatrix} X_2^{-1}\\ X_1\\ X_1X_2+X_2\\ X_1^{-1}+X_1^{-1}X_2+X_2\\ X_1^{-1}+X_1^{-1}X_2^{-1}\end{pmatrix}, \quad \begin{pmatrix} a\\ b\end{pmatrix}\mapsto \begin{pmatrix}-b\\ a\\ \max(b,a+b)\\ \max(-a,-a+b,b)\\ \max(-a,-a-b)\end{pmatrix}$$

and its linear regions are the cones of Δ_{s_0}.



The Fundamental Theorem for Tropical Cluster Varieties in finite type

Theorem 3 (B., in preparation)

Let $A(\Gamma)$ be a $\mathbb{Z}_{\geq 0}$-graded cluster algebra of finite type with full rank exchange matrix and let $\mathcal{B}$ the set of all cluster variables. Then the following three simplicial fans are realizations of the cluster complex Δ_s and agree for every seed s

1. *The image of the Fock–Goncharov tropicalization $\mathcal{A}_\Gamma(\mathbb{R}^T)$ (the cluster complex) under the piecewise linear map $\varphi_\mathcal{B}$ in $\mathbb{R}^\mathcal{B}$.*
2. *The image of the tropicalized positive parametrization $\psi_{\mathcal{B};s}(\Delta_s) \subset \mathbb{R}^\mathcal{B}$.*
3. *The positive part of the tropicalization of $I_\mathcal{B}$ $\mathrm{Trop}^+(I_\mathcal{B}) \subset \mathbb{R}^\mathcal{B}$.*

Comments

Corolario (Ilten–Nájera Chávez–Treffinger, arXiv:2111.02566)

The rays of $\mathrm{Trop}^+(I_{\mathcal{B}})$ *span a maximal cone* C *in the Gröbner fan of* $I_{\mathcal{B}}$*. The Stanley–Reisner complex of the monomial initial ideal of* C *is the cluster complex.*

- Speyer–Williams conjectured that **(2)** has the fan structure of the cluster complex. It has essentially been solved combinatorially by Jahn–Löwe–Stump and Arkani-Hamed–He–Lam.
- Bendle–Boehm–Ren–Schröter use that **(2)=(3)**.
- Beyond finite type choosing an apropriate $\mathcal{B}$ the result generalizes for *finite collections of seeds* defining finite subfans of **(1),(2),(3)**.

References

BBRS Bendle, D. Boehm, J. Ren, Y. and Schröter, B. *Parallel Computation of tropical varieties, their positive part, and tropical Grassmannians.* arXiv:2003.13752

BEW Boretsky, J. Eur, C. and Williams, L. *Polyhedral and Tropical Geometry of Flag Positroids.* arxiv:2208.09131

B Bossinger, L. *Tropical totally positive cluster varieties.* arxiv:2208.01723

BCMN Bossinger, L. Cheung, M. Magee, T. and Nájera Chávez, A. *Newton–Okounkov bodies and minimal models for cluster varieties.* In preparation

FO Fujita, N. and Hironori Oya, H. *Newton-Okounkov polytopes of Schubert varieties arising from cluster structures.* arXiv:2002.09912

FWZ Fomin, S. Williams, L. and Zelevinsky, A. *Introduction to Cluster Algebras. Chapter 6.* arxiv:2008.09189

INT Ilten, N. Nájera Chávez, A. and Treffinger, H. *Deformation Theory for Finite Cluster Complexes.* arxiv:2111.02566

KM Kaveh, K. and Manon, C. *Toric flat families, valuations, and tropical geometry over the semifield of piecewise linear functions.* arxiv:1907.00543

SW Speyer, D. and Williams, L. *The tropical totally positive Grassmannian.* J. Algebraic Combin. 22 (2005)