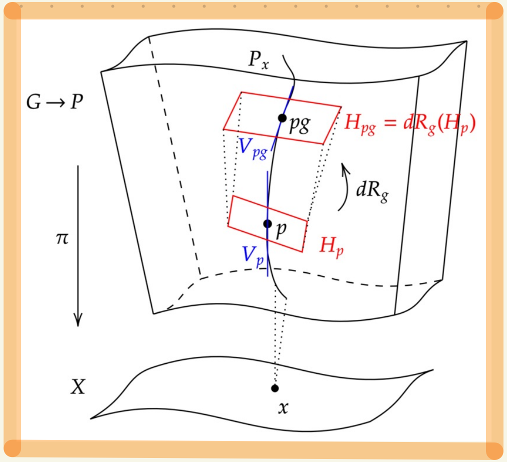
Twisted local systems and Higgs bundles for non-constant groups



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1. Twisted local systems

2. Twisted character varieties

3. Higgs bundles for non-constant groups

Moduli, Motives and Bundles
New Trends in Algebraic Geometry

1. Twisted local systems

a). Local systems (with constant coefficients)

<u>Example</u> $U \subset \mathbb{C}$ open set

$a_1, \dots, a_r : U \to \mathbb{C}$ holomorphic functions

$y : U \to \mathbb{C} \quad | \quad y^{(r)} = a_1(z)\, y^{(r-1)} + \dots + a_r(z)\, y$

linear ODE $\quad \big(Y'(z) = \underbrace{A(z)}_{r \times r \text{ matrix}} Y(z) \big)$

$\forall z \in U, \ \forall v = (v_1, \dots, v_r) \in \mathbb{C}^r$
$\exists !$ germ of solution y
such that $\forall k,\ y^{(k-1)}(z) = v_k$

$\Bigg\}$ defines a local system of rank r $\mathbb{C}$-vector spaces

Exercise

$A : U \to \mathrm{Mat}(r \times r; \mathbb{C})$ holomorphic

(E): $Y'(z) = A(z)\, Y(z)$, $Y : U \to \mathbb{C}^r$

$Sol_E :=$ sheaf of local solutions

$\hookrightarrow$ continuous sections of the étalé space

$$Et(Sol_E) := \bigsqcup_{z \in U} \underbrace{Sol_E(z)}_{\text{germs at } z \text{ (stalk)}}$$

Then the following map is bijective

$$Et(Sol_E) \longrightarrow U \times \mathbb{C}^r$$
$$(z, \varrho_z(Y)) \longmapsto (z, Y(z))$$

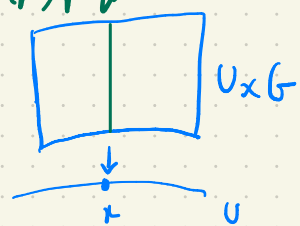
and Sol_E is isomorphic to the sheaf of locally constant sections of $U \times \mathbb{C}^r$.

Definition X a topological space
G a topological group

A G-local system on X is a locally constant sheaf of $\underbrace{G\text{-torsors}}$ on X.

space P with a G-action
$P \times G \to P$
$(p,g) \mapsto p \cdot g$
s.t. $P \times G \xrightarrow{\sim} P \times P$
$(p,g) \mapsto (p, p\cdot g)$

principal homogeneous G-space
(free and transitive action)

Local picture: 

$\rightsquigarrow$ take locally constant sections

$\rightsquigarrow$ A G-local system on X is a sheaf which is locally isomorphic to the sheaf of locally constant sections of $X \times G$

Examples

1. Linear ODE

$\rightsquigarrow$ take <u>bases</u> of solutions to replace

$U \times \mathbb{C}^r$ by $U \times GL(r; \mathbb{C})$

$(z, Y(z))$ $\qquad$ $(z, M(z))$ $\qquad$ $M(z) = \begin{array}{cc} Y_1(z) & \cdots & Y_r(z) \\ \end{array}$ $\begin{pmatrix} | & & | \\ | & \cdots & | \\ | & & | \end{pmatrix}$

2. G-local system associated to a representation

Assume X connected

$$\rho : \pi_1(X, x_0) \longrightarrow G$$

$\rightsquigarrow$ the transition functions of the G-bundle

Flat principal G-bundle $\left\{ \begin{array}{l} (\tilde{X} \times G)/_{\pi_1 X} \\ \text{are locally constant on } X \end{array} \right.$ [where $\gamma \in \pi_1 X$ acts via $\gamma \cdot (\xi, h) = (\gamma \cdot \xi, \rho(\gamma) h)$]

$\rightsquigarrow$ as a consequence the sheaf of locally constant sections of that bundle is well-defined

b). Monodromy and classification

Observation

(i) A morphism between two G-local systems is just a morphism of sheaves between them.

(ii) If $\mathcal{U}$ is a G-local system on X, then $\forall x \in X$, the stalk $\mathcal{U}(x)$ is a G-torsor.

(iii) If $\rho_1, \rho_2 : \pi_1(X, x_0) \to G$ are conjugate, then the associated G-local systems are isomorphic.

$\exists g \in G,\ \forall \gamma,\ \rho_2(\gamma) = g\,\rho_1(\gamma)\,g^{-1}$

Recall the equivalence of categories

$$\{\text{sheaves on } X\} \longleftrightarrow \{\text{étalé spaces over } X\}$$

$$\mathcal{F} \longmapsto Et(\mathcal{F}) := \coprod_{x \in X} \mathcal{F}(x)$$

$\hookrightarrow$ stalk at x

sheaf of continuous sections of E $\longleftarrow\!\shortmid$ $(E \to X)$ local homeomorphism

$\to$ discrete fibres

Through this correspondence, there is an equivalence between the full sub-categories

$$\{\text{locally constant sheaves on } X\} \longleftrightarrow \{\text{topological covering spaces over } X\}$$

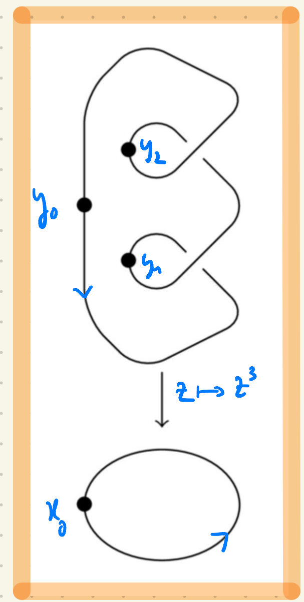
Point: If $E \to X$ is a topological covering space, paths in X and homotopies between them can be lifted to E, and the lifting depends only on the initial point in the fiber.

Assume X locally connected and semi-locally simply connected

$\rightsquigarrow$ there is a notion of parallel transport in E,

hence a group morphism

$$\mu : \pi_1(X, x_0) \to \mathrm{Aut}(E_{x_0})$$



$\leadsto$ A G-local system on X has a well-defined monodromy representation

$$\mu : \pi_1(X, x_0) \longrightarrow \mathrm{Aut}_{G\text{-tors}}(\mathcal{U}(x_0)) \simeq G$$

choose $M_0 \in \mathcal{U}(x_0)$

<u>Example</u> Let $\mathcal{U} := (\tilde{X} \times G) / \pi_1(X, x_0)$ be the local system associated to a representation

$$\rho : \pi_1(X, x_0) \to G$$

Then $\mu \sim_G \rho$ (conjugation by $g \in G$).

Theorem Let $\mathcal{V}$ be a G-local system with monodromy $\mu: \pi_1(X, x_0) \longrightarrow G$.

Then the sheaf of locally constant sections of the flat principal G-bundle $(\tilde{X} \times G)/\pi_1(X, x_0)$ associated to μ is isomorphic to $\mathcal{V}$.

$\mathrm{Hom}(\pi_1 X, G)/G$ is the set of isom. classes of G-local systems

In other words:

$$\left\{ \begin{array}{c} G\text{-local systems on } X \\ \text{locally constant sheaves of } G\text{-torsors} \end{array} \right\} \xrightarrow[\text{monodromy}]{\longleftrightarrow} \left\{ \begin{array}{c} \text{Flat principal } G\text{-bundles on } X \\ \text{principal } G\text{-bundles with locally constant transition functions} \end{array} \right\}$$

c). Local systems for non-constant groups

$\underline{G}_X :=$ sheaf of locally constant G-valued functions on X | sections of $X \times G$

The previous classification theorem says that

$$\underbrace{H^1(X; \underline{G}_X)}_{\substack{\text{flat principal} \\ G\text{-bundles on } X}} \simeq \underbrace{\mathrm{Hom}(\pi_1 X, G) / G}_{\substack{G\text{-representation space} \\ \text{of } \pi_1 X}}$$

$$= H^1(\pi_1 X; G)$$

The group cohomology set $H^1(\Pi; G)$ can be defined in a more general context:

$$\begin{aligned} \varphi : \Pi &\longrightarrow \mathrm{Aut}(G) \quad \text{a group morphism} \\ \gamma &\longmapsto \varphi_\gamma : G \to G \end{aligned}$$

$$Z^1_\varphi(\Pi; G) := \left\{ \underbrace{\rho : \Pi \to G \;\middle|\; \begin{array}{l} \rho(1_\Pi) = 1_G \\ \text{and } \forall (\gamma_1, \gamma_2) \\ \quad \rho(\gamma_1 \gamma_2) = \rho(\gamma_1)\, \varphi_{\gamma_1}(\rho(\gamma_2)) \end{array}}_{\text{a (normalized) crossed morphism [or twisted representation]}} \right\}$$

$e, e' \in Z^1_\varphi(\pi; G)$ are called equivalent if $\exists g \in G$, $\forall \gamma \in \pi$, $e'(\gamma) = g\, e(\gamma)\, \varphi_\gamma(g^{-1})$.

$$H^1_\varphi(\pi; G) := Z^1_\varphi(\pi; G) \big/ G$$

$\rightsquigarrow$ what is the geometric content of this when $\pi = \pi_1 X$?

local systems with "non-constant coefficients"

Basic constructions

$\varphi : \pi_1 X \to \mathrm{Aut}(G)$, $e \in Z^1_\varphi(\pi_1 X ; G)$

$\mathcal{G}_\varphi \times_X \mathcal{G}_\varphi \xrightarrow{m} \mathcal{G}_\varphi$

$e : X \to \mathcal{G}_\varphi$ (section)

+ usual group axioms

(i) $\mathcal{G}_\varphi := (\tilde{X} \times G) / \pi_1 X$

where $\gamma \cdot (\xi, g) := (\gamma \cdot \xi, \underbrace{\varphi_\gamma (g)}_{\pi_1 X\text{-action by group automorphisms}})$

(ii) $\mathcal{V}_e := (\tilde{X} \times G) / \pi_1 X$

where $\gamma \cdot (\xi, h) = (\gamma \cdot \xi, \underbrace{e(\gamma) \varphi_\gamma (h)}_{\pi_1 X\text{-action}})$

(because e is a twisted representation)

<u>Point:</u> The group bundle $G_{\tilde{X}} := \tilde{X} \times G$ acts fiberwise on $\tilde{X} \times G$ (to the right) via $(\xi, h) \cdot (\xi, g) := (\xi, hg)$

The $\pi_1 X$-action on $\tilde{X} \times G$ is compatible with the $G_{\tilde{X}}$-action in the sense that

$$
\begin{tikzcd}
(\xi, h) \arrow[r, mapsto, "\gamma"] \arrow[d, mapsto, "{(\xi, g)}"] & (\gamma \cdot \xi, \rho(\gamma)\varphi_\gamma(h)) \arrow[d, mapsto, "{(\xi, \varphi_\gamma(g)) = \gamma \cdot (\xi, g)}"] \\
(\xi, hg) \arrow[r, mapsto, "\gamma"] & (\gamma \cdot \xi, \rho(\gamma)\varphi_\gamma(hg))
\end{tikzcd}
$$

$\gamma \cdot (h \cdot g)$
$= (\gamma \cdot h) \cdot (\gamma(g))$

$\rightsquigarrow$ the group bundle $\mathcal{G}_\varphi$ acts on $\mathcal{V}_\rho$ (over X)

- $\mathcal{G}_\varphi$ is locally isomorphic to $X \times G$ (but not globally in general) } $\mathcal{G}_\varphi$ is a non-constant group

- The $\mathcal{G}_\varphi$-action is a map $\mathcal{U}_e \times_X \mathcal{G}_\varphi \to \mathcal{U}_e$.
$(h, g) \mapsto hg$

- There is an induced map

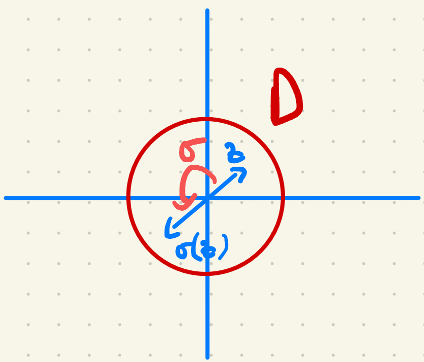
$$\mathcal{U}_e \times_X \mathcal{G}_\varphi \longrightarrow \mathcal{U}_e \times_X \mathcal{U}_e$$
$$(h, g) \longmapsto (h, hg)$$

and this map is a <u>bundle isomorphism over X</u>.

$\rightsquigarrow$ $\mathcal{U}_e$ is a $\mathcal{G}_\varphi$-torsor (principal homogeneous $\mathcal{G}_\varphi$-space).

Examples (i) if $\varphi : \pi_1 X \to \mathrm{Aut}(G)$ is trivial then $\mathcal{G}_\varphi = X \times G$ and a $\mathcal{G}_\varphi$-torsor is a principal G-bundle.

(ii) Linear O.D.E. with symmetry:



$D = D(0,1)$, $z \overset{\sigma}{\longmapsto} -z$

$A : D \to \mathrm{Mat}(r \times r; \mathbb{C})$ such that $\boxed{A(-z) = {}^t A(z)}$

The constant group bundle $D \times GL(r;\mathbb{C})$ has a canonical σ-equivariant structure:

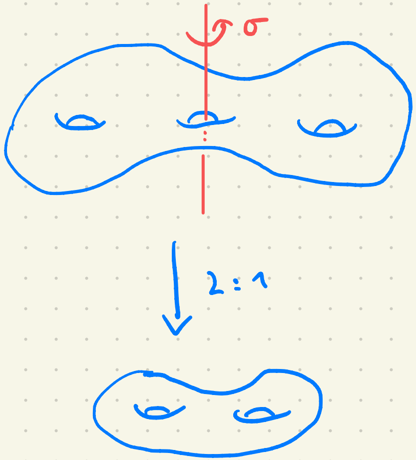
$$(z, g) \longmapsto (z, {}^t g^{-1})$$

$\rightsquigarrow$ non-constant group bundle on $X := [D/\langle\sigma\rangle]$

$\rightsquigarrow$ σ-invariant bases of solutions to $U'(z) = A(z)U(z)$ on D define a twisted local system on X

$M : D \to GL(r;\mathbb{C})$ such that $\boxed{{}^t M(-z)^{-1} = M(z)}$

(iii) Anti-invariant local systems



$X = Y/\langle\sigma\rangle$

(Y, σ) : a Riemann surface with involution

$\mathcal{V}$: a local system of $\mathbb{C}$-vector spaces on Y equipped with

$$\psi_\sigma : \mathcal{V} \xrightarrow{\sim} (\sigma^* \mathcal{V})^\vee$$

such that $(\sigma^* \psi_\sigma)^\vee \circ \psi_\sigma = \mathrm{id}_{\mathcal{V}}$

$\rightsquigarrow$ σ-invariant bases of solutions to the corresponding O.D.E. on Y define a twisted local system on X.

For the group $G_{\langle\sigma\rangle} = (\Gamma \times GL_n(\mathbb{C}))/\langle\sigma\rangle$ where $\sigma(g) = {}^t g^{-1}$.

Classification theorems

For us, $\Gamma = G^{\#}$ (discrete form of a Lie group)

① A group covering $\mathcal{G} \to X$ is necessarily of the form $\mathcal{G}_\varphi = (\tilde{X} \times \Gamma)/_{\pi_1 X}$ where Γ is a discrete group and $\varphi : \pi_1 X \to \mathrm{Aut}(\Gamma)$ is a group morphism

φ-twisted G-local systems

② The non-abelian cohomology set

$$H^1_\varphi(\pi_1 X; \Gamma) = \{\varphi\text{-twisted representations } \rho : \pi_1 X \to \Gamma\}/\Gamma$$

parameterizes $\underbrace{\mathcal{G}_\varphi\text{-torsors on } X}_{\text{covering spaces } E \to X \text{ equipped with } E \times \mathcal{G}_\varphi \xrightarrow{\sim} E \times E}$.

Classification of group coverings

$\mathcal{G} \to X$ group covering $\qquad q : \tilde{X} \to X$ projection

$q^*\mathcal{G} \to \tilde{X}$ is a $\pi_1 X$-equiv. gp covering of $\tilde{X}$

but $\tilde{X}$ is simply connected,

so $q^*\mathcal{G} \cong \tilde{X} \times \Gamma$

discrete gp

$\dashrightarrow$ $\pi_1 X$-action ?

$$\begin{tikzcd}
q^*\mathcal{G} \arrow[r, "\tau_\gamma"] \arrow[d] & q^*\mathcal{G} \arrow[d] \\
\tilde{X} \arrow[r, "\gamma"] & \tilde{X}
\end{tikzcd}$$

$\tau_{\gamma_1\gamma_2} = \tau_{\gamma_1}\tau_{\gamma_2}$

$\tau_\gamma(p_1 p_2) = \tau_\gamma(p_1)\,\tau_\gamma(p_2)$

$\dashrightarrow$ on $\tilde{X} \times \Gamma$, this means $\tilde{X} \times \Gamma \xrightarrow{\tau_\gamma} \tilde{X} \times \Gamma$

$$(\xi, g) \longmapsto (\gamma \cdot \xi, \Phi_\gamma(\xi, g))$$

such that $\Phi_\gamma(\xi, g_1 g_2) \overset{(*)}{=} \Phi_\gamma(\xi, g_1)\,\Phi_\gamma(\xi, g_2)$

Moreover, $\left.\begin{array}{l} \Phi_\gamma : \tilde{X} \times \Gamma \to \Gamma \\ (\xi, g) \mapsto \Phi_\gamma(\xi, g) \end{array}\right]$ Γ discrete, $\tilde{X}$ connected

is continuous

$$\Phi_\gamma(\xi, g) = \varphi_\gamma(g)$$

(indep. of ξ)

and then $(*) \Rightarrow \varphi_{\gamma_1\gamma_2} = \varphi_{\gamma_1} \circ \varphi_{\gamma_2}$

$$\varphi : \pi_1 X \to \mathrm{Aut}(\Gamma).$$

<u>Conclusion:</u> $\mathcal{G} \simeq (q^*\mathcal{G}) / \pi_1 X \simeq (\tilde{X} \times G) / \underset{\text{wrt } \varphi}{\pi_1 X} =: \mathcal{G}_\varphi$

Classification of twisted local systems

$\mathcal{G}_\varphi$ - torsors

$$\left.\begin{array}{ccc} Z^1_\varphi(\pi_1 X; G) & \longrightarrow & \mathcal{G}_\varphi\text{-torsors} \\ e & \longmapsto & V(e) \end{array}\right\} \text{ already seen}$$

Converse? Take $V \to X$ a $\mathcal{G}_\varphi$-torsor

Then $q^* V$ is a $\underbrace{(q^* \mathcal{G}_\varphi)}_{\Gamma_{\tilde{X}}}$-torsor

$\pi_1 X$-equiv

ppal Γ-bdle on $\tilde{X}$ w/ Γ-discrete

$q^* V \simeq \tilde{X} \times G$

$\dashrightarrow$ to see: the $\pi_1 X$-equiv. str. on this has to be given by $e \in Z^1_\varphi(\pi_1 X; G)$

the $\pi_1 X$ - equiv. structure is entirely determined

by $\gamma \cdot (\xi, 1_r) =: \big(\gamma \cdot \xi, \underset{\substack{\cap \\ G}}{e(\gamma)}\big)$ again independent of ξ

to check: e defined in this way

is a crossed morphism.