2. Twisted character varieties

G : reductive affine algebraic group$/\mathbb{C}$

We assume that $\pi_1 X$ is finitely presented.

$$\varphi : \pi_1 X \longrightarrow \mathrm{Aut}(G) \quad \text{group morphism}$$
$$\gamma \longmapsto (\varphi_\gamma : G \to G)$$

$\rightsquigarrow$ gives rise to

$$H^1_\varphi(\pi_1 X; G) = \overbrace{\left\{ \rho : \pi_1 X \to G \;\middle|\; \begin{array}{l} \rho(1_{\pi_1 X}) = 1_G \\ \rho(\gamma_1\gamma_2) = \rho(\gamma_1)\varphi_{\gamma_1}(\rho(\gamma_2)) \end{array} \right\}}^{Z^1_\varphi(\pi_1 X; G)} \Big/ G$$

$\underbrace{\hphantom{H^1_\varphi(\pi_1 X; G)}}_{\text{parameterizes}}$ | φ-twisted local systems on X

$\mathcal{G}^\#_\varphi$ - torsors where $\mathcal{G}^\#_\varphi := (\tilde{X} \times G^\#) / \pi_1 X$, $\gamma \cdot (\xi, g) = (\gamma \cdot \xi, \varphi_\gamma(g))$

a). The Betti moduli space

<u>Observation</u> It is possible to replace twisted representations by usual representations.

<u>Lemma</u> There is a G-equivariant injective map

$$\begin{aligned} Z^1_\varphi(\pi; G) &\longrightarrow Hom(\pi; G \rtimes Aut(G)) \\ \rho &\longmapsto \underbrace{\big(\hat{\rho} : \gamma \longmapsto (\rho(\gamma), \varphi_\gamma)\big)}_{\text{extended representation (def)}} \end{aligned}$$

<u>Claim</u>: $\hat{\rho}$ is a group morphism

Its image is

$$Hom_\varphi(\pi; G \rtimes Aut(G)) := \left\{ \hat{\rho} \in Hom(\pi, G \rtimes Aut(G)) \;\middle|\; \begin{tikzcd} \pi \arrow[r] \arrow[dr, "\varphi"'] & G \rtimes Aut(G) \arrow[d, "pr_2"] \\ & Aut(G) \end{tikzcd} \right\}$$

Point:

The G-action on extended representations is the usual conjugacy action (by G in $G \rtimes \mathrm{Aut}(G)$).

$$\forall g \in G \hookrightarrow G \rtimes \mathrm{Aut}(G)$$
$$g \longmapsto (g, \mathrm{Id}_G)$$

$$\widehat{g \cdot e}(\gamma) = \big((g \cdot e)(\gamma), \varphi_\gamma \big)$$
$$= \big(g\, e(\gamma)\, \varphi_\gamma(g^{-1}), \varphi_\gamma \big)$$
$$= (g, 1) \underbrace{\big(e(\gamma), \varphi_\gamma \big)}_{= \hat{e}(\gamma)} (g, 1)^{-1} \text{ in } G \rtimes \mathrm{Aut}(G)$$

Consequence:

If $F := \mathrm{Im}\,\varphi \subset \mathrm{Aut}(G)$ is a finite group, then

$$H^1_\varphi(\pi_1 X; G) \simeq \mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F) \big/ G$$

with G acting algebraically on the affine variety $\mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F)$.

So, for G reductive, we have an affine GIT quotient

$$\left.\mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F) \,/\!\!/\, G\right\} \text{ The Betti moduli space}$$

Extended representations

$$\varphi : \pi_1 X \to \mathrm{Aut}(G) \qquad F := \mathrm{Im}\,\varphi$$

Denote by $X_\varphi \to X$ the covering space defined by $\pi_1 X_\varphi := \operatorname{Ker}\varphi \lhd \pi_1 X$.

An extended representation $\hat{\varrho} \in \mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F)$ gives rise to a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1 X_\varphi \arrow[r] \arrow[d] & \pi_1 X \arrow[r, "\varphi"] \arrow[d] & F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G \arrow[r] & G \rtimes F \arrow[r, "pr_2"] & F \arrow[r] & 1
\end{tikzcd}$$

Observation $\pi_1 X_\varphi = \operatorname{Ker}\varphi$ acts trivially on G.

The set $H^1_\varphi(\pi_1 X; G) \simeq \mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F) / G$

parameterizes:

① isomorphism classes of $\mathcal{G}^\#_\varphi$-torsors (on X)

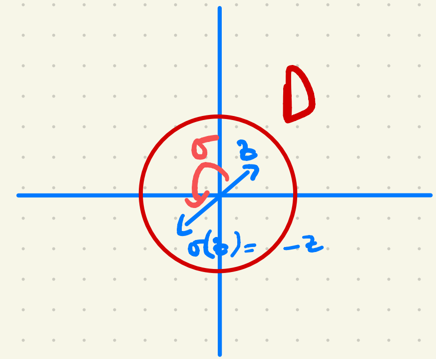
$$\mathcal{G}^\#_\varphi = (\tilde{X} \times G^\#) / \pi_1 X \qquad \gamma \cdot (\xi, g) = (\gamma \cdot \xi, \varphi_\gamma(g))$$

② isomorphism classes of F-equivariant principal $G^\#$-bundles (covering spaces) on X_φ.

$$p : X_\varphi \to \underset{\underset{X_\varphi / F}{\shortparallel}}{X} \qquad p^* \mathcal{G}^\#_\varphi = X_\varphi \times G^\#$$

+ induced F-action

Examples



(i) $A : D \to \mathrm{Mat}(r \times r; \mathbb{C})$

$M : D \to GL(r; \mathbb{C})$

$\hookrightarrow \forall z,\ M(z) = (U_1(z), \dots, U_r(z))$
is a basis of solutions
for the ODE $U'(z) = A(z)\,U(z)$

$X := [D/\langle\sigma\rangle]$ Here, $\pi_1 D = \{1\}$ and $\pi_1^{orb} X \simeq \langle\sigma\rangle$.

symmetry: $\left.\begin{array}{l} {}^tA(-z) = A(z) \\ M'(z) = A(z)M(z) \end{array}\right\} \Rightarrow \left({}^tM(-z)^{-1}\right)' = {}^t\left(\frac{d}{dz}\left(M(-z)^{-1}\right)\right)$

$= {}^tM(-z)^{-1}\ {}^tM'(-z)\ {}^tM(-z)^{-1}$

$= {}^tA(-z)\ {}^tM(-z)^{-1}$

$= A(z)\ {}^tM(-z)^{-1}$

$\forall z$, ${}^tM(-z)^{-1}$ is a basis of solutions

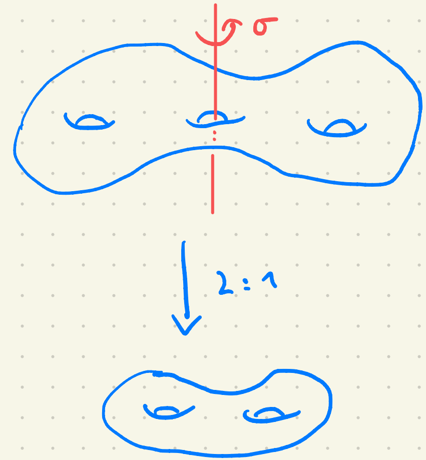
The trivial crossed morphism $\rho : \pi_1 X \to GL(r; \mathbb{C})$ induces a non-trivial extended representation $\hat{\rho} : \underbrace{\pi_1 X}_{\simeq\langle\sigma\rangle} \to GL(r;\mathbb{C}) \rtimes \langle\sigma\rangle$.

(ii) $\langle\sigma\rangle$ acts on $GL(r;\mathbb{C})$ via $g \longmapsto {}^t g^{-1}$.

We set:

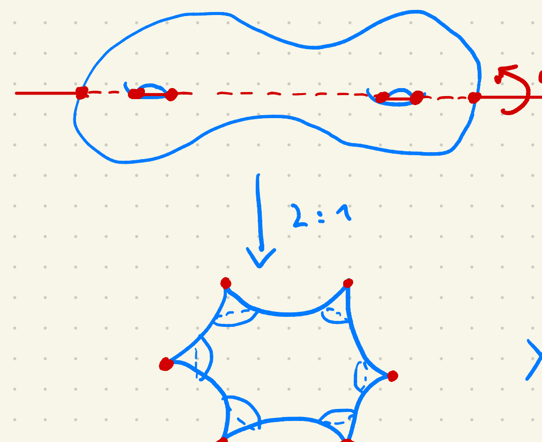
$$\mathcal{G}_{\langle\sigma\rangle} := \left(Y \times GL(r;\mathbb{C})^{\#}\right) \big/ \langle\sigma\rangle$$

[(locally trivial) group covering of X, only isotrivial when $\mathrm{Fix}_\sigma(Y) \neq \emptyset$]



(Y, σ)

$X = Y / \langle\sigma\rangle$



(Y, σ)

$X = [Y / \langle\sigma\rangle]$

{anti-invariant local systems on Y}

$\Updownarrow$

$$\left\{ \begin{tikzcd} 1 \arrow[r] & \pi_1 Y \arrow[r] \arrow[d] & \pi_1 X \arrow[r] \arrow[d] & \langle\sigma\rangle \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & GL(r;\mathbb{C}) \arrow[r] & GL(r;\mathbb{C}) \rtimes \langle\sigma\rangle \arrow[r] & \langle\sigma\rangle \arrow[r] & 1 \end{tikzcd} \right\}$$

b). Stability

Recall that we have a Betti moduli space

$$\mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F) /\!\!/ G \qquad F := \mathrm{Im}\,\varphi \subset \mathrm{Aut}(G)$$

defined as an affine GIT quotient.

$\rightsquigarrow$ points = <u>closed</u> G-orbits in $\mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F)$

$\rightsquigarrow$ what is the representation-theoretic characterization of stability?

ON CHARACTER VARIETIES WITH NON-CONNECTED STRUCTURE GROUPS

CHENG SHU

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G : affine algebraic group$/\mathbb{C}$, reductive (not necessarily connected)

$[G \hookrightarrow GL(V)$
$\dim_{\mathbb{C}} V < +\infty]$

Examples: $GL(r;\mathbb{C})$, $SL(r;\mathbb{C})$, $PGL(r;\mathbb{C})$, $O(r;\mathbb{C})$, $\mu_r(\mathbb{C})$, $\mathfrak{S}_r$

G_0 : neutral component of G

$F \subset \mathrm{Aut}(G)$: finite group of automorphisms of G

$\hat{G} := G \rtimes F$

Observation: $\hat{G}_0 = G_0$

$V := \hat{G} \times \dots \times \hat{G}$

affine variety, with $\hat{G}$ acting diagonally by conjugation

In our context, it is the diagonal G-action on $\hat{G}\times\dots\times\hat{G}$ that matters:

If $\gamma_1, \dots, \gamma_n$ generate $\pi := \pi_1 X$,

then we have a G-equivariant closed embedding

$$Z^1_\varepsilon(\pi; G) \simeq \mathrm{Hom}_\varphi(\pi; \overbrace{G \rtimes F}^{\hat{G}}) \hookrightarrow \hat{G}\times\dots\times\hat{G}$$

$$\begin{array}{ccccc} \rotatebox{90}{$\in$} & & & & \\ e & \longleftrightarrow & \hat{e} = (e, \varphi) & \longmapsto & \left(e(\gamma_1), \varphi_{\gamma_1}, \dots, e(\gamma_n), \varphi_{\gamma_n}\right) \end{array}$$

whose image is an affine sub-variety of $\hat{G}\times\dots\times\hat{G}$.

The Betti moduli space $\mathrm{Hom}_\varphi(\pi_1 X; G\rtimes F)/\!\!/G$ is in bijection with a set of closed G-orbits in $\hat{G}\times\dots\times\hat{G}$.

Complete reducibility

<u>Def</u> A closed subgroup $P \subset G$ is called parabolic if G/P is complete.

There is a splittable short exact sequence with reductive quotient

$$\begin{tikzcd} 1 \arrow[r] & R_u(P) \arrow[r] & P \arrow[r] & L_P \arrow[l, bend right, dashed] \arrow[r] & 1 \end{tikzcd}$$

A closed subgroup $H \subset G$ is called completely reducible if, for all parabolic subgroup $P \subset G$,

$H \subset P \implies \exists$ a Levi factor $L \subset P$ such that $H \subset L$

$\underset{\text{char. } 0}{\iff}$ H is a reductive subgroup of G

Theorem (Richardson)

Take $x = (x_1, \dots, x_n) \in \hat{G} \times \dots \times \hat{G}$ and let $H(x)$ be the Zariski-closure of

$$\langle x_1, \dots, x_n \rangle \subset \hat{G}.$$

Then the following statements are equivalent:

(i) $H(x)$ is completely reducible in $\hat{G}$.

(ii) $\hat{G}_0 \cdot x$ is closed in $\hat{G}^n$

(iii) $\underbrace{\hat{G} \cdot x}_{\text{finite union of copies of } \hat{G}_0 \cdot x}$ is closed in $\hat{G}^n$

Application to extended representations

Recall first that $\mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F)$ embeds onto a G-invariant closed subset $S \subset (G \rtimes F)^n$.

Also, G has finite index in $\hat{G} := G \rtimes F$. In particular, if $x \in S$, $G \cdot x$ closed in $S \iff \hat{G} \cdot x$ closed in $\hat{G}^n$.

<u>Theorem</u> Let $\hat{\rho} \in \mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F)$ be an extended representation.

Let $H(\hat{\rho}) := \overline{\hat{\rho}(\pi_1 X)}^{\mathrm{Zar}} \subset G \rtimes F$.

Then $G \cdot \hat{\rho}$ is closed in $\mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F)$

iff

$H(\hat{\rho})$ is a completely reducible subgroup of $G \rtimes F$.

$\iff \forall P$ parabolic, $\hat{\rho}(\pi_1 X) \subset P \Rightarrow \hat{\rho}(\pi_1 X) \subset L_P$.

Stability

GIT-stability: $G \cdot \hat{\varrho}$ is closed in $S := \mathrm{Hom}_{\varphi}(\pi_1 X; G \rtimes F)$

and $\mathrm{Stab}_G(\hat{\varrho}) \big/ \underbrace{\mathrm{Stab}_G(S)}_{= \bigcap_{\hat{\varrho} \in S} \mathrm{Stab}_G(\hat{\varrho})}$ is finite.

Richardson's results also give a representation-theoretic characterization of GIT stability:

$\hat{\varrho}$ is GIT-stable iff $\underbrace{\nexists \text{ proper parabolic } P \subset G,\ H(\hat{\varrho}) \subset P.}_{\text{such a subgroup } H \subset G \text{ is called irreducible.}}$

Observations:

(i) H irreducible $\Rightarrow$ H completely reducible.

and

$\hat{\rho}$ stable $\Rightarrow$ $\hat{\rho}$ polystable

(ii) When $\mathrm{Hom}_\varphi(\pi_1 X; G \rtimes F)$ is identified with a closed sub-variety of $\hat{G} \times \dots \times \hat{G}$,

$$\mathrm{Stab}_G(\hat{\rho}) \simeq \underbrace{\mathrm{Stab}_G(x_1, \dots, x_n)}_{\simeq\, Z_G(H(\hat{\rho}))} \subset Z_{\hat{G}}(x_1, \dots, x_n)$$

$$\simeq Z_G(H(\hat{\rho})) \subset Z_{\hat{G}}(H(\hat{\rho}))$$

since $H(\hat{\rho}) = \overline{\langle x_1, \dots, x_n \rangle}^{Zar}$ in $\hat{G}$.

c). Integrable connections

Starting with a complex algebraic group G and an action $\varphi : \pi_1 X \to \mathrm{Aut}(G)$, one can construct:

(i) a group covering $\mathcal{G}^\#_\varphi := (\check{X} \times G^\#) / \pi_1 X$

discrete form of G

$\hookrightarrow$ acts via $\gamma \cdot (\xi, g) = (\gamma\xi, \varphi_\gamma(g))$

$\mathcal{G}^\#_\varphi$-torsors $=:$ φ-twisted G-local systems on X

(ii) a group bundle $\mathcal{G}_\varphi := (\tilde{X} \times G) / \pi_1 X$

$\rightsquigarrow$ which $\mathcal{G}_\varphi$-torsors "come from" $\mathcal{G}^\#_\varphi$-torsors ?

Connection on a $\mathcal{G}_\varphi$-torsor

First, given a $\mathcal{G}_\varphi$-torsor $\mathcal{E}$, define its adjoint bundle

$$\mathrm{ad}(\mathcal{E}) := \left(\mathcal{E} \times_X \mathrm{Lie}(\mathcal{G}_\varphi)\right) \big/ \mathcal{G}_\varphi$$

where $\mathrm{Lie}(\mathcal{G}_\varphi) = \left(\tilde{X} \times \underline{\mathfrak{g}}\right) \big/ \pi_1 X$

with $\pi_1 X$ acting on $\underline{\mathfrak{g}}$ via

$$\begin{aligned} \pi_1 X &\xrightarrow{\varphi} \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(\underline{\mathfrak{g}}) \\ \gamma &\longmapsto \varphi_\gamma \longmapsto d_e \varphi_\gamma \end{aligned}$$

Second, denote by $\mathrm{At}(\mathcal{E})$ the bundle of $\mathfrak{g}_\gamma$-invariant vector fields on $\mathcal{E}$.

Then there is a short exact sequence of vector bundles

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(\mathcal{E}) \arrow[r] & \mathrm{At}(\mathcal{E}) \arrow[r] & TX \arrow[r] \arrow[l, bend right, dashed, "\nabla"'] & 0 \end{tikzcd}$$

Definition A connection on the $\mathfrak{g}_\gamma$-torsor $\mathcal{E}$ is a splitting of the short exact sequence above.

Theorem (// Atiyah) There is a connection on $\mathcal{E}$ if and only if the class of the extension above is trivial in $H^1(X; \underbrace{\mathrm{Hom}(TX; \mathrm{ad}(\mathcal{E}))}_{\Omega^1_X \otimes \mathrm{ad}(\mathcal{E})}) \simeq H^0(X; \mathrm{ad}(\mathcal{E}))^*$.

↳ $\dim_{\mathbb{C}} X = 1$

Let us relate the short exact sequence

$$0 \to \mathrm{ad}(\mathcal{E}) \to \mathrm{At}(\mathcal{E}) \to TX \to 0$$

Locally, we have:

$$0 \to U \times \underline{\mathfrak{g}} \to \underset{\mathrm{At}(\mathcal{E}|_U)}{\underset{\shortparallel}{\mathrm{At}(\mathcal{E})\big|_U}} \to TU \to 0$$

so, a $\underbrace{\text{local trivialization of } \mathcal{E}|_U}_{\text{section of } \mathcal{E}|_U}$ defines a connection on $\mathcal{E}|_U$.

If one can define a connection on $\mathcal{E}$ by gluing such connections on $\mathcal{E}|_U$, then the resulting connection on $\mathcal{E}$ is called <u>integrable</u>. [automatic if $\dim_{\mathbb{C}} X = 1$].

Torsors defined by twisted representations

$\varphi : \pi_1 X \to \mathrm{Aut}(G)$

$\rho : \pi_1 X \longrightarrow G$ a φ-twisted representation

$\mathcal{V}(\rho) := (\tilde{X} \times G) / \pi_1 X$ where $\pi_1 X$ acts via

$\gamma . (\xi, h) = (\gamma . \xi, \rho(\gamma) \varphi_\gamma(h))$

has a canonical integrable connection, which is $\pi_1 X$-invariant

$\longrightarrow$ it descends to $\mathcal{V}(\rho)$

Theorem (// Atiyah) Conversely, a $\mathcal{G}_\varphi$-torsor with integrable connection comes from a φ-twisted representation $\rho : \pi_1 X \to G$.

Riemann–Hilbert correspondence

given $\varphi : \pi_1 X \to \mathrm{Aut}(G)$,

there is a correspondence

$$\underbrace{\left\{ \mathcal{G}_\varphi^\# \text{-torsors} \right\}}_{\text{“}\varphi\text{-twisted } G\text{-local systems”}} \longleftrightarrow \underbrace{\left\{ \mathcal{G}_\varphi\text{-torsors with integrable connection} \right\}}_{\text{“Flat } \mathcal{G}_\varphi\text{-torsors”}}$$

[topological objects] [analytic objects]

$\rightsquigarrow$ both are parameterized by $H^1_\varphi(\pi_1 X; G)$