# An introduction to matroid theory

- The definition and some basic examples
- Elementary operations on matroids
- Some important theorems

http://www.math.LSU.edu/~oxley/survey4.pdf

# A fundamental example

$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$E = \{1, 2, 3, 4, 5, 6\}$ column labels

$\mathcal{I}$ : lin. independent sets of columns

$A_3$ : view $A$ over $GF(3)$

$A_2$ : view $A$ over $GF(2)$

$$\mathcal{I}_3 = \{ X \subseteq E : |X| \leq 3 \text{ and } X \neq \{1,2,4\}, \{1,3,5\}, \{2,3,6\} \}$$

$\mathcal{I}_2$ : same except $X \neq \{4,5,6\}$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
A = & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}
\end{array}
$$

$\mathcal{I}_3 = \{X \subseteq E : |X| \leq 3$ and $X \neq \{1,2,4\}, \{1,3,5\}, \{2,3,6\}\}$

$\mathcal{I}_2$ : same except $X \neq \{4,5,6\}$

$(E, \mathcal{I}_3)$ and $(E, \mathcal{I}_2)$ are matroids.

ground set    independent sets

(I1) $\mathcal{I}$ is non-empty.

(I2) Every subset of an indep't set is indep't.

(I3) If $X \subseteq E$, all max'l indep't subsets of $X$ have the same size.

(I1) $\mathcal{I}$ is non-empty.

(I2) Every subset of an indep't set is indep't.

(I3) If $X \subseteq E$, all max'l indep't subsets of $X$ have the same size.

A **matroid** $M = (E, \mathcal{I})$ is a finite set $E$ and a collection $\mathcal{I}$ of subsets of $E$ satisfying (I1) - (I3).

**Examples** Vector matroids of matrices

$M[A_3]$ ternary $(GF(3))$

$M[A_2]$ binary $(GF(2))$

(I1) $\mathcal{I}$ is non-empty.

(I2) Every subset of an indept set is indep't.

(I3) If $X \subseteq E$, all max'l indep't subsets of $X$ have the same size.

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maximal independent sets : **bases**

dependent set : not indep't

minimal dependent sets : **circuits**

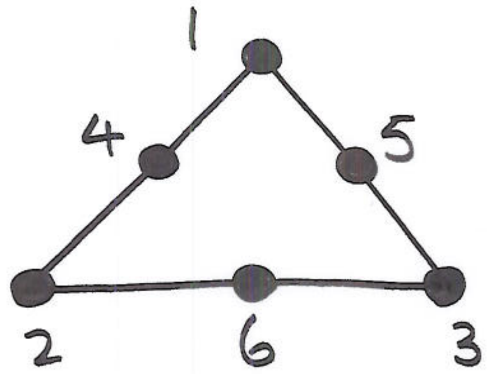
**rank** $r(X)$ of $X$ : size of a max'l indep't subset of $X$.

## Example

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$r(\{4,5,6\}) = \begin{cases} 3 & \text{in } M[A_3] \\ 2 & \text{in } M[A_2] \end{cases}$$

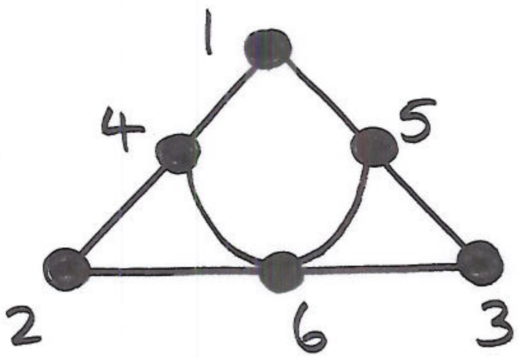
# A geometric example




E : a set of points in the plane

Set of lines : sets of $\geq 2$ points

- two distinct lines have at most one common point.



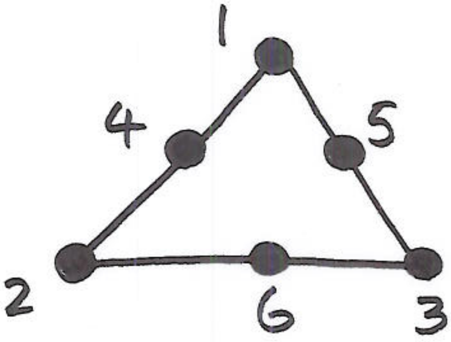

Dependent sets:

- 3 collinear points
- 4 coplanar points

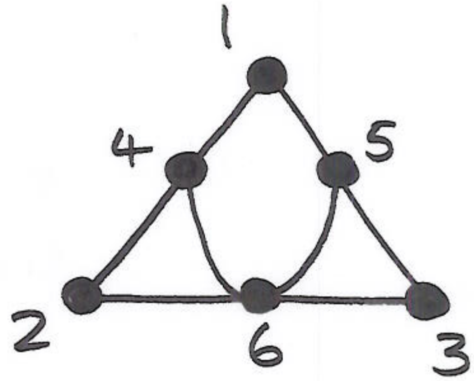
This gives a matroid.

$$\mathcal{I}_{top} = \{X \subseteq \{1,2,\ldots,6\} : |X| \leq 3 \text{ and } X \neq \{1,2,4\}, \{1,3,5\}, \{2,3,6\}\}$$

$\mathcal{I}_{bottom}$ = same except $X \neq \{4,5,6\}$




$M_t$




$M_b$

For a field $F$, a matroid $M$ is $F$-<u>representable</u> if $M \cong M[A]$ for some matrix $A$ over $F$.

$GF(2)$-rep. binary

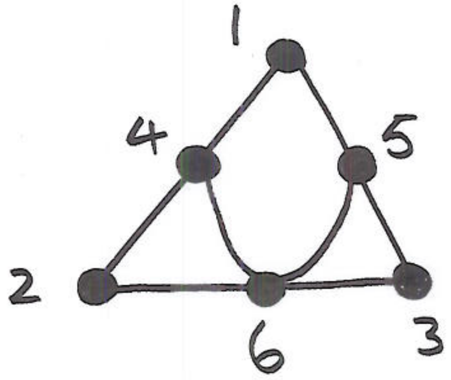
$GF(3)$-rep. ternary

$M_t$ is ternary

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{GF(3)} \end{array}$$

$M_b$ is binary

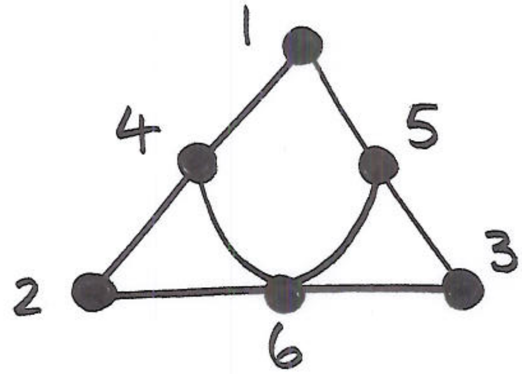
$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{GF(2)} \end{array}$$




$M_b$

$M_b$ is binary.

Is $M_b$ ternary?




$M_b$ is binary.

Is $M_b$ ternary?

$$A_2 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \qquad D_3 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

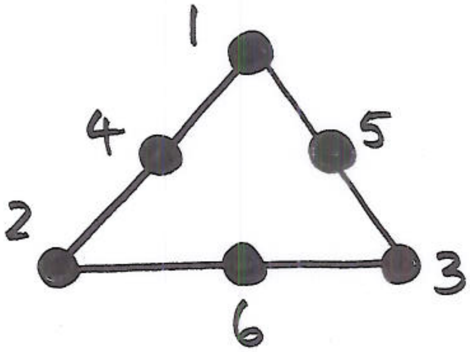
binary rep'n ternary rep'n.

$M_b$ is regular (representable over every field).

$M_b = M[D_3]$ where $D_3$ is interpreted over the field given.

$D_3$ is a <u>totally</u> <u>unimodular</u> <u>matrix</u>

- all its subdeterminants calculated over $\mathbb{R}$ are in $\{0, 1, -1\}$.



$M_t$

$M_t$ is ternary.

Is $M_t$ binary?

Try to build a binary rep'n.

$\{1,2,3\}$ is a basis :

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \left[\begin{matrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{matrix}\right] \end{array}$$

$\{1,2,4\}$ is a circuit :

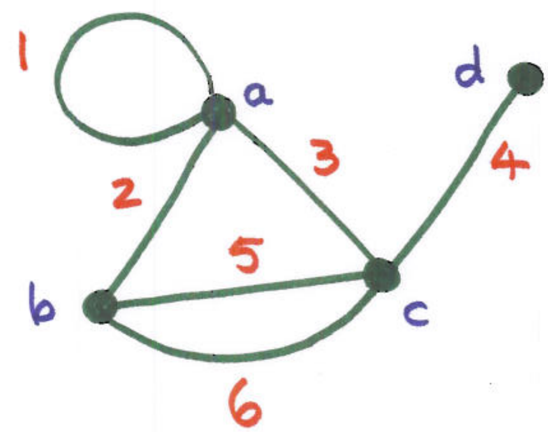
$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \left[\begin{matrix} 1 & 0 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & & \\ 0 & 0 & 1 & 0 & & \end{matrix}\right] \end{array}$$

$\{1,3,5\}$ and $\{2,3,6\}$ are circuits :

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \left[\begin{matrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}\right]_{GF(2)}. \end{array}$$

But $M_t \neq M[A_2]$, so $M_t$ is non-binary.

# Matroids from graphs




$G$

$$
\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
$$

Vertex-edge incidence matrix over $GF(2)$

$A_G$

$M(G)$, cycle matroid of $G$, is $M[A_G]$.

circuits of $M(G)$ : cycles of $G$

# Some questions

**Q1.** Is every matroid $F$-rep. for some field $F$?

**Q2.** If not, what can be said about $\dfrac{\#\ F\text{-rep. } n\text{-element matroids}}{\#\ n\text{-element matroids}}$?

**Q3.** For a specific field $GF(q)$, which matroids are $GF(q)$-representable?

Which graphs are planar?

# Minors of matroids

$$A_3 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$M[A_3] \setminus 1$ (delete 1)

is represented by $\begin{array}{c} \begin{array}{ccccc} 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$

For an element $e$ of a matroid $M = (E, \mathcal{I})$, the <u>deletion</u> $M \setminus e$ is the matroid $(E - \{e\}, \mathcal{I}')$ where

$$\mathcal{I}' = \{X \subseteq E - \{e\} : X \in \mathcal{I}\}.$$

- Easy to check this is a matroid.

$$A_3 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

What is $M[A_3] / 1$ (contract 1)?

$$A_3 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$M[A_3]/1$ is rep. by

$$\begin{array}{c} \begin{array}{ccccc} 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

What about contracting 4 instead?

Pivot on the circled entry.

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Row2 - Row 1

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$M[A_3]/4$ is rep. by

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

# Contraction

If $\{e\}$ is a loop (one-element circuit) of a matroid $M$, then

$$M/e = M \backslash e.$$

If $\{e\}$ is independent and $M = (E, \mathcal{I})$, then $M/e = (E - \{e\}, \mathcal{I}'')$ ($M$ contract $e$) where $\mathcal{I}'' = \{X \subseteq E - \{e\} : X \cup \{e\} \in \mathcal{I}\}$.

- $M/e$ is a matroid.
- $M \backslash e \backslash f = M \backslash f \backslash e$

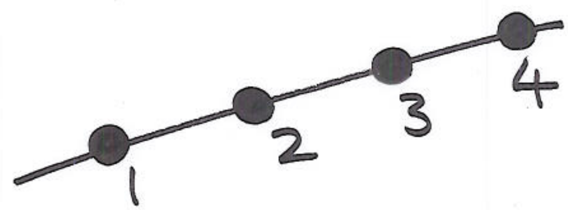
  $M/e/f = M/f/e$

  $M \backslash e/f = M/f \backslash e$
- $M \backslash X / Y$ is a minor of $M$, for disjoint subsets $X$ and $Y$ of $E$.

**Lemma** Every minor of an F-rep. matroid is F-rep.

**excluded minors** for F-rep.: matroids that are not F-rep. but every deletion and every contraction is F-rep.

**Example**




$E = \{1, 2, 3, 4\}$

$\mathcal{I} = \{X \subseteq E : |X| \leq 2\}$ $\qquad (E, \mathcal{I}) = U_{2,4}$

$U_{2,4}$ is not binary:

a 2-dimensional vector space over $GF(2)$ has $2^2 - 1$ non-zero vectors.

**Theorem** (Tutte 1958) A matroid is binary iff it has no $U_{2,4}$-minor.

Binary matroids have many other characterizations.

$$X \triangle Y = (X-Y) \cup (Y-X)$$

<u>Theorem</u> The following are equivalent for a matroid $M$:

(i) $M$ is binary.

(ii) For all distinct circuits $C_1$ and $C_2$, the set $C_1 \triangle C_2$ contains a circuit.

(iii) For all distinct circuits $C_1$ and $C_2$, $C_1 \triangle C_2$ is a disjoint union of circuits.

# Duality

$$A_3 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

- a standard rep. matrix for $M[A_3]$
- a standard-form generator matrix for the $[6,3]$-code over $GF(3)$, the code $C$ is the vector space generated by the rows of $A_3$
- $A_3 = [\, I_3 \mid D \,]$

  $[\, -D^T \mid I_{6-3} \,]$ is a parity-check matrix for $C$, a generator matrix for $C^\perp$, the dual code.

- The vector matroid of $[\, -D^T \mid I_{6-3} \,]$

i.e. of

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

is the dual matroid of $M[A_3]$.

For a matroid $M$ with ground set $E$ and with $\mathcal{B}$ as its set of bases, the <u>dual matroid</u> $M^*$ has ground set $E$ and its set of bases is $\{E-B : B \in \mathcal{B}\}$.

<u>Lemma</u> $[I_r \mid D]$ represents an $n$-element matroid $M$ over a field $F$ iff $[-D^T \mid I_{n-r}]$ represents $M^*$ over $F$.

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- $(M^*)^* = M$
- $M / T = (M^* \setminus T)^*$
- $M \setminus T = (M^* / T)^*$
- A matroid $N$ is a minor of $M$ iff $N^*$ is a minor of $M^*$.

- $M$ is an excluded minor for $F$-rep. $\Leftrightarrow$ $M^*$ is an excluded minor for $F$-rep.

## Terminology

| $M^*$ | $M$ |
|---|---|
| circuit | cocircuit |
| basis | cobasis |

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A set is spanning if it contains a basis.

(i) $X$ is a basis of $M \Leftrightarrow E-X$ is a basis of $M^*$.

(ii) $X$ is spanning in $M \Leftrightarrow X$ contains a basis of $M$

$\Leftrightarrow E-X$ is contained in a basis of $M^*$.

$\Leftrightarrow E-X$ is indep't in $M^*$.

(iii) $X$ is non-spanning in $M \Leftrightarrow E-X$ is dep't in $M^*$.

(iii) $X$ is non-spanning in $M$ $\Longleftrightarrow$ $E-X$ is dependent in $M^*$.

(iv) $X$ is a max'l non-spanning set in $M$ $\Longleftrightarrow$ $E-X$ is a minimal dep't set in $M^*$.

$\Longleftrightarrow$ $E-X$ is a circuit of $M^*$

$\Longleftrightarrow$ $E-X$ is a cocircuit of $M$.

maximal non-spanning set : hyperplane

<u>Lemma</u> The hyperplanes of $M$ are the complements of its cocircuits.

<u>Corollary</u> A circuit and a cocircuit in a matroid cannot have exactly one common element.

# Circuit - cocircuit intersections

**Theorem** The following are equivalent for a matroid $M$.

(i) $M$ is binary.

(ii) $|C \cap C^*|$ is even for every circuit $C$ and cocircuit $C^*$.

(iii) $|C \cap C^*| \neq 3$ for every circuit $C$ and cocircuit $C^*$.

# Cocircuits and weights

**Lemma** Let $A$ be a matrix over a field $F$, and let $M = M[A]$. The cocircuits of $M$ coincide with the minimal non-empty supports of vectors from the row space of $A$.

- If $A$ is a generator matrix of a code $C$, then the minimum weight of $C$ is the minimum cocircuit size of $M[A]$.

# Uniform matroids

Let $E = \{1, 2, \ldots, n\}$ $\quad 0 \le r \le n$

$U_{r,n} = (E, \mathcal{I})$ where $\mathcal{I} = \{X \subseteq E : |X| \le r\}$.

$$U_{r,n}^* = U_{n-r,n}$$

- If $U_{r,n}$ is $GF(q)$-rep., we have an $r \times n$ matrix $A$ over $GF(q)$ such that every $r$ columns are lin. indep't.

The $[n,r]$-code over $GF(q)$ generated by $A$ is a maximum-distance-separable (MDS) code.

# Main conjecture for MDS codes

**Conjecture** For all prime powers $q$, $U_{r,q+2}$ is an excluded minor for the class of $GF(q)$-rep. matroids

$$\begin{cases} \text{for all } r \text{ in } \{2,3,\ldots,q\} & \text{for } q \text{ odd;} \\ \text{for all } r \text{ in } \{2,3,\ldots,q\}-\{3,q-1\} & \text{for } q \text{ even.} \end{cases}$$

- Holds for $r \leq 5$.
- Holds for $q \leq 27$.

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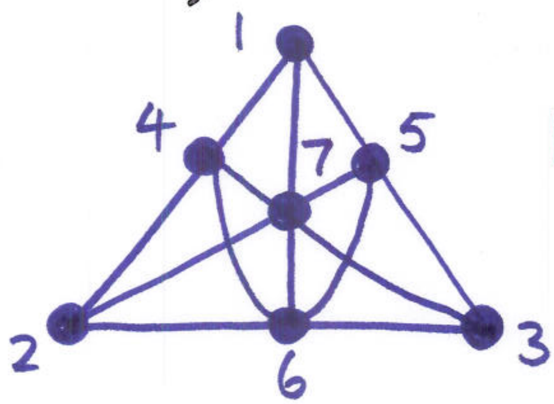
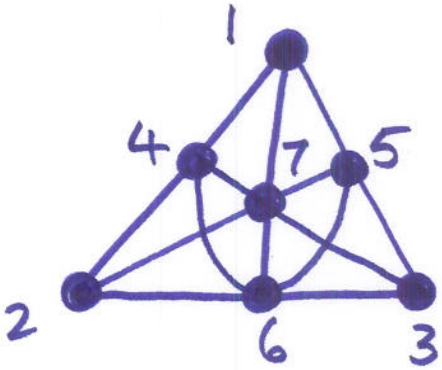
## Excluded minors for ternary matroids

- $U_{2,5}$ and its dual, $U_{3,5}$
- 

$F_7$, the Fano matroid, and its dual, $F_7^*$




$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}_{GF(2)}$$

$F_7$

The dual code is the binary Hamming code $\mathrm{Ham}(3,2)$; a rep'n for $F_7^*$ is a generator matrix for $\mathrm{Ham}(3,2)$.

**Theorem** (Reid 1971; Bixby 1979; Seymour 1979)

A matroid is ternary iff it has no minor isomorphic to any of $U_{2,5}$, $U_{3,5}$, $F_7$, or $F_7^*$.

# Rota's Conjecture (1970)

For all finite fields $GF(q)$, there are only finitely many excluded minors for the class of $GF(q)$-rep. matroids.

$q = 2$ $\quad U_{2,4}$

$q = 3$ $\quad U_{2,5}, U_{3,5}, F_7, F_7^*$

**Theorem** (Geelen, Gerards, Kapoor 2000)

There are exactly seven excluded minors for the class of $GF(4)$-rep matroids.

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$q \geqslant 5$ $\quad$ OPEN

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- For all infinite fields $F$, there are infinitely many excluded minors for $F$-rep.

# Regular matroids

(representable over every field)

**Theorem** (Tutte)

The following are equivalent for a matroid $M$.

(i) $M$ is regular.

(ii) $M$ is binary and ternary.

(iii) $M$ is binary and $\mathbb{F}$-rep. for some field of characteristic not 2.

| Class | Excluded minors |
| --- | --- |
| binary | $U_{2,4}$ |
| ternary | $U_{2,5}, U_{3,5}, F_7, F_7^*$ |
| regular = binary $\cap$ ternary | minor-minimal members of $\{U_{2,4}, U_{2,5}, U_{3,5}, F_7, F_7^*\}$ = $\{U_{2,4}, F_7, F_7^*\}$ |

Tutte (1958)

# Non-representable matroids

**Proposition** For a fixed field F,

$$\frac{\#\text{ non-iso. F-rep. n-elt matroids}}{\#\text{ non-iso. n-element matroids}} \to 0$$

as $n \to \infty$.

**Conjecture** (Mayhew, Newman, Welsh, Whittle 2009)

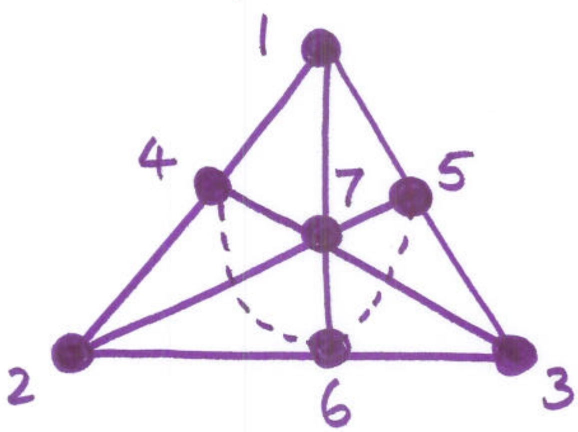
$$\frac{\#\text{ non-iso. rep. n-elt matroids}}{\#\text{ non-iso. n-element matroids}} \to 0$$

as $n \to \infty$

(rep: representable over some field)

**Proposition** Every matroid with at most 7 elements is representable; so is every matroid with 8 elements unless its rank is 4.

**Example**



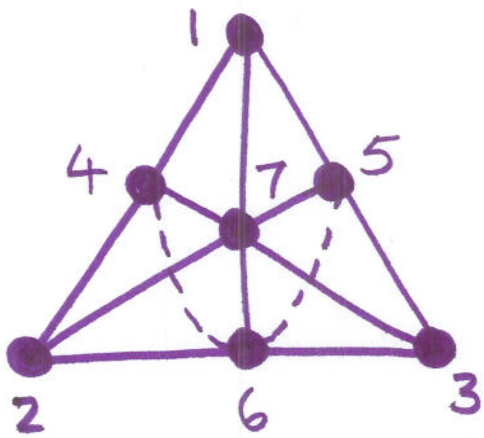

$F_7$ (Fano)

The dotted line encodes the equation

$1+1=0$

__Proposition__ Every matroid with at most 7 elements is representable; so is every matroid with 8 elements unless its rank is 4.

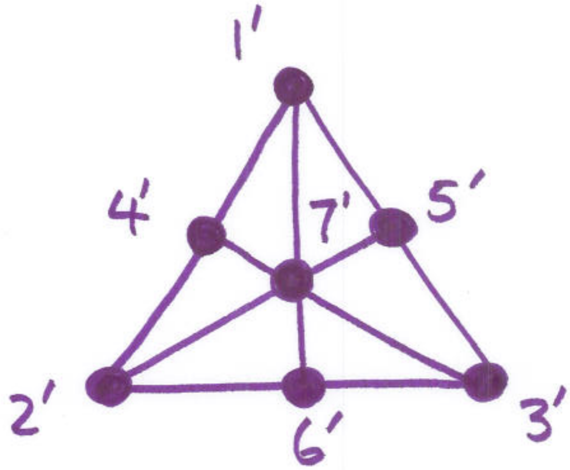
__Example__




$F_7$ (Fano)

$\{4, 5, 6\}$ is a circuit

$\Rightarrow 1+1=0$




$F_7^-$ (non-Fano)

$\{4', 5', 6'\}$ is not a circuit

$\Rightarrow 1+1 \neq 0$

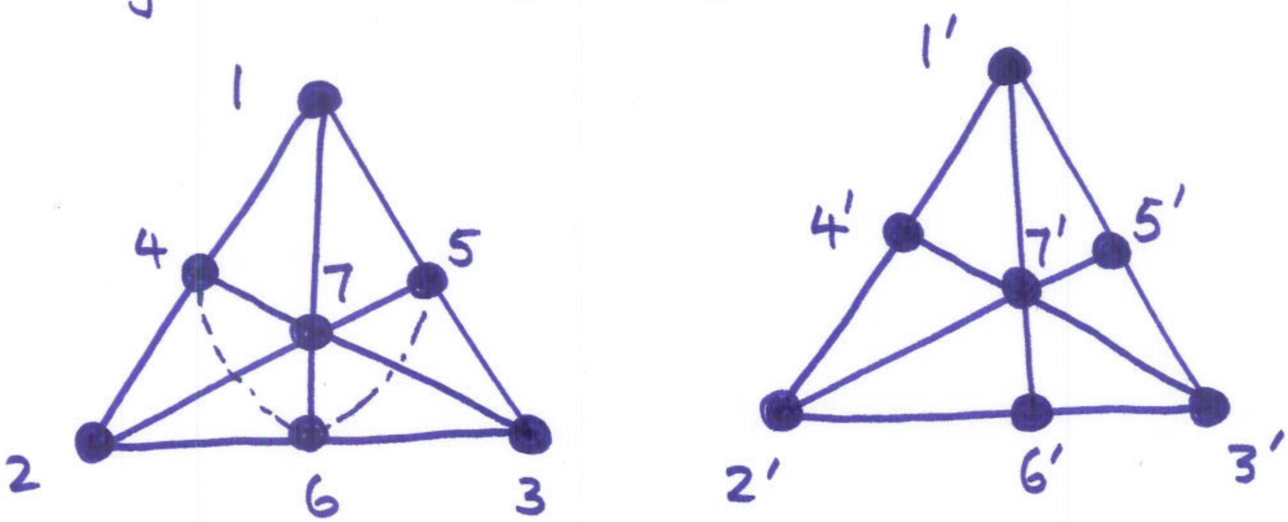
$F_7 \oplus F_7^-$ is not rep'ble.

This matroid has ground set $E(F_7) \cup E(F_7^-)$.

Its indep't sets : $\{I_1 \cup I_2 : I_1 \in \mathcal{I}(F_7),$
$I_2 \in \mathcal{I}(F_7^-)\}$

- $F_7 \oplus F_7^-$, the direct sum of $F_7$ and $F_7^-$, has 14 elements and rank $= 3+3 = 6$.

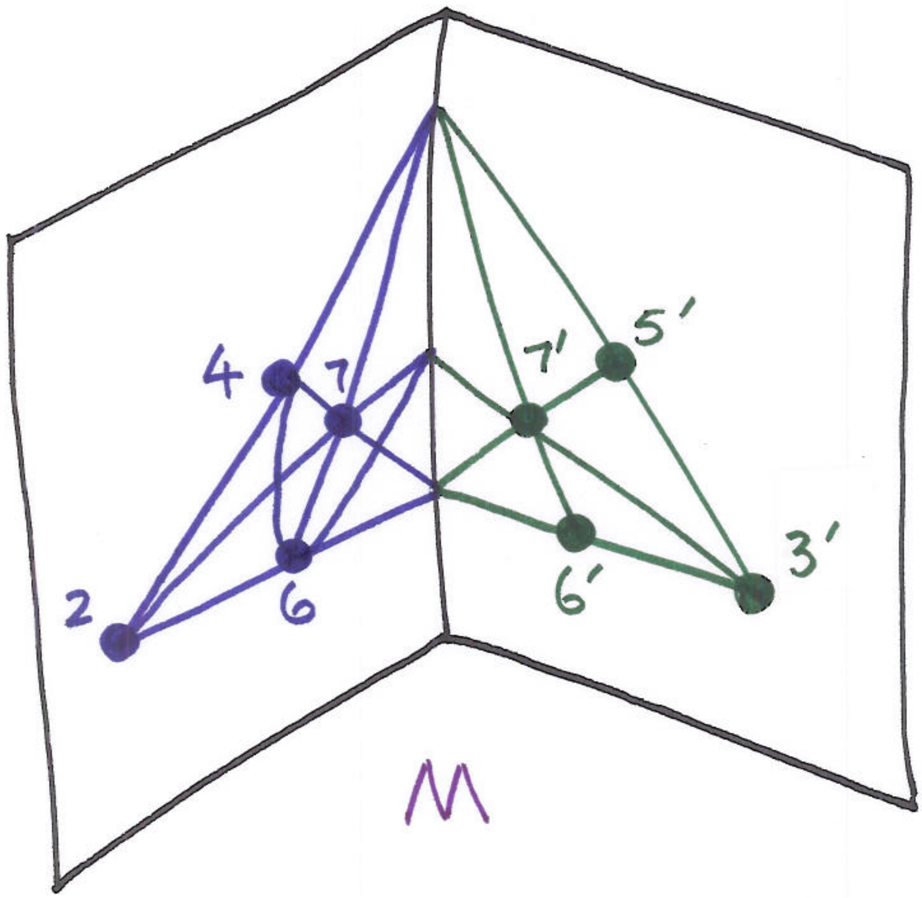
A different way to put $F_7$ and $F_7^-$ together: <u>in</u> <u>rank</u> <u>3</u>




$E = \{1, 2, \dots, 7\} \cup \{1', 2', \dots, 7'\}$

$\mathcal{I} = \{X \subseteq E : |X| \leq 3$, elements of $X$ are not collinear$\}$

# Another way to join $F_7$ and $F_7^-$ in rank 4




$M$

- 8 elements, rank 4

- $M/2 \cong F_7^-$

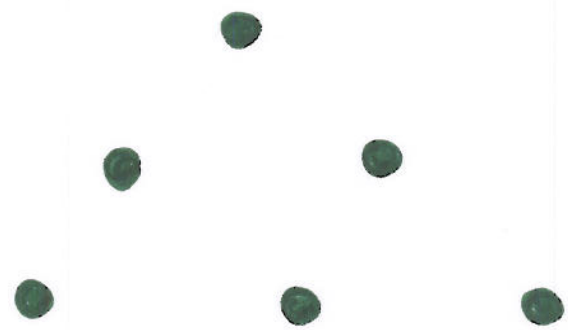
  $M/3' \cong F_7$

To contract 2, project from 2, so 4, 7, and 6 plug the holes in $F_7^-$.

$U_{r,n}$ : $n$-element matroid

- every $r$-element set is a basis

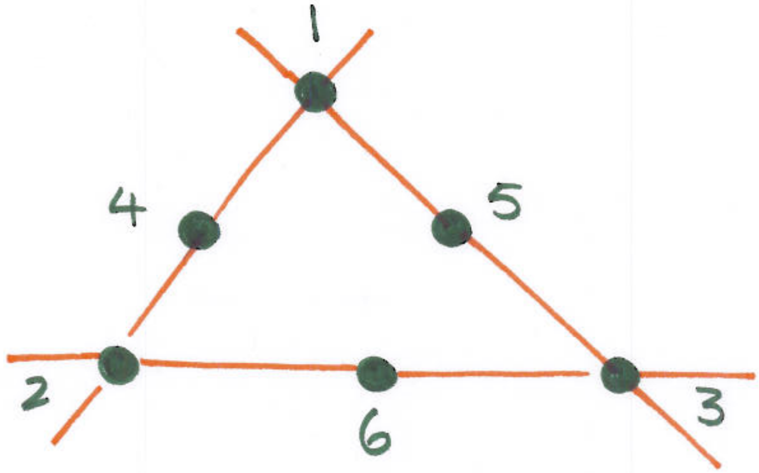
$U_{3,6}$



6 points in the plane, no three collinear.

$U_{r,n}$ : n-element matroid

- every r-element set is a basis




Three orange lines correspond to circuits and hyperplanes: circuit-hyperplanes.

In general, take a subset $\mathcal{X}$ of the bases of $U_{r,n}$ so that no two members of $\mathcal{X}$ have more than $r-2$ common elements.

$\mathcal{X}$-sets are non-bases – they become circuit-hyperplanes.

This gives a <u>sparse</u> <u>paving</u> <u>matroid</u>

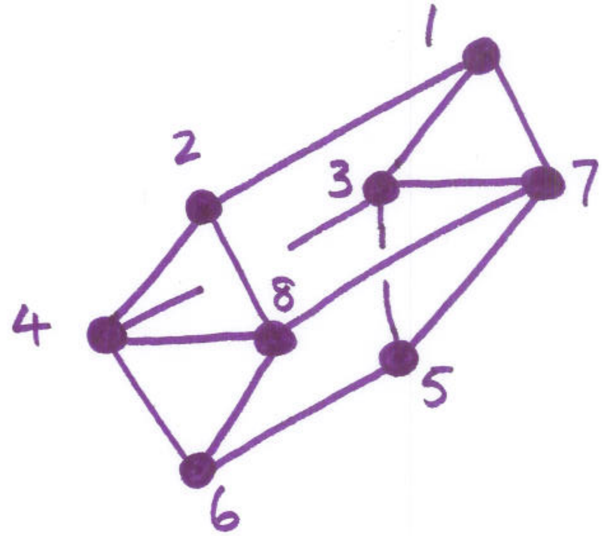
**Example** Start with $U_{4,8}$.

Let $\mathcal{X} = \{ \{1,2,3,4\}, \{3,4,5,6\}, \{5,6,7,8\}, \{1,2,7,8\}, \{3,4,7,8\} \}$

Matroid $V_8$, the Vámos matroid

$E = \{1, 2, \ldots, 8\}$

bases : all 4-element sets except those in $\mathcal{X}$.




This matroid is not representable.

# Rank axioms

**Lemma** Let $E$ be a finite set. A function $r: 2^E \to \mathbb{Z}^+ \cup \{0\}$ is the rank function of a matroid on $E$ iff

(R1) If $X \subseteq E$, then $r(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.
(increasing)

(R3) If $X, Y \subseteq E$, then
(submodular)

$$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$$

Independent sets: $|X| = r(X)$

(R1) If $X \subseteq E$, then $r(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

(R3) If $X, Y \subseteq E$, then

$$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y).$$

Weaken (R1) to

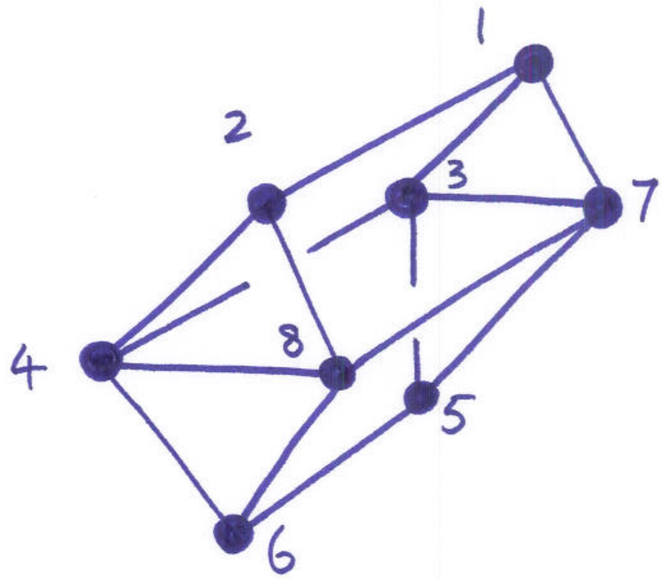
(R1)' $\quad r(\emptyset) = 0.$

Resulting structures are polymatroids.

Example In a vector space $V$, let $E$ be some set of subspaces

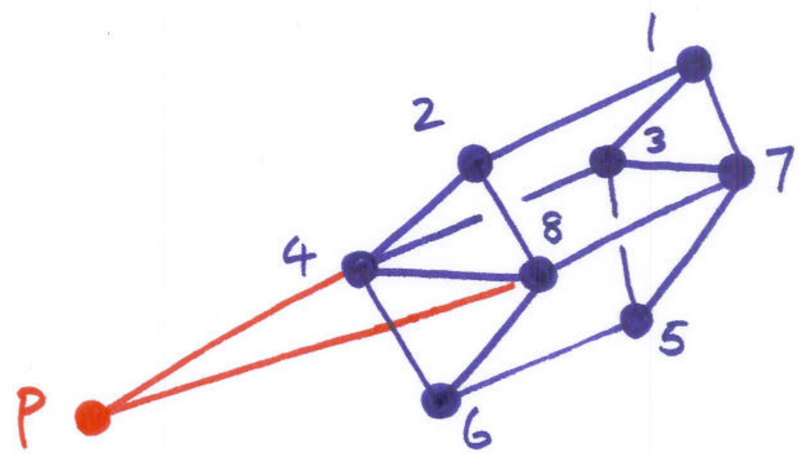
– if $E$ consists of one-dim'l subspaces only, then we have a matroid.

$V_8$ is not representable




<u>Proof</u>

Suppose $V_8$ is representable. Then it sits in a projective space



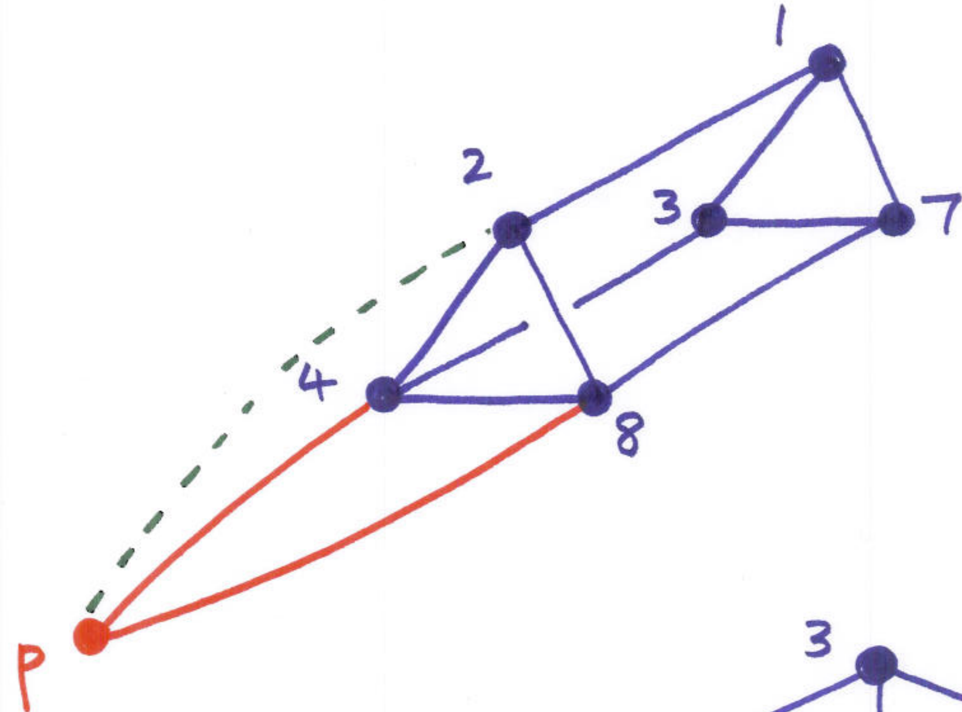

$$r(\{1,2,3,4,p\}) + r(\{1,2,7,8,p\})$$

$$\geq r(\{1,2,3,4,7,8,p\}) + r(\{1,2,p\})$$

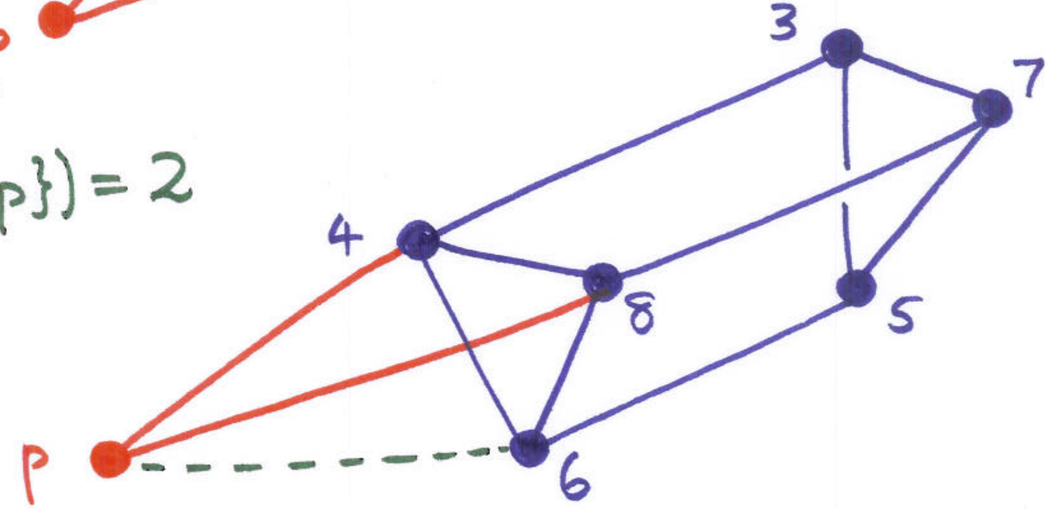
$$3 + 3 \geq 4 + r(\{1,2,p\})$$

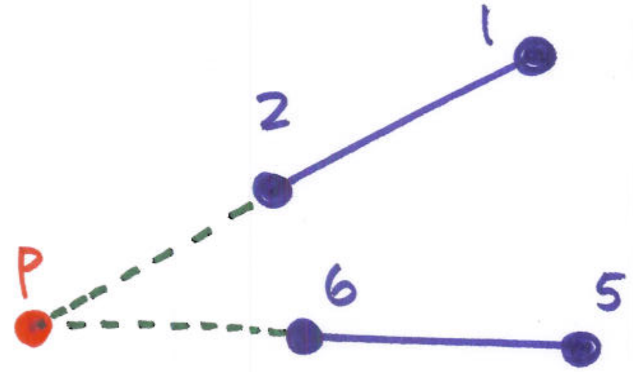
$$3 + 3 \geq 4 + r(\{1, 2, p\})$$

so $r(\{1,2,p\}) = 2$.




$r(\{5,6,p\}) = 2$




So $\{1,2,5,6\}$ are coplanar; not so in $V_8$.

$\square$

# Summary

## Basic matroid operations

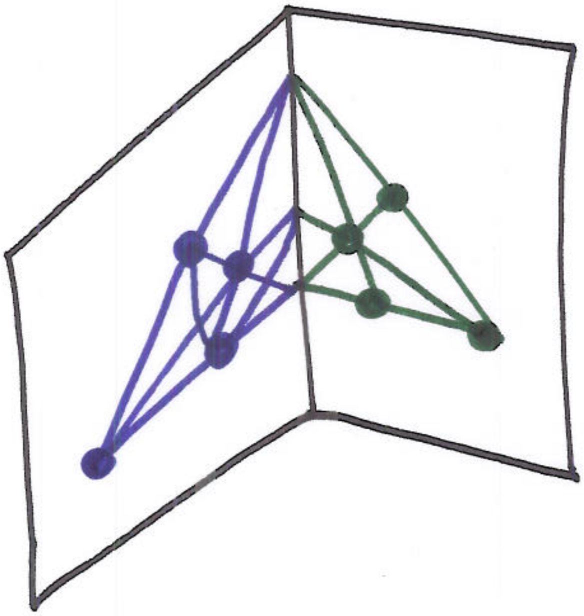
- duality
  - generalizes vector space orthogonality
  - matches up with coding theory duality
- deletion
- contraction
  - dual of deletion
  - corresponds geometrically to projection.
- minors
  - sequences of deletions and contractions

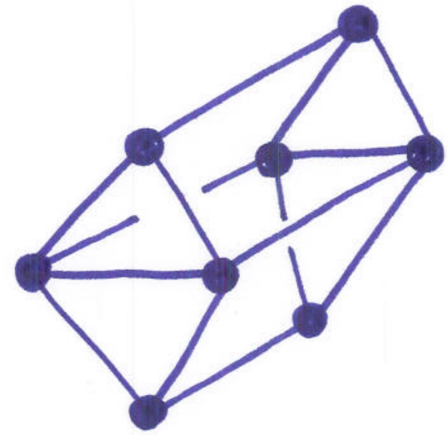
**Basic example** F-rep matroids

**Rota's Conj.** For all q, the set of excluded minors for GF(q)-rep is finite

# Non-representable matroids

## Examples





$V_8$, Vámos matroid

**Conjecture** (Mayhew, Newman, Welsh, Whittle)

Asymptotically almost every $n$-element matroid:

- is not representable
- is sparse paving
- has rank in $\{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$
- has $V_8$ as a minor
- has any fixed sparse non-paving matroid as a minor