Metastability and NIP

ACVF / MS / NIP

Abelian groups.

NIP assumed throughout.

Haskell, Loeser , Macpherson, Pillay, Simon.

A *generically stable* measure is a definable measure $p(x)$, such that $p(x) \otimes p(y) = p(y) \otimes q(x)$.

Equivalent forms include: if $a = (a_1, \ldots, a_n)$, let $f(\phi, a) = |\{i : \phi(a_i))\}|/n$. Then:

For appropriate sequences a^n (in fact, with high probability, an realization of p^n will do),

(fim) $p(\phi(x,b)) = \lim_{n \to \infty} f(\phi, a^n)$

For types, (fim) is Shelah's "majority rule".

Will begin with $\widehat{V}$ =generically stable (global) *types* on a definable set V.

$\widehat{V}(A) =$ elements of $\widehat{V}$ definable over A.

Stability: all types, measures are generically stable. Fundamental theorem: properties of generically stable types, and:

$\widehat{V} \to S_A(V)$ is bijective, $A = \text{acl}(A)^{eq}$.

In general, $\widehat{V}(A) \to S_A(V)$ injective.

What could replace surjectivity?

In metastable case: consider $\widehat{V}$ as fundamental space; find an arbitrary type at the limit of a path on this space.

$\widehat{V}$ as a pro-definable set

The definable $\phi(v, y)$ types on V form a $\bigwedge\bigvee$-definable set. For some $\theta(y, c)$, the type has the form: $\phi(v, y) \iff \theta(y, c)$; and the set of c that work is $\bigwedge$-definable.

Iteration is a $\bigwedge\bigvee$-definable function: $p(x), q(y) \mapsto p(x) \otimes q(y)$.

The definable ϕ-types extending to a generically stable type can always be defined by a bounded Boolean combination of instances (majority rule, fim).

Moreover, the generically stable types can be recognized via: $p(x) \otimes p(y) = p(y) \otimes p(x)$, a $\bigwedge$-condition. This removes the $\bigvee$.

Example: uniform families of normal subgroups.

A group G is *generically stable* if it admits a generically stable (left) translation invariant type. In this case, the type is unique.

Let G be a definable group. Let N_i be a family of generically stable normal subgroups. Then there exists a generically stable group containing them all.

Proof. For $p, q \in \widehat{V}$, let $p * q = m_*(p \otimes q)$.

p is the generic of a subgroup $A(p)$ iff $p * p = p$.

We have $A(p) \subseteq A(q)$ iff $p * q = q$.

Seeking q with $q * q = q$ and $p_i * q = q$, where p_i is the generic of N_i.

For any finite subfamily, $i_1, \ldots, i_k$, take the generic of $N_{i_1} \ldots N_{i_k}$.

Compactness.

$\square$

Corollary 1. *Among the generically stable subgroups, there exists a cofinal* uniform *family* C_t.

Proof. Let A_i be a family of generically stable groups, containing an instance of each $Aut(\mathbb{U})$-conjugacy class of such groups. Find a generically stable $C = C_e$ containing each A_i, $q = tp(e)$. Then $\{C_t : t \models q\}$ is such a family. $\square$

Define $t \leq t'$ if $C_t \leq C_{t'}$; a pro-definable partial ordering.

Results initially obtained in metastable setting. Assuming metastability, Q is Γ-internal. (And with additional conditions, definable.)

$L(G)$, the limit group = union of all generically stable subgroups. If $G = L(G)$, say G is limit metastable.

The group structure of $L(G)$ is decomposed into: a partial ordering; and: a uniform family of generically stable groups.

(*) What about $G/L(G)$?

(**) What happens in C_e, below the generic?

metastability over Γ

Let Γ be stably embedded. Assume $\widehat{U} = U$ for all definable $U \subset \Gamma^{eq}$.

T is *generically stable over* Γ if any type in V over A has the form $f_*(q)|A$, q a type of Γ^*, $f : \Gamma^* \to \widehat{V}$ a ($\bigwedge$)-definable function.

Equivalently: for $c \in V$,
$tp(c/A, \Gamma) \in Im(\widehat{V} \to S_A(V))$.

In particular $\widehat{V}$= definable types $\perp \Gamma$.

This is a notion of "relative stability" (quite different from "stability over a predicate".)

Metastability: in addition, generically stable = stably dominated.

Question: how far are generically stable types from being stably dominated? Is non-genericity caused by a stable relation in a reasonable logic?
Present examples show that Ind-definable equivalence relations must be considered.

Metastability gives a way to impose finite dimensionality conditions. We'll be interested in: Γ o-minimal, stable part of of finite Morley rank. This gives in particular *finite weight* for $p \in \widehat{V}$.

This makes it possible to try Zilber's indecomposability. It works in Abelian case.

Note that $G/L(G)$ has no nontrivial generically stable subgroups. By "groupification" lemma 2 below, it has no generically stable types. By generic metastability over Γ, it follows that:
(*) $G/L(G)$ *is* Γ*-internal*.

Lemma 2. *Let H be a piecewise definable, or even piecewise *-definable, Abelian group, p a symmetric definable type of elements of H. Assume H has p-weight $< 2n$, in the sense that:*

Whenever $b \in H$, $(a_1, \dots, a_{2n}) \models p^{\otimes 2n}$, $a_i \models p|b$ for some i.

Then there exists an ∞-definable subgroup G of H with generic type $p^{\pm 2n}$. p is contained in a coset of G.

Proof. Let $(a_1, a_2, \dots, a_{2n}) \models p^{\otimes 2n}$, and let $b = a_1^{-1} a_2 \cdot \dots \cdot a_{2n}$.

By the weight assumption, $a_i \models p|b$ for some i. Since H is commutative, $tp(a_1, a_2, \dots, a_{2n}/b)$ is $Sym(n)$-invariant,so $a_1 \models p|b$.

Let G be the stabilizer of $p^{\pm 2n}$, and $C = Stab(p^{\mp 2n-1}, p^{\pm 2n})$. Then $a_1^{-1} \in C$, so $p^{\pm 1}$ is a type of elements of C. It follows that $p^{\pm 2}$ and hence also $p^{\pm 2n}$ is a type of elements of G. Being invariant, it shows G is generically stable.

$\square$

Question: What about the non-Abelian case? A limit metastable K with $K\backslash G/K$ Γ-internal?

Inside a stably dominated group:

Proposition 3. *Let G be a generically stable group. Assume the generic p of G is stably dominated. Then there exists a *-definable stable group $\mathfrak{g}$, and a *-definable homomorphism $g : G \to \mathfrak{g}$, such that the generics of G are stably dominated via g.*

"Groupification of domination"; to be discussed later. If one specifies that $\mathfrak{g}$ is as large as possible, then $(g, \mathfrak{g})$ are canonical. Let K be the kernel.

(**)

Proposition 4. *K is limit metastable.*

Factoring out $L(K)$ we may assume K is Γ-internal. to obtain H with $0 \to K \to G \to \mathfrak{g} \to 0$. Also a map to Γ^{eq} with stable fibers; an almost section $S \to H$. Contradicts domination by $G \to \mathfrak{g}$.

Picture: chain of “closed” (generically stable) subgroups, going to ∞; for each one a canonical maximal “open” subgroup, with a chain of closed subgroups approaching it; etc.

ACVF, picture with topology.

V has a definable topology (Zariski), with a definable sheaf of functions into Γ_∞ ($f = \mathrm{val}\phi$, ϕ regular.) Γ_∞ too has a definable topology (o).

Topology on $\widehat{V}$: $\{p \in W : f(p) \in U\}$ basic open, with W open in V, U open in Γ_∞.

Notions of definable compactness, definable connectedness; $\widehat{V}$ definably connected for V a ball (but not the union of two), $\widehat{V}$ definably compact for V a closed ball (but not an open ball.)

$\widehat{V}$ admits a definable contraction to a closed subspace, homeomorphic to a subset of Γ_∞^n.

Question: Contractibility of generically stable groups.

Proof in affine case.

Proposition 5. *Let G be a generically stable $\wedge$- definable subgroup of an affine algebraic group. Then there exists a group scheme $\mathcal{G}$ over $\mathcal{O}$ such that $G \cong \mathcal{G}(\mathcal{O})$.*

Proof: p the unique translation invariant generically stable type of G; $G \leq H$, H affine, defined over some $K_0 = (K_0)^a$. Let $R_0 := K_0[H]$ be the affine coordinate ring of H. Define

$$R = \{f \in K_0[G] : (d_p x)\mathrm{val} f(x) \geq 0\}$$

This is an $\mathcal{O}$-subalgebra of R_0. Show: if $f \in R$, then $f(xy) = \sum g_i(x)h_i(y)$ with $g_i, h_i \in R$; finite generations; a group scheme structure on $SpecR$. $(\ldots.)$

Identify $g : G \to \mathfrak{g}$ as $G(\mathcal{O}) \to G(\mathcal{O}/\mathcal{M})$
$\mathcal{M} = \{x : \mathrm{val}(x) > 0\}$
(this works only over a model!)

A chain of ideals of $\mathcal{O}$, $\mathcal{M}_\alpha = \{x : \mathrm{val}(x) \geq \alpha\}$.

Obtain a continuous path $p \to 1$, $\alpha \mapsto \ker(G \to \mathcal{G}(\mathcal{O}/\mathcal{M}_\alpha))$.

To what extent can we generalize this picture beyond metastability?

Generically stable *measures* , $m\widehat{V}$.

The properties of generically stable types generalize in full.

Review 1-3 for measures.

o-minimal Abelian groups: (say $G = \mathbb{R}$) : $G/L(G)=$ maximal definably compact quotient.

$L(G)$ is the union of Ind-definable generically stable groups.

In stable case, fundamental theorem admits two equivalent forms:

a) $\widehat{V}(A) \to S_A(V)$ is bijective, $A = \text{acl}(A)^{eq}$.

(acl, eq developed, in large part, for this statement!)

or,

b) $m\widehat{V}(A) \to mS_A(V)$ is bijective, any A.

In NIP context, even for types in the image of (b), analogue of first approach is not (presently?) available, since going up to A^{bdd} can destroy generic stability.

If p is a type over A, μ the unique generically stable measure μ defined over A and extending p, then μ is the integral over the compact Lascar group of certain invariant types; but these are not generically stable.

The notion of domination uses only the measure-0 ideal and not the full measure.

Proposition 3 has been generalized to this setting: a symmetric ideal of ∞-definable sets with certain definability properties.

A generalization in a different direction replaces the family of stable formulas (or types) with an arbitrary family $\mathbb{C}$ of hyperimaginary sorts. This allows a uniform treatment of compact domination and stable domination.

Let E be an inf-definable equivalence relation on X, and let $\pi : X \to Y$ be a map with kernel E. We define a measure $\pi_*\mu$ on Y: U is measurable iff $\pi^{-1}(U)$ is μ-measurable; and then $\pi_*\mu(U) = \mu(\pi^{-1}U)$. Similarly, given an ideal $\mathcal{I}$ we define $\pi_*\mathcal{I} = \{U : \pi^{-1}(U) \in \mathcal{I}\}$.

Let $\mathcal{C}$ be a class of hyperdefinable sets.

Definition 6. *Let* $f : X \to Y = X/E$ *and let* $\mathcal{I}_Y$ *be an ideal on* Y*. Then* $(f, \mathcal{I}_Y)$ *is* $\mathcal{C}$*- dominating if for any base set* A*, for* $\mathcal{I}_Y$*-almost every* $b \in Y$*, all elements of* $f^{-1}(b)$ *have the same type over* $A \cup \mathcal{C}$*.*

(E) for any $A = \text{acl}(A)$, any type over A extends to an A-invariant type p.

Equivalent to: (1) types over A do not for over A;

(2) elimination of bounded hyperimaginaries = the (compact) Lascar group is profinite = $\mathbb{R}/\mathbb{Z}$ is not a subquotient of $Aut(\mathbb{U}/A)$.

Inductive proof of density of A-definable types in A-topology.

Existence of invariant extensions follows from density, since the set of A-invariant types is a closed subspace of $S_x(\mathbb{U})$, so the projection to $S_x(A)$ is closed.

Descent/ non-descent.