Whittaker functions and Topological Field Theories

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2. A. Gerasimov, D. Lebedev, S. Oblezin, *Archimedean L-factors and topological field theories I, II*, [math.RT/0906.1065], and [math.AG/0909.2016], 2009.
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• **Whittaker functions.** Let G be a (quasi)-split real connected reductive group. The Iwasawa decomposition:

$$G \,=\, U_- \cdot A \cdot K \qquad\qquad A = \left\{e^{x_1}, \ldots, e^{x_N}\right\}$$

The Gauss (Bruhat) decomposition:

$$G_0 \,=\, U_- \cdot A \cdot U_+ \,.$$

Character of $B_- = U_- A$ with $\underline{\gamma} \,=\, (\gamma_1, \ldots, \gamma_N)$:

$$\chi_{\underline{\gamma}} : \quad B_- \longrightarrow \mathbb{C}\,, \qquad\qquad \chi_{\underline{\gamma}}(na) \,=\, \prod_{i=1}^{N} e^{(\gamma_i+\rho_i)\,x_i}\,.$$

The principal series representation $(\pi_{\underline{\gamma}},\, \mathcal{V}_{\underline{\gamma}})$:

$$\mathrm{Ind}_{B_-}^{G}\,\chi_{\underline{\gamma}} = \left\{\, f \in C^\infty(G) \,\middle|\, f(bg) = \chi_{\underline{\gamma}}(b)\, f(g)\,, \quad b \in B_- \right\}$$

Definition 1 *The Whittaker function $\Psi_{\underline{\gamma}}(z)$ is a smooth function on $X = G/B_+$ given by*

$$\Psi_{\underline{\gamma}}(\underline{x}) \,=\, e^{-\sum x_i\rho_i}\big\langle \psi_L\,,\, \pi_{\underline{\gamma}}(e^{-\sum x_i H_i})\,\psi_R \big\rangle\,, \tag{1}$$

the Whittaker vectors ψ_L, $\psi_R \in \mathcal{V}_{\underline{\gamma}}$ are defined by

$$E_i\psi_R \,=\, -\frac{1}{\hbar}\,\psi_R\,, \qquad F_i\psi_L \,=\, -\frac{1}{\hbar}\,\psi_L\,, \qquad i = 1, \ldots, N{-}1\,.$$

Remark 1 *The Whittaker function can be lifted either to a B_+-invariant function $\Psi_{\underline{\gamma}}(g)$, or to a K-invariant function $\Phi_{\underline{\gamma}}(g)$; the following relation holds:*

$$\Psi_{\underline{\gamma}}(\underline{x}) \,=\, \Phi_{\underline{\tilde{\gamma}}}(\underline{\tilde{x}})\,, \qquad \tilde{x}_i \,=\, \frac{1}{2}\,x_i\,, \qquad \tilde{\gamma}_i \,=\, 2\gamma_i\,,$$

for $i = 1, \ldots, N$.

• In 1996 Givental proposed an integral representation:

$$\Psi^{\mathfrak{gl}_N}_{\underline{\gamma}}(\underline{x}) \;=\; \int_{\mathcal{C}} e^{\frac{1}{\hbar}\mathcal{F}_N(x)} \prod_{k=1}^{N-1}\prod_{i=1}^{k} dx_{k,i}\,, \qquad (2)$$

$$\mathcal{F}_N \;=\; \imath\sum_{k=1}^{N}\gamma_k\Big(\sum_{i=1}^{k}x_{k,i}-\sum_{i=1}^{k-1}x_{k-1,i}\Big)-\sum_{k=1}^{N-1}\sum_{i=1}^{k}\Big(e^{x_{k+1,i}-x_{k,i}}+e^{x_{k,i}-x_{k+1,i+1}}\Big)$$

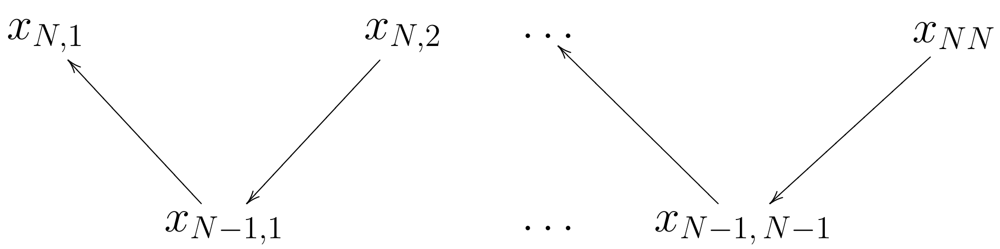
$$\begin{array}{ccccccc}
x_{N,1} & & & & & & \\
\uparrow & & & & & & \\
x_{N-1,1} & \longleftarrow & x_{N,2} & & & & \\
\uparrow & & \uparrow & & & & \\
\vdots & & \vdots & & & & \\
x_{2,1} & \longleftarrow & x_{3,2} & \longleftarrow & \ldots\; x_{N,N-1} & & \\
\uparrow & & \uparrow & & \uparrow & & \\
x_{1,1} & \longleftarrow & x_{2,2} & \longleftarrow & \ldots x_{N-1,N-1} & \longleftarrow & x_{NN}
\end{array}$$

• Iteration over the rank N:

$$\Psi^{\mathfrak{gl}_N}_{\gamma_1,\ldots,\gamma_N}(\underline{x}_N) \;=\; \int\limits_{\mathbb{R}^{N-1}} d\underline{x}_{N-1}\; Q^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}(\underline{x}_N\,;\,\underline{x}_{N-1}|\,\gamma_N)\;\Psi^{\mathfrak{gl}_{N-1}}_{\gamma_1,\ldots,\gamma_{N-1}}(\underline{x}_{N-1})$$

$$Q^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}} \;=\; \exp\Big\{\,\imath\gamma_N\Big(\sum_{i=1}^{N}x_{N,i}-\sum_{i=1}^{N-1}x_{N-1,i}\Big)-\sum_{i=1}^{N-1}\Big(e^{x_{N,i}-x_{N-1,i}}+e^{x_{N-1,i}-x_{N,i+1}}\Big)\Big\}$$

corresponding to the Givental diagram:



• In 2000 Kharchev and Lebedev proposed the Mellin-Barnes representation:

$$\Psi^{\mathfrak{gl}_N}_{\underline{\gamma}_N}(\underline{x}) \;=\; \int\limits_{\mathcal{S}} \prod_{n=1}^{N-1} \frac{\prod\limits_{m=1}^{n+1}\prod\limits_{k=1}^{n}\Gamma\Big(\frac{\imath\gamma_{n+1,m}-\imath\gamma_{n,k}}{\hbar}\Big)}{\prod\limits_{s\neq p}\Gamma\Big(\frac{\imath\gamma_{n,s}-\imath\gamma_{n,p}}{\hbar}\Big)}\,\cdot \tag{3}$$

$$\cdot\exp\Big\{\imath x_n\Big(\sum_{j=1}^{n}\gamma_{n,j}\;-\;\sum_{j=1}^{n-1}\gamma_{n-1,j}\Big)\Big\}\prod_{\substack{n=1\\ j\leq n}}^{N-1} d\gamma_{nj}$$

where $\min\limits_{j}\{\mathrm{Im}(\gamma_{kj})\} > \max\limits_{m}\{\mathrm{Im}(\gamma_{k+1,m})\}$, $k=1,\ldots,N-1$

$$\begin{array}{ccccccccc}
\gamma_{N,1} & & \gamma_{N,2} & & \cdots & & \gamma_{N,N-1} & & \gamma_{NN}\\
& \gamma_{N-1,1} & & \gamma_{N-1,2} & & \cdots & & \gamma_{N-1,N-1} & \\
& & \ddots & & \cdots & & \cdots & & \\
& & & \gamma_{21} & & \gamma_{22} & & & \\
& & & & \gamma_{11} & & & &
\end{array}$$

• Iteration over the rank N:

$$\Psi^{\mathfrak{gl}_N}_{\underline{\gamma}_N}(x_1,\ldots,x_N) \;=\; \int\limits_{\mathbb{R}^{N-1}} d\underline{\gamma}_{N-1}\; \overset{\vee}{Q}{}^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}(\underline{\gamma}_N,\underline{\gamma}_{N-1}|\,x_N)\;\Psi^{\mathfrak{gl}_N}_{\underline{\gamma}_N}(x_1,\ldots,x_{N-1})\,,$$

$$\overset{\vee}{Q}{}^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}} = \prod_{j=1}^{N}\prod_{k=1}^{N-1}\Gamma\Big(\frac{\imath\gamma_{N,j}-\imath\gamma_{N-1,k}}{\hbar}\Big)\;\exp\Big\{\imath x_N\Big(\sum_{j=1}^{N}\gamma_{N,j}-\sum_{k=1}^{N-1}\gamma_{N-1,k}\Big)\Big\}$$

corresponding to the Gelfand-Zetlin pattern

$$\begin{array}{ccccccccc}
\gamma_{N,1} & & \gamma_{N,2} & & \cdots & & \gamma_{N,N-1} & & \gamma_{NN}\\
& \gamma_{N-1,1} & & \gamma_{N-1,2} & & \cdots & & \gamma_{N-1,N-1} &
\end{array}$$

• Introduce the dual Baxter operator with the kernel

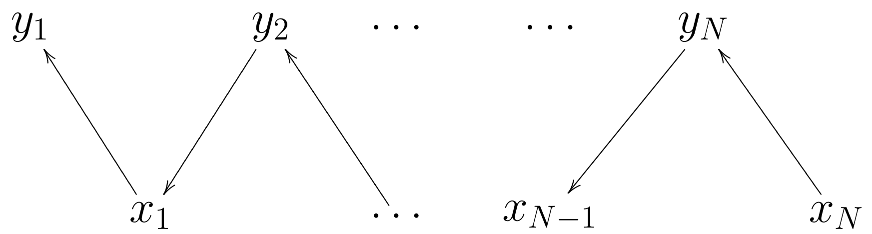
$$\overset{\vee}{Q}{}^{\mathfrak{gl}_N}(\underline{\lambda};\underline{\gamma}|z) \;=\; e^{\imath z\sum\limits_{i=1}^{N}(\gamma_i-\lambda_i)}\prod_{i,j=1}^{N}\Gamma(\imath\gamma_i-\imath\lambda_j)\,.$$

The following holds:

$$\overset{\vee}{\mathcal{Q}}{}^{\mathfrak{gl}_N}(z)*\Psi^{\mathfrak{gl}_N}_{\underline{\gamma}}(\underline{x}) \;=\; \int\limits_{\mathcal{S}_N} d\underline{\gamma}\;\overset{\vee}{Q}{}^{\mathfrak{gl}_N}(\underline{\lambda};\underline{\gamma}|z)\,\Psi^{\mathfrak{gl}_N}_{\underline{\gamma}}(\underline{x}) \;=\; e^{-e^{x_N-z}}\,\Psi^{\mathfrak{gl}_N}_{\underline{\gamma}}(\underline{x})\,.$$

• Introduce the Baxter Q-operator with integral kernel

$$Q(\underline{x},\underline{y}|\lambda) \;=\; 2^N\exp\Big\{\sum_{i=1}^{N}\imath\lambda\big(y_i-x_i\big)-\frac{1}{\hbar}\sum_{i=1}^{N-1}\Big(e^{2(y_i-x_i)}+e^{2(x_i-y_{i+1})}\Big)$$
$$-\frac{1}{\hbar}e^{2(x_N-y_N)}\Big\}\,,$$



Theorem 1 *Consider the K-biinvariant function on G*

$$\phi_{Q(\lambda)}(g) \;=\; 2^N\,|\det(g)|^{\,\imath\lambda-\frac{N-1}{2}}\;\exp\Big\{-\frac{1}{\hbar}\,\mathrm{Tr}(g^tg)\Big\}\,.$$

Then for the K-invariant Whittaker function $\Phi^{\mathfrak{gl}_N}_{\underline{\lambda}}(g)$:

$$\int\limits_G dg\;\phi_{Q(\lambda)}(g^{-1}h)\,\Phi^{\mathfrak{gl}_N}_{\underline{\lambda}}(g) \;=\; \mathcal{Q}(\lambda)*\Phi_{\underline{\gamma}}(\underline{x}) \;=\; L_Q(\lambda)\,\Phi^{\mathfrak{gl}_N}_{\underline{\gamma}}(\underline{y})\,,$$

where

$$L_Q(\lambda) \;=\; \prod_{i=1}^{N}\hbar^{\frac{\lambda-\gamma_i}{2}}\;\Gamma\Big(\frac{\lambda-\gamma_i}{2}\Big)$$

• **Macdonald polynomials.** Given a partition $\Lambda = (\Lambda_1 \geq \ldots \geq \Lambda_N \geq 0) \in \mathbb{Z}_+^N$, it labels two types of bases in $\mathfrak{S}_N(\underline{x})$:

1. $$\mu_\Lambda \;=\; \sum_{\sigma\in\mathfrak{S}_N} x_{\sigma(1)}^{\Lambda_1} x_{\sigma(2)}^{\Lambda_2} \cdot \ldots \cdot x_{\sigma(N)}^{\Lambda_N}$$

2. $$\pi_\Lambda \;=\; \prod_{i=1}^{N} \pi_{\Lambda_i} \qquad \text{with} \qquad \pi_n \;=\; \sum_{i=1}^{N} x_i^n$$

On symmetric functions over $\mathbb{Q}(q,t)$ define a pairing:

$$\big\langle \pi_\Lambda,\, \pi_{\Lambda'} \big\rangle_{q,t} \;=\; \delta_{\Lambda,\,\Lambda'} \prod_{n\geq 1} n^{m_n} m_n! \prod_{\lambda_k\neq 0} \frac{1-q^{\Lambda_k}}{1-t^{\Lambda_k}}\,,$$

where $m_n = \big|\{k \,|\, p_k = n\}\big|$.

Definition 2 *Macdonald polynomials* $P_\Lambda(\underline{x}|q,t)$ *are the symmetric functions over* $\mathbb{Q}(q,t)$ *such that*

$$P_\Lambda(\underline{x}|q,t) \;=\; \mu_\Lambda \,+ \sum_{\Lambda'\preceq\Lambda} u_{\Lambda',\,\Lambda}\, \mu_{\Lambda'}\,, \qquad\qquad u_{\Lambda',\,\Lambda} \in \mathbb{Q}(q,t)\,,$$

$$\big\langle P_\Lambda(\underline{x}),\, P_{\Lambda'}(\underline{x}) \big\rangle_{q,t} \;=\; 0\,, \qquad\qquad \Lambda' \neq \Lambda$$

Basic properties:

1. The (q,t)-deformation of the Cauchy identity:

$$\prod_{i=1}^{N}\prod_{j=1}^{N}\prod_{n=0}^{\infty} \frac{1-tx_iy_jq^n}{1-x_iy_jq^n} \;=\; \sum_{\underline{p}} P_\Lambda(\underline{x})\, P_\Lambda^*(\underline{y})\,, \qquad (4)$$

where

$$P_\Lambda^*(\underline{y}) \;=\; \frac{P_\Lambda(\underline{y})}{\langle P_\Lambda,\, P_\Lambda\rangle_{q,\,t}};$$

2. Eigenvalue property:

$$H_r \cdot P_\Lambda(\underline{x}) = c_r(q^\Lambda)\, P_\Lambda(\underline{x})\,, \qquad c_r(q^{\underline{\lambda}}) = \sum_{I_r} \prod_{i\in I_r} q^{p_i}\; t^{\varrho_i}\,,$$

with the Macdonald difference operators:

$$H_r = \sum_{I_r} t^{\frac{r(r-1)}{2}} \prod_{i\in I_r,\, j\notin I_r} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I_r} q^{x_i \frac{\partial}{\partial x_i}}\,, \tag{5}$$

where $I_r = (i_1 < i_2 < \ldots < i_r)$ for $r = 1, \ldots, N$.

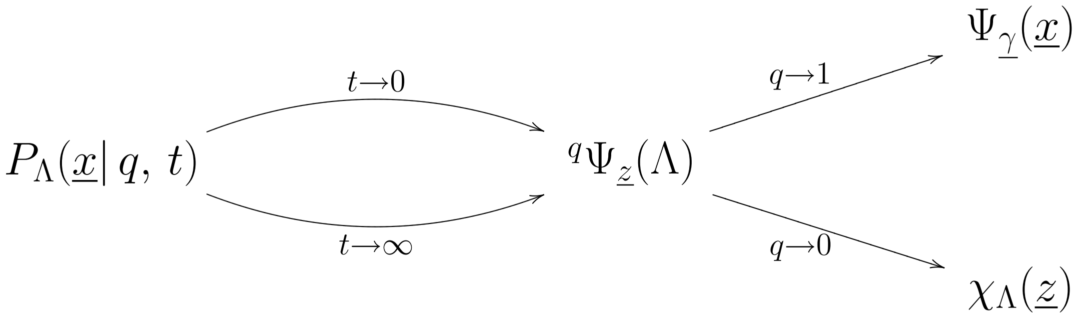
3. For a normalized Macdonald function

$$\mathcal{P}_\Lambda(\underline{x}|q,t) \;:=\; t^{\sum_{i=1}^{N}\Lambda_i\rho_i} \prod_{i<j} \prod_{n=0}^{\infty} \frac{1 - t^2 q^{\Lambda_i - \Lambda_j + n}}{1 - tq^{\Lambda_i - \Lambda_j + n}}\;\; P_{\underline{\lambda}}(\underline{x};q,t)\,,$$

the following self-duality holds:

$$\mathcal{P}_\Lambda(q^{\Lambda' - k\rho}; q,t) \;=\; \mathcal{P}_{\Lambda'}(q^{\Lambda - k\rho}; q,t)\,. \tag{6}$$

Relation to (q-deformed) Whittaker functions:



with

$$x_i \;=\; q^{\Lambda_i}\,, \qquad z_i \;=\; q^{\gamma_i}\,, \qquad\qquad i = 1, \ldots, N\,.$$

• **q-Whittaker function: the First formula.**

For $q<1$ consider the limit $t=q^{-k}\to\infty,\ k\to\infty$.

$$\mathcal{P}^{(N)}\ :=\ \Big\{p_{k,i}\,\Big|\,p_{k+1,i}\geq p_{k,i}\geq p_{k+1,i+1}\Big\}\subset\mathbb{Z}^{N(N-1)/2}$$

$$\mathcal{P}_{N,N-1}:=\Big\{\underline{p}_{N-1}\,\Big|\,p_{N,i}\geq p_{N-1,i}\geq p_{N,i+1}\Big\}\subset\mathcal{P}^{(N)}$$

Theorem 2 *The q-Whittaker function ${}^q\Psi_{\underline{z}}(\underline{p}_N)$ reads*

(I) For $\underline{p}_N$ being in the dominant domain $p_{N,1}\geq\ldots\geq p_{NN}$

$$ {}^q\Psi_{\underline{z}}(\underline{p}_N)\ =\ \sum_{\underline{p}_k\in\mathcal{P}^{(N)}}\ \prod_{k=1}^{N} z_k^{|\underline{p}_k|-|\underline{p}_{k-1}|} \tag{7}$$

$$\cdot\frac{\prod\limits_{k=2}^{N-1}\prod\limits_{i=1}^{k-1}(p_{k,i}-p_{k,i+1})_q!}{\prod\limits_{1\leq i\leq k\leq N-1}(p_{k+1,i}-p_{k,i})_q!\ (p_{k,i}-p_{k+1,i+1})_q!},$$

(II) When $\underline{p}_N$ is outside the dominant domain ${}^q\Psi_{\underline{z}}(\underline{p})=0$.

$${}^q\Psi_{z_1,\ldots,z_N}(\underline{p}_N)\ =\sum_{\underline{p}_{N-1}\in\mathcal{P}_{N,\,N-1}}\Delta(\underline{p}_{N-1})\ z_N^{|\underline{p}_N|-|\underline{p}_{N-1}|}\ \ {}^qQ^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}\ \ {}^q\Psi_{z_1,\ldots,z_{N-1}}(\underline{p}_{N-1}),$$

$$Q^{\mathfrak{gl}_N}_{\mathfrak{gl}(N-1)}(\underline{p}_N,\underline{p}_{N-1}|\,q)\ =\ \frac{1}{\prod\limits_{i=1}^{N-1}(p_{N,i}-p_{N-1,i})_q!\ (p_{N-1,i}-p_{N,i+1})_q!}$$

$$\Delta(\underline{p}_{N-1})\ =\ \prod_{i=1}^{N-2}(p_{N-1,i}-p_{N-1,i+1})_q!$$

Example: The case $U_q(\mathfrak{gl}_2)$.

$${}^q\Psi^{\mathfrak{gl}_2}_{z_1,\,z_2}(p_1,\,p_2)\ =\ \begin{cases}(z_1z_2)^{p_2}\sum\limits_{p=p_2}^{p_1}\frac{z_1^{p_1-p}\,z_2^{p-p_2}}{(p_1-p)_q!\,(p-p_2)_q!}\,, & p_1\geq p_2\\[2ex] 0\,, & p_1<p_2\end{cases}$$

• Gelfand-Zetlin formula for dominant $\underline{p} = (p_1 \geq \ldots \geq p_N)$:

$$\lim_{q\to 0} {}^q\Psi_{\underline{z}}(\underline{p}) \;=\; \sum_{\underline{p}_k \in \mathcal{P}^{(N)}} \; \prod_{k=1}^{N} z_k^{|\underline{p}_k| - |\underline{p}_{k-1}|} \;=\; \chi_{\underline{p}}(\underline{z})\,. \qquad (8)$$

• In the limit $q \to 1$ consider ${}^q\widetilde{\Psi}_{\underline{z}}(\underline{p}) := \Delta(\underline{p}_N)\,{}^q\Psi_{\underline{z}}(\underline{p})$

$$\lim_{q\to 1} {}^q\widetilde{\Psi}_{\underline{z}}(\underline{p}) \;=\; \chi_N(\underline{z})^{p_{NN}} \prod_{k=1}^{N-1} \chi_k(\underline{z})^{p_{N,k} - p_{N,\,k+1}} \;=\; \mathrm{Tr}_{\,V_{\mathrm{f}}} \prod_{i=1}^{N} z_i^{E_{ii}}$$

$$V_{\mathrm{f}} \;=\; V_{\varpi_N}^{\otimes p_{NN}} \bigotimes_{k=1}^{N-1} V_{\varpi_k}^{\otimes (p_{N,k} - p_{N,\,k+1})} \qquad (9)$$

• For $w \in \widehat{W}$ the subspace $V^{[w\cdot\varpi]}(\varpi) \subset V(\varpi)$ is 1-dim. Demazure module $V_w(\varpi) := U(\mathfrak{b}) \cdot V^{[w\cdot\varpi]}(\varpi) \subset V(\varpi)$

$$\mathrm{ch}_{V_w(\varpi)} \;=\; \sum_{\mu\in P} \dim V_w^{[\mu]}(\varpi)\, e^{\mu} \;=\; \mathcal{D}_{s_{i_1}} \cdot \ldots \cdot \mathcal{D}_{s_{i_m}} \cdot e^{\varpi}$$

where $\mathcal{D}_{s_i} = \frac{1 - e^{-\alpha_i}\, s_i}{1 - e^{-\alpha_i}}$, and $w = s_{i_1} \cdots s_{i_m}$. Identify a set of $\widehat{W}$-orbits in $\dot{P}$ with $\mathbb{Z} \times (\mathbb{Z}/\mathbb{Z}_N)$, and for $0 < i < N$ let

$$\varpi_{k,i} \;:=\; k\varpi_N \,+\, \varpi_i \;=\; (\underbrace{k+1, \ldots, k+1}_{i}, k, \ldots, k)$$

be the representatives of the $\widehat{W}$-orbits. Besides, let us introduce the homomorphism $\pi:\, \mathbb{Z}[A] \to \mathbb{Z}[q; z_1, \ldots, z_N]$ defined by $\pi(e^{\varpi_0}) = 1$, $\pi(e^{\varpi_k}) = z_1 \cdots z_k$, $0 < k \leq N$, and $\pi(e^{\delta}) = q$.

Theorem 3 *Let $\widehat{\varpi}_{k,i} := \varpi_0 + \varpi_{k,i}$, and let $w_{k,i} \in \widehat{W}$ be such that the projection of $w_{k,i} \cdot \widehat{\varpi}_{k,i}$ onto P is anti-dominant. Let $\Lambda_{k,i} := w_0 \cdot \varpi_{k,i}$; then the following holds.*

$${}^q\widetilde{\Psi}_{\underline{z}}(\underline{p}_N) \;=\; q^{\frac{1}{2}(\varpi_{k,i},\,\varpi_{k,i}) - \frac{1}{2}(\Lambda_{k,i},\,\Lambda_{k,i})}\, \pi\big(\, \mathrm{ch}_{V_{w_{k,i}}(\Lambda_{k,i})}\big)$$

where $\underline{p}_N = \Lambda_{k,i}$.

• **q-Whittaker function: the Second formula.** Consider the limit $t \to 0$. Let $\{z_{k,i};\ 1 \le i \le k \le N\}$ with $z_{N,i} := z_i$, $1 \le i \le N$.

Theorem 4 *In the dominant domain* $p_{N,1} \ge \ldots \ge p_{NN}$

$$ {}^{q}\Psi_{\underline{z}_N}(\underline{p}_N) \;=\; \Gamma_q(q)^{\frac{(N-1)(N-2)}{2}} \prod_{1\le j\le n\le N-1} \frac{1}{2\pi\imath}\oint \frac{dz_{n,j}}{z_{n,j}} $$

$$ \cdot \prod_{i\le k}\Big(\frac{z_{k,i}}{z_{k-1,i}}\Big)^{p_{N,k}} \prod_{n=1}^{N-1} \frac{\prod\limits_{i=1}^{n+1}\prod\limits_{j=1}^{n} \Gamma_q\big(z_{n,j}^{-1}z_{n+1,i}\big)}{n!\,\prod\limits_{j\ne m}\Gamma_q\big(z_{n,m}^{-1}z_{n,j}\big)} \qquad (10) $$

where

$$ \Gamma_q(z) \;=\; \prod_{n=0}^{\infty}\frac{1}{1\,-\,zq^n} \;=\; \sum_{n=0}^{\infty}\frac{z^n}{(n)_q!}\,. $$

The second formula also admits a recursion w.r.t. the rank N:

$$ {}^{q}\Psi_{\underline{z}_N}(\underline{p}_N) \;=\; \Gamma_q(q)^{N-2}\prod_{k=1}^{N-1}\oint\frac{dz_{N-1,k}}{2\pi\imath z_{N-1,k}}\Big(\frac{z_{N,1}z_{N,2}\,\cdot\ldots\cdot\,z_{NN}}{z_{N-1,1}\cdots z_{N-1,N-1}}\Big)^{p_{NN}} $$

$$ \cdot{}^{\vee}\Delta(\underline{z}_{N-1})\ {}^{\vee}Q^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}(\underline{z}_N,\underline{z}_{N-1}|q)\ \cdot\ {}^{q}\Psi_{\underline{z}_{N-1}}(\underline{p}'_N) $$

where

$$ {}^{\vee}\Delta(\underline{z}_{N-1}) \;=\; \prod_{k\ne j}\ \Gamma_q\Big(\frac{z_{N-1,k}}{z_{N-1,j}}\Big)^{-1} $$

$$ {}^{\vee}Q^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}(\underline{z}_N,\underline{z}_{N-1}|q) \;=\; \prod_{i=1}^{N}\prod_{k=1}^{N-1}\ \Gamma_q\Big(\frac{z_{N,i}}{z_{N-1,k}}\Big) $$

Example: $\underline{p} \,=\, (n+k,\ k,\ldots,k)$**.**

$$ {}^{q}\Psi^{\mathfrak{gl}_N}_{\underline{z}}(n+k,\ k,\ldots,k) \;=\; (z_1z_2)^{p_2}\frac{1}{2\pi\imath}\oint_{t=0}\frac{dt}{t}\ t^{-n}\prod_{j=1}^{N}\Gamma_q(z_jt)\,. $$

- **Counting holomorphic sections.**

$$\mathcal{M}_d(\mathbb{P}^1,\,\mathbb{P}^{N-1}) \;:=\; \Big\{\text{holomorphic maps } \mathbb{P}^1 \longrightarrow \mathbb{P}^{N-1} \text{ of degree } d\Big\}$$
$$=\;\Big\{\, f^1(y_1,y_2),\,\ldots\,,f^N(y_1,y_2)\,\Big|\,\deg\, f^i(\underline{y}) = d\Big\}\Big/\mathbb{C}^* \;\subseteq\; \mathbb{P}^{N(d+1)-1}$$

Let $G = S^1\times U(N)$, and $\mathcal{L}_k$ is such that $E_{ii}\cdot\mathcal{L}_k = k\mathcal{L}_k$, then

$$\chi_G\big(\mathcal{L}_k(n)\big) \;=\; \sum_{k=0}^{N(d+1)-1} (-1)^k\,\mathrm{Tr}_{H^k(\mathbb{P}^{N(d+1)-1},\mathcal{L}_k(n))}\, e^{\hbar L_0 + \sum\limits_{j=1}^{N}\lambda_j E_{jj}}$$

$$=\; A^{(d)}_{n,\,k} \;=\; (z_1z_2)^k\frac{1}{2\pi\imath}\oint\limits_{t=0}\frac{dt}{t}\,t^{-n}\prod_{j=1}^{N}\prod_{m=0}^{d}\frac{1}{1-z_jtq^m}\,. \qquad (11)$$

Proposition 1 *For $q=e^{\hbar}$ and $z_i=e^{\lambda_i}$ one has*

$$A^{(\infty)}_{n,\,k} \;:=\; \lim_{d\to\infty}\,A^{(d)}_{n,k} \;=\; {}^q\Psi_{\underline{z}}(n+k,\,k,\ldots,k) \qquad (12)$$

For the equivariant cohomology

$$H^*_G\big(\mathbb{P}^{N(d+1)-1}\big) \;=\; \mathbb{C}\big[x,\hbar\big]\Big/\prod_{j=1}^{N}\prod_{m=0}^{d}(x-\lambda_j-\imath\hbar m)$$

the Riemann-Roch-Hirzebruch formula reads

$$\chi_G\big(\mathcal{L}_k(n)\big) \;=\; \big\langle \mathrm{Ch}_G(\mathcal{L}_k(n))\,\mathrm{Td}_G,\;\big[\mathbb{P}^{N(d+1)-1}\big]\big\rangle \qquad (13)$$

where

$$\mathrm{Ch}_G\big(\mathcal{L}_k(n)\big) \;=\; e^{nx\,+\,k(\lambda_1+\ldots+\lambda_N)}\,,$$

$$\mathrm{Td}_G(\mathcal{T}\mathbb{P}^{N(d+1)-1}) \;=\; \prod_{j=1}^{N}\prod_{m=0}^{d}\frac{x-\lambda_j-\imath\hbar m}{\big(1-e^{\lambda_j+\imath\hbar m-x}\big)}\,.$$

Problem: Construct a relevant semiinfinite cohomology theory for (q-deformed) Whittaker functions $\Psi_{\underline{\gamma}}(\underline{x})$ and ${}^q\Psi_{\underline{z}}(\underline{p})$.

• **Parabolic Whittaker functions.** Let $N = \ell + m$ and

$$\mathfrak{b}_+ = \mathfrak{h}^{(m,\,\ell+m)} \oplus \mathfrak{n}_+^{(m,\,\ell+m)}$$

with the N-dimensional commutative subalgebra spanned by

$$H_1 = E_{11} + \ldots + E_{mm}\,, \qquad H_k = E_{1,\,k}, \quad k = 2, \ldots, m\,;$$
$$H_{m+n} = E_{m+k,\,\ell+m}, \qquad n = 1, \ldots, \ell - 1\,,$$
$$H_{\ell+m} = E_{m+1,\,m+1} + \ldots + E_{\ell+m,\,\ell+m}\,,$$

and the subalgebra generated by

$$E_{1,\,m+1}\,, \qquad E_{1,\,\ell+m}\,, \qquad E_{m,\,\ell+m}\,;$$
$$E_{kk}\,, \quad k = 2, \ldots, N-1\,; \qquad E_{j,j+1}\,, \quad j = 2, \ldots, N-2\,.$$

Definition 3 *The* $(m,\, \ell+m)$*-Whittaker function associated to* $(\pi_{\underline{\gamma}},\, \mathcal{V}_{\underline{\gamma}})$ *is given by*

$$\Psi^{(m,\,\ell+m)}_{\underline{\gamma}}(\underline{x}) = e^{-\rho_1 x_1 - \rho_N x_N} \Big\langle \psi_L\,,\, \pi_{\underline{\gamma}}\big(e^{-\sum\limits_{i=1}^{N} x_i H_i}\big)\psi_R \Big\rangle \qquad (14)$$

the Whittaker vectors $\psi_L,\, \psi_R \in \mathcal{V}_{\underline{\gamma}}$ *are defined by*

$$E_{n+1,\,n}\,\psi_L = \frac{1}{\hbar}\,\psi_L\,, \qquad n = 1, \ldots, N-1\,,$$

$$\begin{cases} E_{kk}\psi_R = 0\,, & k = 2, \ldots, N-1 \\ E_{j,j+1}\psi_R = 0\,, & j = 2, \ldots, N-2 \\ E_{1,\,m+1}\psi_R = E_{m,\,\ell+m}\psi_R = 0 & \\ E_{1,\,\ell+m}\psi_R = \frac{(-1)^{\varepsilon(\ell,m)}}{\hbar}\,\psi_R & \end{cases}$$

The Casimir elements $C_r \in \mathcal{ZU}(\mathfrak{gl}_N)$, $r = 1, \ldots, N$ act by

$$\mathcal{H}_r(x, \partial_x)\cdot\Psi^{(m,\,\ell+m)}_{\underline{\gamma}}(x) = \hbar^r\, e^{-\rho_1(x_1 - x_{\ell+m})}\Big\langle \psi_L\, \pi_{\underline{\gamma}}\big(C_r\, g(x)\big)\psi_R\Big\rangle\,,$$
$$= \sigma_r(\underline{\gamma})\, \Psi^{(m,\,\ell+m)}_{\underline{\gamma}}(x)\,.$$

The $\mathcal{L}$-operator

$$\mathcal{L}(x,\partial_x)\cdot\Psi^{(m,\,\ell+m)}_{\underline{\gamma}}(x)\;=\;\sum_{i,j=1}^{N}e_{ij}\,e^{-\rho_1(x_1-x_{\ell+m})}\big\langle\psi_L\,\pi_{\underline{\gamma}}\big(E_{ij}\,g(x)\big)\psi_R\big\rangle\,,$$

Let $P_j=-\hbar\frac{\partial}{\partial x_j}$, $j=1,\ldots,N$, the limit $\hbar\to 0$ implies

$$L(x,P)\;=\;\begin{pmatrix} P_1 & \ldots & P_m & 0 & \ldots & (-1)^{\varepsilon}e^{x_1-x_N}\\ -1 & 0 & \ldots & 0 & \ddots & 0\\ 0 & \ddots & \ddots & \vdots & & \vdots\\ \vdots & \ddots & -1 & 0 & & P_{m+1}\\ \vdots & & \ddots & -1 & 0 & \vdots\\ 0 & \ldots & \ldots & 0 & -1 & P_N \end{pmatrix}$$

- Consider the case $\mathrm{Gr}(1,\,\ell+1)\;=\;\mathbb{P}^{\ell}$ and $N=\ell+1$

Theorem 5 *The* $(1,\,\ell+1)$*-Whittaker function with* $x=x_1,\ x_{>1}=0$ *and* $\varepsilon(1,\,\ell+1)=1+\frac{\ell(\ell-1)}{2}$ *coincides with the generating function of* $S^1\times U(\ell+1)$*-equivariant quantum cohomology of* $\mathbb{P}^{\ell}$*:*

$$\Psi^{(1,\,\ell+1)}_{\underline{\lambda}}(x,0,\ldots,0)\;=\;\frac{1}{2\pi\hbar}\int\limits_{\mathbb{R}+\imath\sigma}dH\;e^{-\frac{\imath}{\hbar}xH}\prod_{j=1}^{\ell+1}\hbar^{\frac{\imath H-\lambda_j}{\hbar}}\,\Gamma\Big(\frac{\imath H-\lambda_j}{\hbar}\Big)$$

$$=\int\limits_{\mathcal{C}}\prod_{k=1}^{\ell}dt\;\;\exp\Big\{-\frac{1}{\hbar}\lambda_{\ell+1}x-e^{x-\sum\limits_{k=1}^{\ell}t_k}-\frac{1}{\hbar}\sum_{k=1}^{\ell}\Big((\lambda_k-\lambda_{\ell+1})t_k+e^{t_k}\Big)\Big\}$$

The corresponding D-module $QH^*_G(\mathbb{P}^{\ell})$ is defined by

$$\Big\{\prod\Big(\hbar\frac{\partial}{\partial x}\,+\,\lambda_j\Big)\,+\,e^{x}\Big\}\cdot\Psi^{(1,\,\ell+1)}_{\underline{\lambda}}(x,0,\ldots,0)\;=\;0$$

• **L-funciton as an equivariant symplectic volume.**

An element of $\mathcal{M}_d(\mathbb{P}^1,\,\mathbb{C}^N)$ is a collection

$$f^i(y)\;=\;\sum_{m=0}^{d} f^i_m\,w^m\,, \qquad\qquad i=1,\dots,N\,.$$

Symplectic form on $\mathcal{QM}_d(\mathbb{C}^N)$ is

$$\Omega\;=\;\frac{\imath}{2}\sum_{i=1}^{N}\sum_{m=0}^{d} df^i_m\wedge d\overline{f^i_m}\;=\;\frac{\imath}{4\pi}\sum_{i=1}^{N}\int_0^{2\pi}\!d\theta\;\chi^i(\theta)\,\overline{\chi^i(\theta)}\,.$$

The momenta of Hamiltonian action of $G=S^1\times U(1)^N$ are

$$H_{U(1)_i}\;=\;-\frac{1}{2}\sum_{m=0}^{d}|f^i_m|^2\;=\;-\frac{1}{4\pi}\int_0^{2\pi}\!d\theta\;|f^i(\theta)|^2\,,\qquad i=1,\dots,N\,;$$

$$H_{S^1}\;=\;-\frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N}\sum_{m=0}^{d} m|f^i_m|^2\;=\;-\frac{1}{4\pi\imath}\sum_{i=1}^{N}\int_0^{2\pi}\!d\theta\;\overline{f^i(\theta)}\cdot\partial_\theta f^i(\theta)\,.$$

The G-equivariant symplectic volume of $\mathcal{M}_d(\mathbb{P}^1,\,\mathbb{C}^N)$:

$$Z^{(d)}(\underline{\lambda},\,\hbar)\;=\;\int_{\mathcal{M}_d} e^{\;\Omega+\hbar H_{S^1}+\sum\limits_{i=1}^{N}\lambda_i H_{U(1)_i}}$$

$$=\;\mathrm{vol}\,U(1)^N\prod_{i=1}^{N}\prod_{m=0}^{d}\frac{1}{\lambda_i+\hbar m}\,. \qquad (15)$$

This is Gaussian integral, when $d\to\infty$ is defined using the Hurwitz zeta for $\mathrm{Re}(s)>1$ and $\arg(an+b)\in(-\pi;\,\pi]$:

$$\zeta_a(s;\,b)\;=\;\sum_{n=0}^{\infty}\frac{1}{\left(an+b\right)^s}\,,\qquad \zeta_a(0;\,b)\;=\;\frac{1}{2}-\frac{b}{a},\quad (16)$$

$$\zeta'_a(0;\,b)\;=\;-\Big(\frac{1}{2}-\frac{b}{a}\Big)\ln a\;+\;\ln\frac{1}{\sqrt{2\pi}}\,\Gamma\Big(\frac{b}{a}\Big)\,.$$

Theorem 6 *The G-equivariant volume $Z(\underline{\lambda},\,\hbar;\,\mu)$ reads:*

$$\int\limits_{\Pi\mathcal{TM}_{hol}(\mathbb{C}^N)} [D^2\chi][D^2\varphi]\; e^{\mu\, S_{\rm eff}} \;=\; \prod_{i=1}^{N} AB^{\frac{\lambda_i}{\hbar}}\;\Gamma\Big(\frac{\lambda_i}{\hbar}\Big)\,, \qquad (17)$$

where $A = A(\hbar;\mu)$, $B = B(\hbar;\mu)$, and

$$S_{\rm eff} \;=\; \frac{1}{4\pi}\int\limits_0^{2\pi} d\theta\;\sum_{i=1}^{N}\Big[\imath\chi^i(\theta)\,\overline{\chi^i(\theta)} \;-\; \overline{\varphi^i(\theta)}\big\{\lambda_i + \hbar\partial_\theta\big\}\varphi^i(\theta)\Big]\,.$$

• **Fixed point calculation.** To the equivariant volume:

$$Z(\lambda_i,\hbar;\,\mu) \;=\; 2\pi\mu\int\limits_{\mathbb{R}} dx\, e^{\mu\,\lambda_i x}\int\limits_{\mathbb{P}\mathcal{M}(D,\mathbb{C})} e^{\,\mu\,(\tilde\Omega(x)\,+\,\hbar\tilde H_{S^1})}$$

apply the Duistermaat-Heckman formula:

$$\int\limits_{\mathbb{P}\mathcal{M}(D,\mathbb{C})} e^{\,\mu\,(\tilde\Omega(x)\,+\,\hbar\tilde H_{S^1})} \;=\; \sum_{p_n\in\mathbb{P}\mathcal{M}^{S^1}} \frac{e^{\,\mu\hbar\,\tilde H_{S^1}(p_n)}}{\det_{T_{p_n}\mathbb{P}\mathcal{M}}\big(\frac{\hbar}{2\pi}\,\hat v_{S^1}\big)}\,. \qquad (18)$$

$$p_n \;=\; \varphi_n e^{\imath\, n\theta}\,, \qquad \tilde H_{S^1}(p_n) \;=\; nx\,, \qquad n = 0,\ldots,\infty\,,$$

$$\ln\det_{T_{p_n}\mathbb{P}\mathcal{M}}\Big(\frac{\hbar}{2\pi}\,\hat v_{S^1}\Big) \;=\; \Big[\ln\prod_{\substack{m=0\\ m\neq n}}^{\infty}\frac{\hbar}{2\pi}(m-n)\Big]_a$$

$$:= \;-\frac{\partial}{\partial s}\Big[\sum_{m=1}^{n} e^{-\imath\pi\, s}\Big(\frac{a\hbar m}{2\pi}\Big)^{-s} \;+\; \sum_{m=1}^{\infty}\Big(\frac{a\hbar m}{2\pi}\Big)^{-s}\;\Big]_{s\to 0}$$

Using the Riemann zeta values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = \ln\frac{1}{\sqrt{2\pi}}$

$$Z(\lambda_i,\hbar;\,\hbar^{-1}) \;=\; \mu\sqrt{a\hbar}\int\limits_{\mathbb{R}} dx\, e^{\mu\,\lambda_i x}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{2\pi}{a\hbar}\,e^{\mu\hbar x}\Big)^n$$

$$= \;\sqrt{\frac{2\pi}{\hbar}}\int_{\mathbb{R}} dx\;\exp\Big\{\,\frac{1}{\hbar}\big(\lambda_i x \;-\; e^{\,x}\big)\Big\} \;=\; \sqrt{\frac{2\pi}{\hbar}}\;\hbar^{\frac{\lambda_i}{\hbar}}\;\Gamma\Big(\frac{\lambda_i}{\hbar}\Big)\,.$$

• **Parabolic Whittaker function as a symplectic equivariant volume.**

The above considerations can be extended to parabolic Whittaker function.

Theorem 7 *The following holds:*

$$\Psi^{(1,\,\ell+1)}_{\underline{\lambda}}(x) \;=\; \int\limits_{\mathcal{M}(D,\mathbb{P}^{\ell})} e^{\tilde{\Omega}(x)\,+\,\hbar\tilde{H}_{S^1}\,+\,\sum\limits_{i=1}^{\ell+1}\lambda_i\tilde{H}_i}\,. \qquad (19)$$