# Embedding into BD spaces and spaces with very few operators.

S.A. Argyros, D. Freeman, R. Haydon, E. Odell, Th. Raikoftsalis, Th. Schlumprecht, and D. Zisimopoulou

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**Theorem (Lewis, Stegall 1973)**

*If a $\mathcal{L}_\infty$ space $X$ has a separable dual $X^*$, then $X^*$ is isomorphic to $\ell_1$.*

# Embedding into isomorphic preduals of $\ell_1$

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3. *If $X$ is reflexive then $X$ embeds into an isomorphic predual of $\ell_1$ which is somewhat reflexive.*

# Embedding into spaces with very few operators

**Theorem (Argyros, F, Haydon, Odell, Raikoftsalis, Schlumprecht, Zisimopoulou 2012)**

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*Let $X$ be a separable uniformly convex Banach space. $X$ embeds into a Banach space $Z$ such that $Z$ is an isomorphic predual of $\ell_1$, and $Z$ has very few operators. That is, every operator on $Z$ is equal to a scalar times the identity plus a compact operator.*

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Assume there exists some constant $C \geq 1$ such that

$$\left\|(x, U_n(x), U_{n+1}(x, U_n(x)), U_{n+2}(...), ...)\right\|_\infty \leq C\left\|x\right\|_\infty \quad \forall x \in \ell_\infty(\cup_{i=1}^n \Delta_i)$$

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# General Bourgain Delbaen construction continued

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*For all $\gamma \in \Delta_{n+1}$ there exists constants $a_\gamma, b_\gamma \in \mathbb{R}$, an integer $1 \leq k < n$, an element $\eta \in \Delta_k$ and a functional $b^* \in B_{\ell_1(\cup_{i=1}^{n-1})}$ such that:*

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2. $|a_\gamma| \leq 1$ *and* $|b_\gamma| \leq 1/4$ *or* $a_\gamma = 0$ *and* $|b_\gamma| \leq 1$
3. $b^*(J_n(x)) = 0$ *for all* $x \in \ell_\infty(\cup_{i=1}^k \Delta_i)$

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We use the $c$-decomp. of a countable subset of $B_{X^*}$ to create a B-D space containing $X$.

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We define the embedding $\phi : X \to Y \subset \ell_\infty(\cup_{i=1}^{\infty}\Delta_i)$ by:

$$\phi(x)(\gamma) = \sum_{i=1}^{m} x_i^*(x) \quad \text{where } \gamma = (x_1^*, ..., x_m^*)$$

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Define: $e_\gamma^* = u_\gamma^* + d_\gamma^*$. Note that $u_\xi^*$ has the same form as $u_\gamma^*$!

After repeatedly substituting, we obtain the evaluation analysis of $\gamma$:

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We may choose $m_1 = c$. If $X$ is uniformly convex then there exists $n_1 \in \mathbb{N}$ such that if $x^* \in B_{X^*}$ then $x^*$ has a c-decomposition $(x_1^*, x_2^*, ..., x_a^*)$ with $a \leq n_1$.

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$$e^*_{(cx_1^*, cx_2^*, \ldots, cx_a^*)} = \sum_{i=1}^{a} d^*_{(cx_1^*, cx_2^*, \ldots, cx_i^*)} + c \sum_{i=1}^{a} \frac{\|x_i^*\|}{c} e^*_{(cx_i^* / \|x_i^*\|)}$$

We may choose $m_1 = c$. If $X$ is uniformly convex then there exists $n_1 \in \mathbb{N}$ such that if $x^* \in B_{X^*}$ then $x^*$ has a c-decomposition $(x_1^*, x_2^*, ..., x_a^*)$ with $a \leq n_1$.Thus FOS fits the setup:

$$\begin{aligned} u_\gamma^* &= m_j^{-1} b^* && \text{or} \\ u_\gamma^* &= e_\xi^* + m_j^{-1} b^* && \text{and weight of } \xi \text{ is } m_j \end{aligned}$$

After repeatedly substituting, we obtain the analysis of $\gamma$:

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