# Spectral theory for differential operators with indefinite weights

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BIRS Workshop

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$r, p^{-1}, q \in L^1_{\mathrm{loc}}(a,b)$ real functions, $r \neq 0$ a.e.

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$$\Psi(z,\nu) = e^{-\lambda z}f(\nu) \quad\Rightarrow\quad \frac{1}{r}\left(-\frac{d}{d\nu}p\frac{d}{d\nu}f + qf\right) = \lambda f$$

# Differential operators associated to $\ell = \frac{1}{r}\left(-\frac{d}{dx} p \frac{d}{dx} + q\right)$

Weighted $L^2$ **Hilbert space**:

$$L^2_{|r|}(a,b) := \left\{ f : (a,b) \to \mathbb{C} \text{ measurable } : \int_a^b |f|^2 |r| dx < \infty \right\}$$

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**The maximal and minimal operator are defined as**

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- $r \not> 0$: $T_{\min}$ **NOT** symmetric in Hilbert space $L^2_{|r|}(a,b)$

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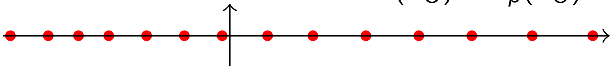

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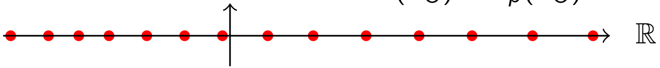


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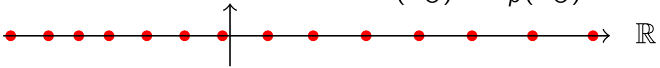


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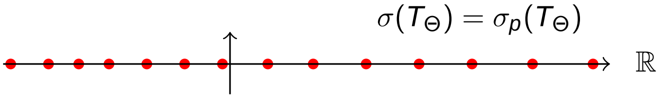


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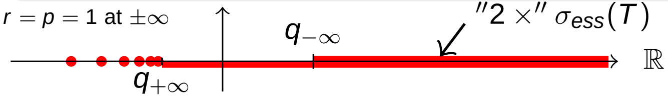

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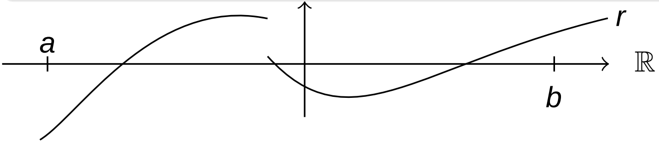

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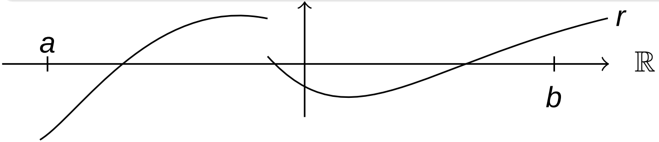


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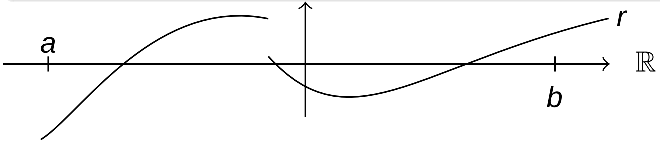


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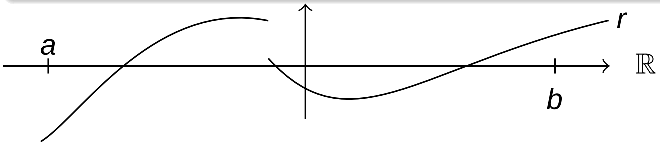


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- $L^2_r(a,b) := (L^2_{|r|}(a,b), [\cdot,\cdot])$ is a **Krein space**

Regular case: $(a, b)$ finite interval, $\frac{1}{p}, q, r \in L^1(a, b)$

# Regular indefinite Sturm-Liouville operators

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**Proposition**

For $2 \times 2$-matrices $\Theta = \Theta^*$,

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- $\lambda \in \sigma(B) \Rightarrow \bar{\lambda} \in \sigma(B)$

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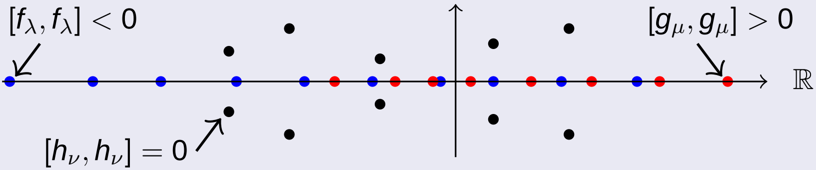
**Theorem [Curgus, H. Langer 89]: YES. YES.**



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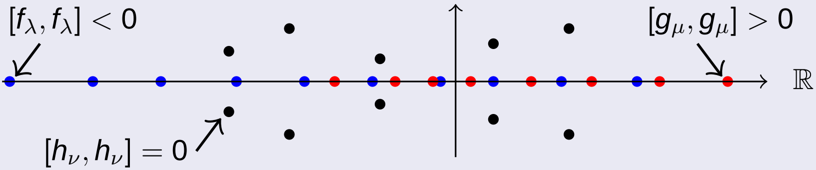


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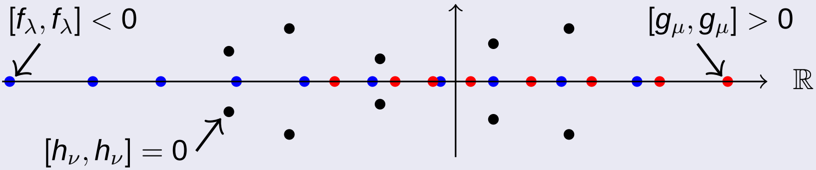


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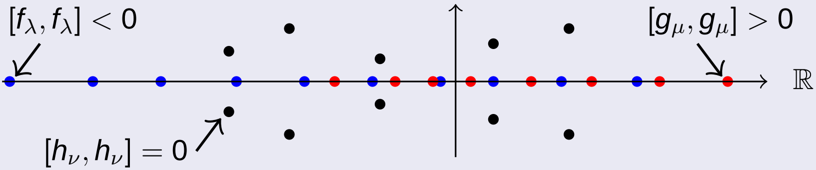


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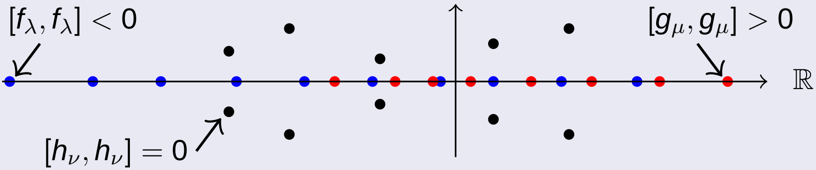


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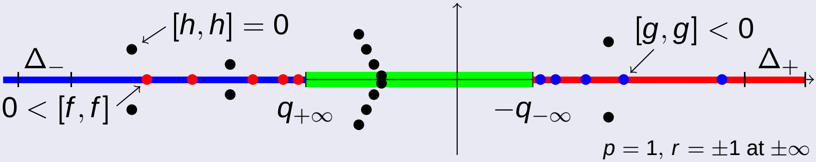

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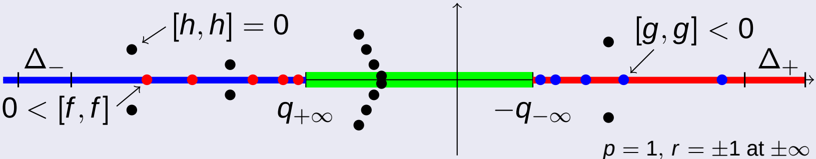


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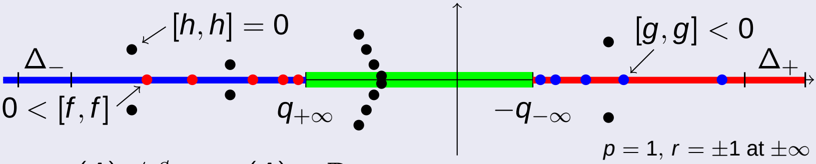


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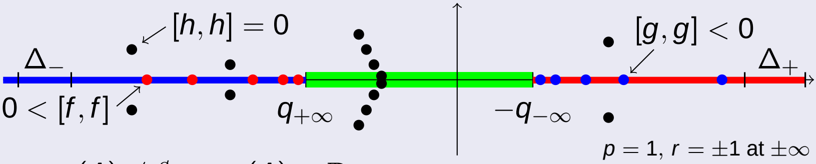


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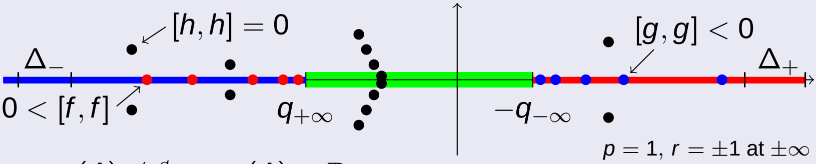


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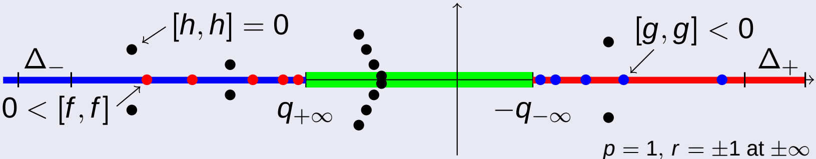


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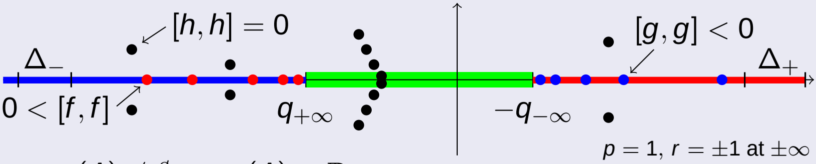


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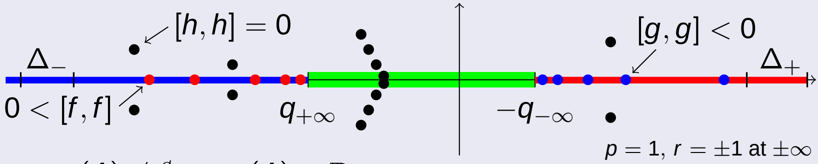


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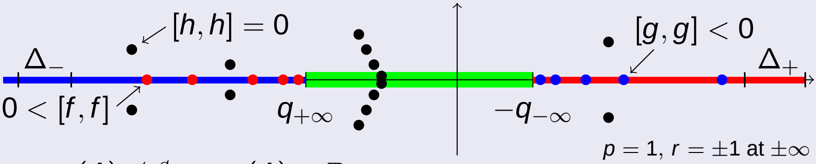


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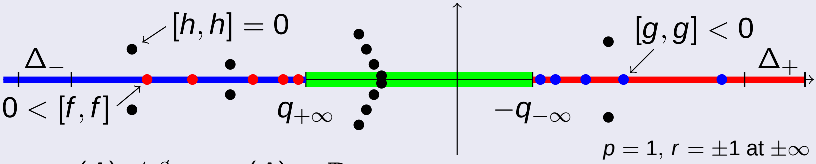


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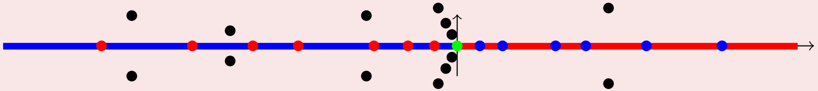

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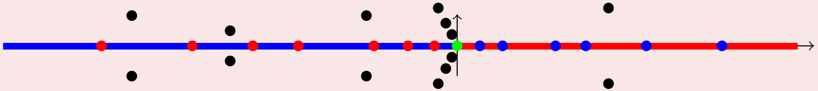

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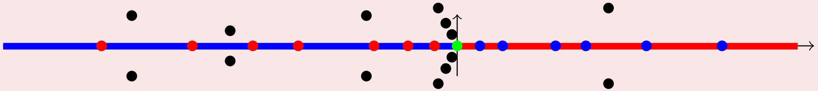

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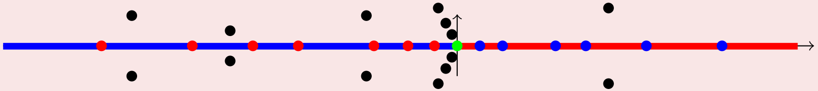

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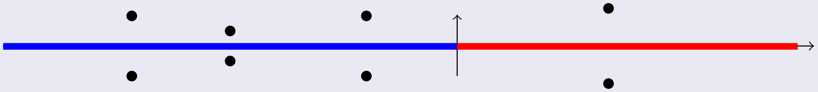

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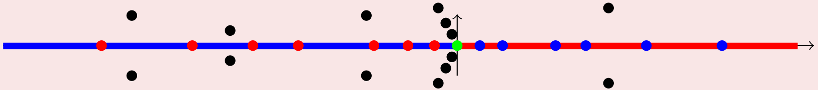

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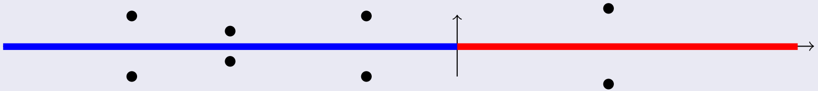

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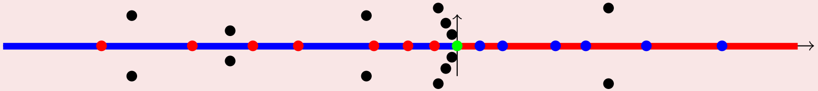

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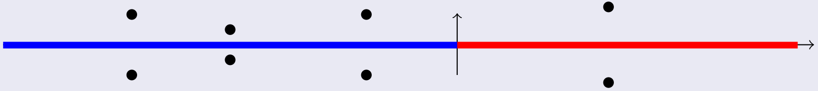

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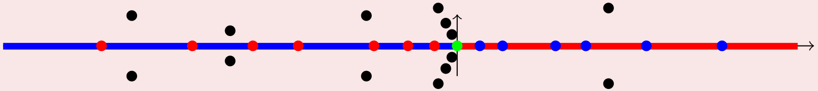

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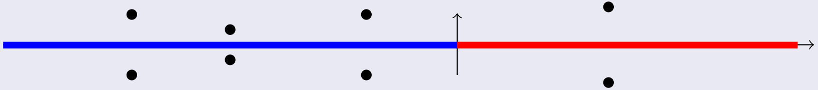

- Nonreal EVs do **NOT** accumulate to 0
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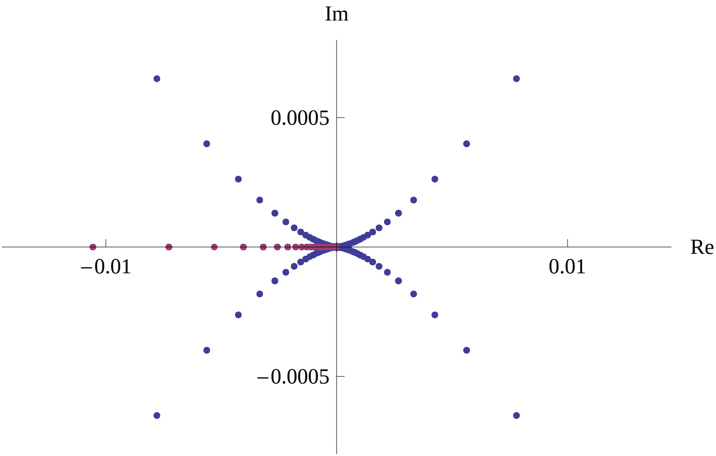
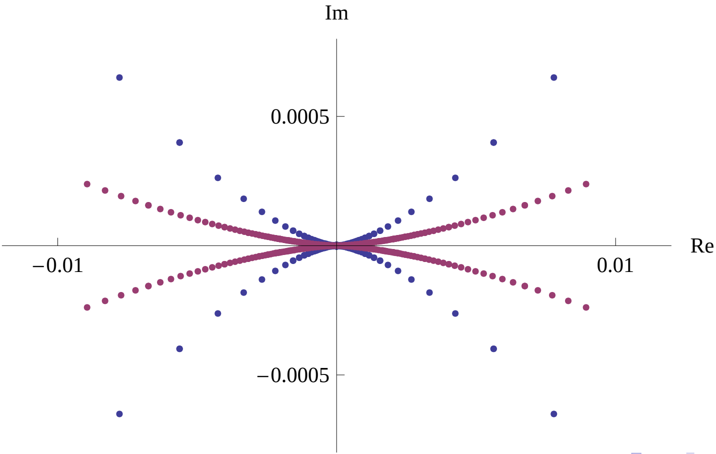
- Number of nonreal EVs can be described exactly with Titchmarsh-Weyl functions

$$\begin{aligned} Af &= \operatorname{sgn}(\cdot)\big(-f'' + qf\big) \\ Tf &= -f'' + qf \end{aligned}, \qquad q(x) = \frac{-1}{1+|x|}$$



# Examples: Nonreal eigenvalues [B., Katatbeh, Trunk 08]

$$Af = \operatorname{sgn}(\cdot)\big(-f'' + qf\big), \quad q(x) = \begin{cases} \frac{-1}{1+|x|} & \text{blue} \\ \frac{-5}{1+|x|} & \text{purple} \end{cases}$$



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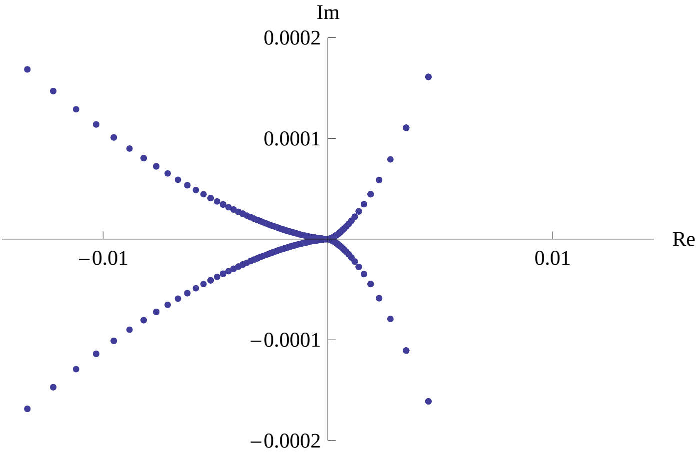

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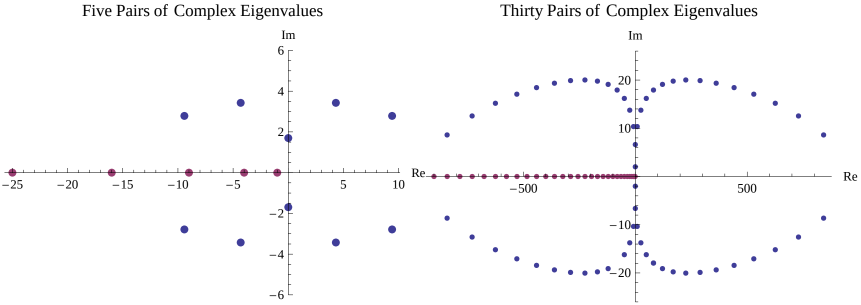


- Blue points: Nonreal eigenvalues of $A$ for $n = 5$ and $n = 30$
- Purple points: Negative eigenvalues of $Tf = -f'' + qf$

# Spectrum of $A = \frac{1}{r}(-\frac{d}{dx}(p\frac{d}{dx}) + q)$ with gap

**Theorem: [Curgus,Langer 89], [B.,Möws,Trunk 11]**

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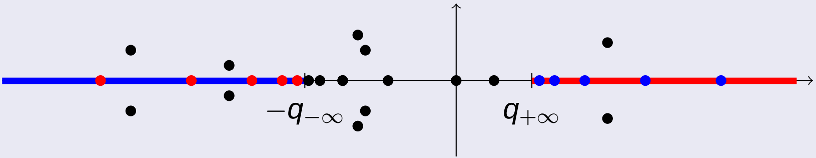

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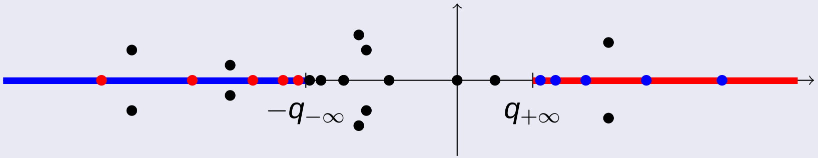


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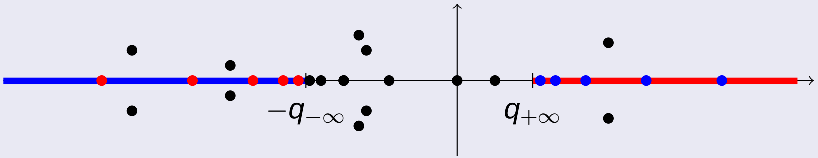


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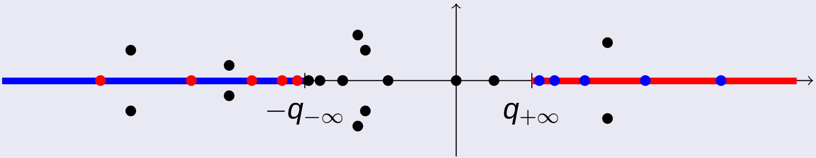


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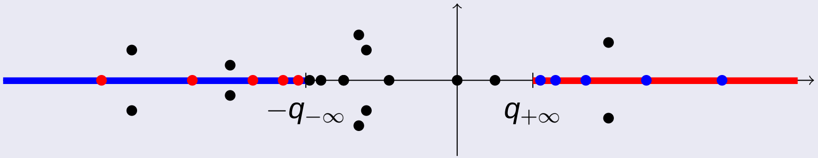


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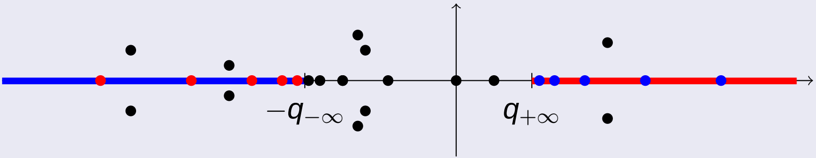


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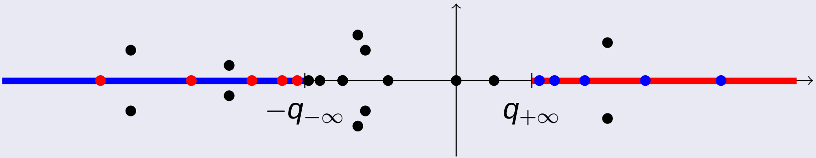


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[Curgus,Najman 95],[Binding,Browne 99],[Binding,Browne,Watson 02] [Karabash 06]

# Typical spectra: Definite vs. indefinite case

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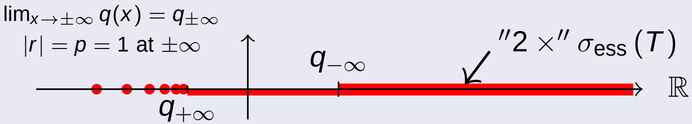


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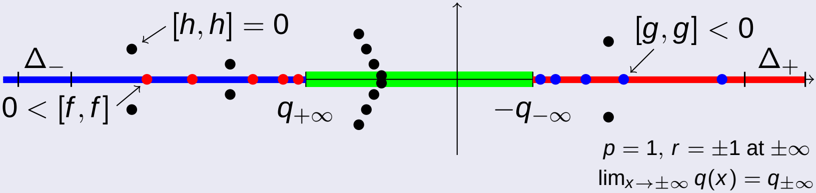

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Nevertheless, it turns out that the **Conjecture is true**

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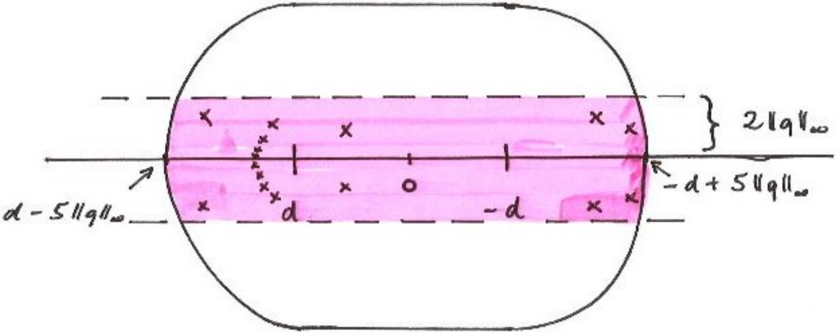

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...can also be applied to elliptic PDOs with indefinite weights...

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