Stability Analysis and Bifurcations of the Hip-Hop orbit

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Outline

Newtonian symmetric N-body problem

- Configuration space (without collisions)

$$\mathcal{X} := \{q \in R^{3N} \mid q_i \neq q_j, \forall i \neq j, q_1 + \ldots + q_N = 0\}$$

- Newtonian gravitational potential

$$U(q) := \sum_{1 \leq i < j \leq N} \frac{\mathcal{G} m^2}{||q_i - q_j||}$$

- Set $p := m(Id)\dot{q}$, then

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, N$$

where H is the *Hamiltonian*:

$$H(q,p) = \sum_{i=1}^{N} \frac{||p_i||^2}{2m} - U(q).$$

Symmetric Hamiltonian systems

Consider a smooth G-reversible equivariant (convex and superlinear) Hamiltonian ODE

$$\dot{x} = J\nabla H(x); \quad H : \mathbb{R}^{2n} \to \mathbb{R}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Let $x = (q, p)$, $q =$configuration, $p =$momentum and define

$$X_H(x) := J\nabla H(x).$$

There exists a representation $\chi : G \to \{\pm 1\}$ such that

$$X_H(g.x) = \chi(g)g.X_H(x).$$

- $\Gamma = \ker \chi$ consists of spatial symmetries
- $G \setminus \Gamma$ are the time-reversing symmetries and

$$G/\Gamma \simeq \mathbb{Z}_2.$$

Lagrangian formulation of Hamiltonian dynamics

Lagrangian: $L(q,\dot{q}) = \sum_{i=1}^{N} \frac{||\dot{q}_i||^2}{2} + U(q)$.

Action functional

$$\mathcal{A}(c(t)) := \int_0^T L(c(t), \dot{c}(t))\, dt.$$

with $c(t) \in H^1([0,T], \mathcal{X})$ with boundary conditions

$$(c(0), c(T)) \in V \subset \mathbb{R}^n \times \mathbb{R}^n; \quad e.g. \quad c(T) = Sc(0).$$

1. $\widehat{c}(t)$ is a critical point of $\mathcal{A}$ if $\delta\mathcal{A}(\widehat{c}(t))[h] = 0$.
2. Critical points of $\mathcal{A}$ are solution trajectories of $\dot{x} = X_H(x)$;

$$H(q,p) = p^T \dot{q} - L(q,\dot{q}).$$

Collisionless Symmetric Periodic Orbits: Hip-Hop

Chenciner-Venturelli (1999): 4-body
Terracini-Venturelli (2007): $2n$-bodies

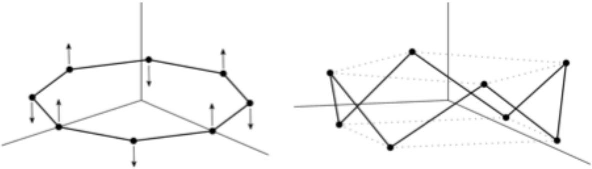
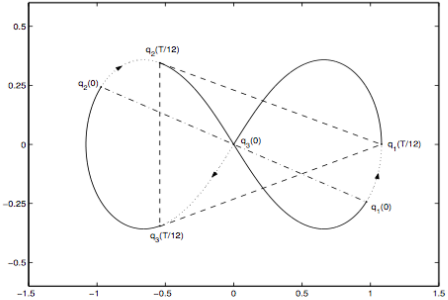


FIGURE 1. Qualitative diagram of the hip-hop configuration for eight masses.

Collisionless Symmetric Periodic Orbits: Figure-eight

Chenciner-Montgomery (2000): Planar 3-body.



Collisionless Symmetric Periodic Orbits

Marchal (2003), Ferrario and Terracini (2004): General criteria for collisionless periodic orbits using symmetry condition.

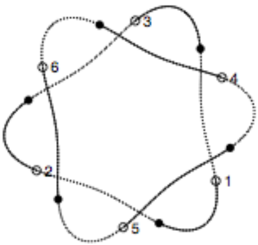


Fig. 7. The planar equivariant minimizer of Example (11.7)

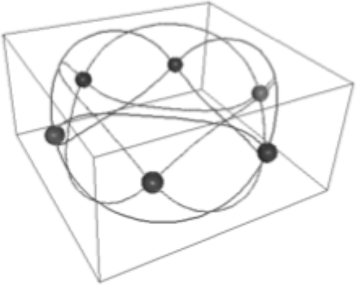
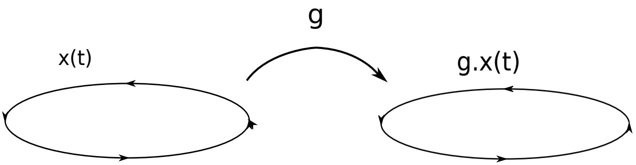


Fig. 8. The three-dimensional equivariant minimizer of Example (11.7)

Symmetries of periodic orbits

1. $x(t)$ is a T-periodic orbit.
2. By unicity of solutions of ODEs

$$\forall g \in G : g.\{x(t)\} \cap \{x(t)\} = \emptyset \quad \text{or} \quad g.\{x(t)\} \cap \{x(t)\} = \{x(t)\}.$$



3. Symmetry group $\Sigma_{x(t)}$: let $\widetilde{G} = G \times \mathbb{R}/[0, T)$

$$\Sigma_{x(t)} := \{(g, \theta) \in \widetilde{G} \mid g.x(t) = x(\chi(g)t + \theta(g))\}.$$

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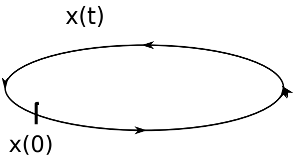

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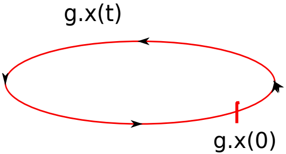

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Stability of Periodic Orbits

1. Offin (1994, 2000): Time-reversing T-periodic orbits.
2. Offin and Cabral (2009): Isosceles three-body problem: spatio-temporal symmetry.
3. G. Roberts (2007): combination of analytic and numerics.
4. Hu and Sun (2009): Maslov index methods: criteria for instability. Figure-eight orbit argument for stability.
5. B. and Offin, in revision.

4-body problem: the Hip-Hop orbit

- Configuration space: $\mathcal{X} \simeq \mathbb{R}^{12}$.
- Thm (Chenciner-Venturelli (1999)): There exists a collisionless $4T$-periodic orbit $\widehat{c}(t)$ minimizing

$$\mathcal{A}_{[-T,T]}(c(t)) = \int_{-T}^{T} L(c(t), \dot{c}(t))\, dt,$$

given $\Lambda = \{c \in H^1([-T,T), \mathcal{X}) \mid c(t-T) = -c(t+T)\}$.

The Hip-Hop orbit

The Hip-Hop orbit is obtained as a special realization of the above minimizer as follows.

- Let $A(x, y, z) = (-y, x, -z)$ and

$$\rho.(q_1, q_2, \ldots, q_{2n}) = (Aq_{2n}, Aq_1, \ldots, Aq_{2n-1})$$

- $\mathbb{Z}_{2n} := \langle (A, \rho) \rangle$ and set $\mathcal{C} := \mathrm{Fix}(\mathbb{Z}_{2n})$.
- Lift symplectically (A, ρ) to $T^*\mathcal{X}$.

Reduced Hip-Hop orbit

Thm (CV (1999), Terracini-Venturelli (2007)): There exists a collisionless $4T$-periodic orbit $\widehat{q}(t)$ minimizing

$$\min \mathcal{A}(q(t)), \quad \text{over}$$

$$\Lambda_{\mathbb{Z}_{2n}} = \{q \in H^1(\mathbb{R}/4T\mathbb{Z}, \mathcal{C}) \mid q(t-T) = q(t+T)\}.$$

The orbit $\widehat{q}(t)$ is not a relative equilibrium, has nonzero angular momentum μ and is not planar.

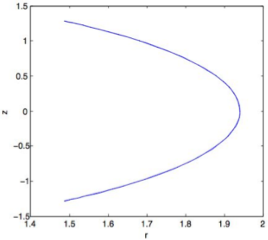
1. On $\mathcal{C}$, the dynamics of all bodies follows the first one.
2. X_H restricted to $T^*\mathcal{C}$ is a 3-degrees of freedom system.
3. **Conjecture**: It is a brake orbit.

Symmetry and Reduction

- $X_H(q,p)$ has reversing-symmetry $G = \mathbf{O}(3) \times \mathbf{S}_{2n} \times \mathbb{Z}_2(S_2)$.
- $W(\mathbb{Z}_{2n}) \simeq (\mathbf{SO}(2) \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2(k) \times \mathbb{Z}_2(S_2)$ acts on $T^*\mathcal{C}$.
- Momentum map: $J : T^*\mathcal{C} \to T_1^*\mathbf{SO}(2)$
- $J^{-1}(\mu)/\mathbf{SO}(2)$ is 4-D with amended potential: $U_\mu(x)$
- $\mathbb{D}_2(S_1, S_2)$ acts on $J^{-1}(\mu)/\mathbf{SO}(2)$ where

$$S_1 = \mathrm{diag}(\sigma, -\sigma) \quad \sigma = \mathrm{diag}(1,-1) \quad \text{and} \quad S_2 = \mathrm{diag}(I, -I).$$

$$\mathcal{X} \to \mathcal{X}/\mathbf{SO}(2) \qquad (r(t), \theta(t), d(t)) \mapsto (r(t), d(t)).$$



Numerical algorithm:

- Truncated Fourier series for $q_i(t)$, $i = 1, 2, 3, 4$.
- Minimisation of a function (discretized integral) depending on Fourier coefficients α, β:

$$\mathcal{G}(\alpha, \beta) := \sum_{j=1}^{k} L(q^f(t_i, \alpha, \beta), \dot{q}^f(t_i, \alpha, \beta)).$$

Brake orbit

Consider $H^{-1}(h)$, the Hill's region is

$$N_h = \{x \in \mathcal{S}/\mathbf{S}^1 \mid U_\mu(x) \leq h\}.$$

A *brake orbit* is an orbit of $X_H|_{H^{-1}(h)}$ which projects to a trajectory of $\mathcal{X}$ in N_h which intersects ∂N_h in two distinct points only.

Theorem (Lewis *et al.* online DCDS-A (2013))

If $\widehat{q}(t)$, $-T \leq t \leq T$, minimizes the action $\mathcal{A}_{[-T,T]}$ on the function space $H^1([-T,T],\mathcal{C})$, then the corresponding loop $x(t)$ in reduced configuration space with $-2T \leq t \leq 2T$, is a brake orbit in the Hill's region

Idea of the proof

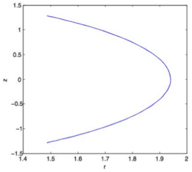


Figure: Symmetric across the horizontal r-axis

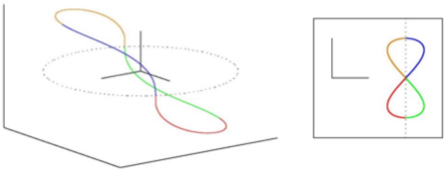


FIGURE 2. An anti-symmetric loop in $T\mathcal{C}$ whose projection in $T(\mathcal{C}/S^1)$ has a transverse self-intersection (inset).

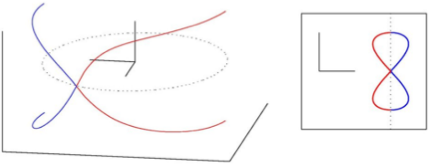


FIGURE 3. The non-smooth curves q_1 and q_2 constructed from the loop of Figure 2, and their projections in $T(\mathcal{C}/S^1)$ (inset).

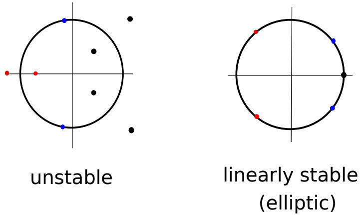
Symplectic matrices

Consider the linearisation of X_H near $x(t)$:

$$\dot{\xi} = dX_H(x(t))\xi, \qquad \xi(0) = Id \tag{1}$$

and let $\gamma(t)$ be the fundamental matrix solution of (1).

- $\gamma(t)$ is symplectic for all $t \in \mathbb{R}$.
- Eigenvalues of symplectic matrices come in quadruplets: $\{\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}\}$.



Lagrangian subspaces

- Consider, $(\mathbb{R}^{2n}, \omega)$ a symplectic space with $\omega(u, v) := u^* Jv$
- A subspace $W \subset V$ is *Lagrangian*

$$\omega|_{W^2} = 0 \quad \text{and} \quad \dim W = n.$$

- Lagrangian Grassmanian $\Lambda(n)$: manifold of all Lagrangian subspaces in $\mathbb{R}^2 n$.
- A symplectic matrix, W Lagrangian subspace $\Rightarrow AW$ Lagrangian subspace.

Maslov index and focal points

- Let $\gamma : [a, b] \to \Lambda(n)$ be a continuous path.
- Maslov index: for $0 < \epsilon << 1$

$$\mu(\alpha, \gamma(t)) := [e^{-\epsilon J}\gamma(t), \overline{\Lambda^1(\alpha)}].$$

- Let W be a Lagrangian subspace, a point $\tau \in (a, b)$ is a *focal point* if

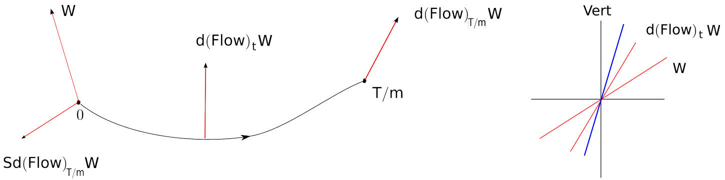
$$d\phi_\tau W \cap Ver \neq \{0\}, \quad \text{where} \quad Ver = \{(0, v)^* \mid v \in \mathbb{R}^n\}.$$

  Ver is a Lagrangian subspace.
- If $\alpha = Ver := \{(0, v)^* \mid v \in \mathbb{R}^n\}$,

$$\mu(d\phi_t W, \alpha) = \#\,\text{focal points}.$$

General Idea

- Offin (1994,2000), Offin and Cabral (2009), B and Offin.



1. Choice of W is crucial.
2. $d\phi_t W \cap V = \{0\},\ t \in [0, T/m]$: Tool - Comp. Thm (Arnol'd 1985)
3. Comp Thm (Offin 2000) + $\delta^2 \mathcal{A}(\widehat{c}(t)) \geq 0$:

$$d\phi_t W \cap V = \{0\} \implies d\phi_t(Sd\phi_{T/m} W) \cap V = \{0\} \text{ on } [0, T/m]$$

Result of Hu and Sun does not apply

Theorems (Hu and Sun (2009))

1. For a critical point $\widehat{c}(t)$ of a variational problem with BC $\overline{S}c(t) = c(t + T/m)$

$$\text{Morse index}(\widehat{c}(t)) + \ker(\overline{S} - I) = \mu(\mathrm{Gr}(S^T), \mathrm{Gr}(\gamma(t)))$$

2. Let $z(t)$ be a periodic solution with spatio-temporal symmetry $Sz(t) = z(t + T/m)$, $S = \mathrm{diag}(\overline{S}, \overline{S})$. If

$$\mu(\mathrm{Gr}(S), \mathrm{Gr}(\gamma(t)))$$

is odd then $z(t)$ is unstable.

Hip-Hop orbit: Morse index$= 0$, $\overline{S} = -I \Rightarrow \mu(\mathrm{Gr}(S), \mathrm{Gr}(\gamma(t))) = 0$.

Instability Theorem

Theorem (Lewis *et al.* online DCDS-A (2013))

The reduced hip hop orbit $z(t)$ is hyperbolic in the energy surface $H^{-1}(h)$ when it is dynamically non-degenerate. If the unreduced variational problem is non-degenerate then the reduced hip hop orbit is (linearly) unstable.

Focal points of Hip-Hop orbit

1. Choice of Lag. subspace: $W = X_H(z(0)) \oplus u$, where $u \in Fix(S_1)$. $W \in TH^{-1}(h)$.

2. The only focal points of W on $[-T, T]$ are the brake point

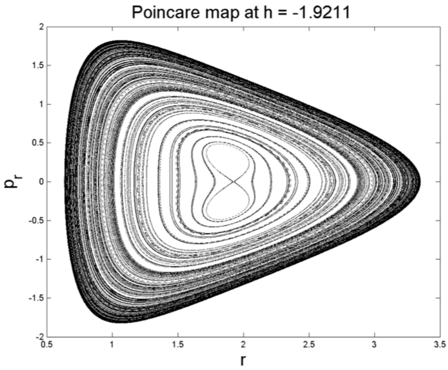
$$X_H(-T) = X_H(T).$$

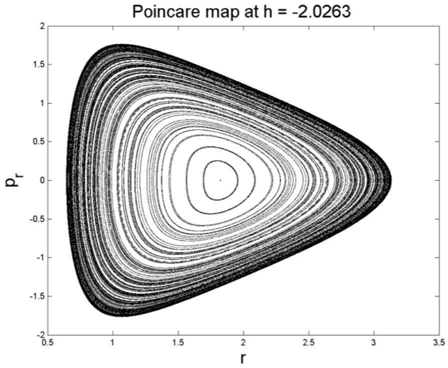
3. $Y = T_{z(t)}H^{-1}(h)/X_H(z)$ is a symplectic space and W' projection of W to Y is a $1D$ Lagrangian subspace of Y with no focal points on $[0, 2T]$. v

4. Consecutive $(Sd\phi_{2T})^n W'$ are transverse and have no focal points in $[0, 2T]$

$$\Rightarrow \quad d\phi_t W' \cap V = \{0\} \qquad 0 \leq t < \infty.$$

Numerical Poincaré map





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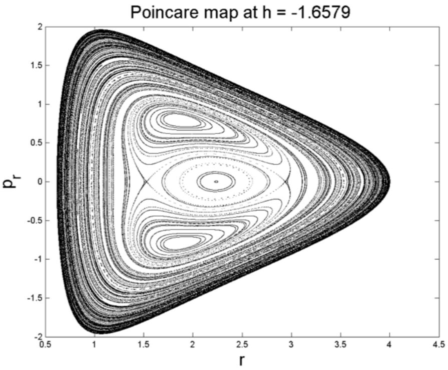
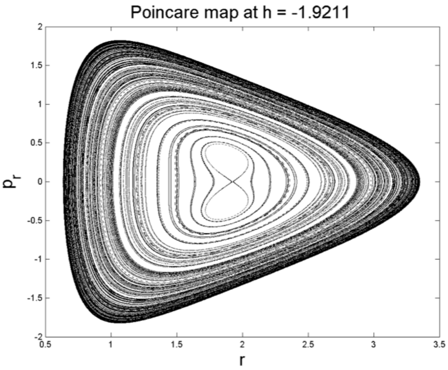
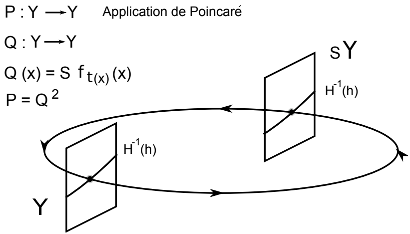




Bifurcations of the Hip-Hop orbit: Buono *et al.* (in prep)



- Y is a S_2-invariant Poincaré section.
- $(dP) = (dQ)^2$ implies suppression of period doubling.
- Q is S_2-reversible: $Q \circ S_2 = S_2 \circ Q_2^{-1}$

$$dQ(0) = \begin{pmatrix} a & b \\ c & a \end{pmatrix}, \qquad \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \text{ or } \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}.$$

1. Classification of symmetry-breaking bifurcations of periodic orbits with $\mathbb{D}_2$-reversing symmetry group (Lamb et al (2003)):

Rev	L_0	K	Δ^{bif}	Σ^{bif}	σ^{bif}	ρ^{bif}
Y	$+1$	$\mathbb{D}_2(S_2, L_0)$	1	$\mathbb{D}_2(S_2, S)$	S	S_2
N	$+1$	$\mathbb{Z}_2(L_0)$	1	$\mathbb{Z}_2(S)$	S	1
Y	-1	$\mathbb{Z}_2(S_2)$	1	$\mathbb{Z}_2(S_2)$	1	S_2
Y	-1	$\mathbb{Z}_2(S_2 L_0)$	1	$\mathbb{Z}_2(S_1)$	1	S_1

2. Bifurcation diagrams: $+1$ eigenvalue (left), -1 eigenvalue (right).

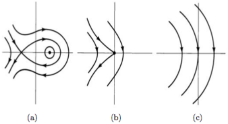
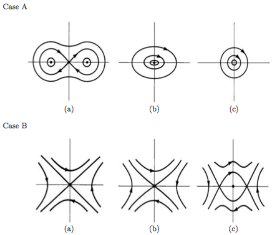
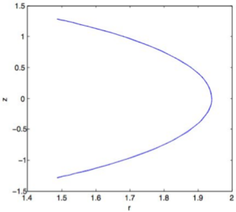


Figure 11.3. Extremal fixed point.





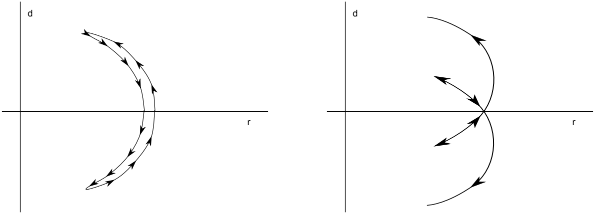


Figure: Left: $\mathbb{Z}_2(S_1)$-symmetric orbit. Right: $\mathbb{Z}_2(S_2)$-symmetric orbit