Stability of Regularized Shock Solutions in Coating Flows

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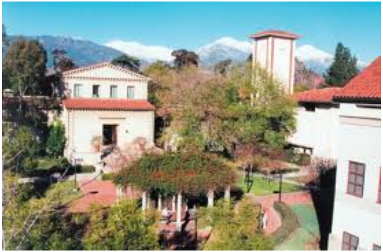

Outline

Thin-films in everyday life for breakfast.

Coating and rimming flows

Consider a horizontal cylinder, rotating about its axis. If there is a fluid on the outside of the cylinder, this is called a coating flow. If the fluid is on the inside of the cylinder, this is called a rimming flow.

"It is a matter of common experience that if a knife is dipped in honey and then held horizontally, the honey will drain off; but that the honey may be retained on the knife by simply rotating it about its length. The question arises: what is the maximum load of honey that can be supported per unit length of knife for a given rotation rate?" — H.K. Moffatt, [*Journal de Méchanique, 1977*].

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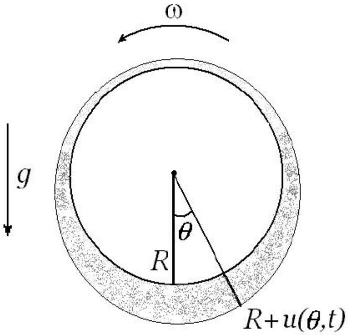
Model parameters

Consider a thin liquid film on the outer surface of a cylinder:

- R is the radius of the cylinder.
- ω is the rate of rotation.
- g is the acceleration due to gravity.
- ν is the kinematic viscosity.
- ρ is the density.
- σ is the surface tension.



Lubrication approximation model

Three dimensionless quantities: $\mathrm{Re} = \frac{R^2\omega}{\nu}$, $\gamma = \frac{g}{R\omega^2}$, and $\mathrm{We} = \frac{\rho R^3\omega^2}{\sigma}$.

Modelling assumptions:

- The fluid flow is modelled by the Navier Stokes equations
- There is no slip at the liquid/solid interface
- There is surface tension at the liquid/air interface
- If $\bar{u}$ is the average thickness of the fluid then $\varepsilon = \bar{u}/R$ is small
- $\chi = \frac{\mathrm{Re}}{We}\varepsilon^3$ and $\mu = \gamma\ \mathrm{Re}\ \varepsilon^2$ have finite, nonzero limits as $\varepsilon \to 0$.

Lubrication approximation model

Assume the flow is constant along the length of the cylinder

Moffatt, [*J. de Mécanique*, 1973] found an evolution equation neglecting surface tension ($\chi = \sigma = 0$):

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \theta}\left(u - \frac{\mu}{3}u^3 \sin(\theta)\right) = 0.$$

Pukhnachov, [*Journal of Applied Mechanics and Technical Physics*, 1977] included surface tension into the model:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \theta}\left(u - \frac{\mu}{3}u^3 \sin(\theta)\right) + \frac{\chi}{3}\frac{\partial}{\partial \theta}\left(u^3\left[\frac{\partial u}{\partial \theta} + \frac{\partial^3 u}{\partial \theta^3}\right]\right) = 0$$

where $\mu = \gamma \, \mathrm{Re} \, \varepsilon^2 = \frac{gR}{\omega\nu}\varepsilon^2$ and $\chi = \frac{\mathrm{Re}}{\mathrm{We}}\varepsilon^3 = \frac{\sigma}{\nu\rho R\omega}\varepsilon^3$.

Both models used periodic boundary conditions.

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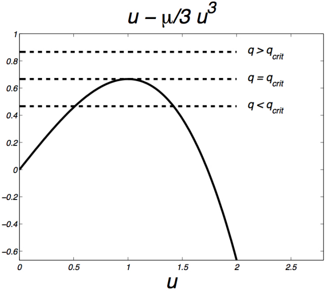
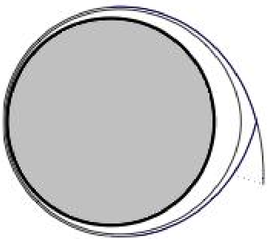
Steady states in Moffatt's model

$$\frac{\partial}{\partial\theta}\left(u - \frac{\mu}{3}u^3 \sin(\theta)\right) = 0 \quad \Longrightarrow \quad u - \frac{\mu}{3}u^3 \sin(\theta) = q$$

At $\theta = \pi/2$: $u(\pi/2)$ is a root of $u - \frac{\mu}{3}u^3 = q$ there might be no positive root if q is too big.





If $q < q_{crit}$ there is a smooth positive steady state, at $q = q_{crit}$ the steady state develops a cusp, if $q > q_{crit}$ a positive steady state does not exist.

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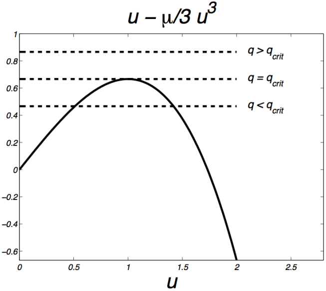

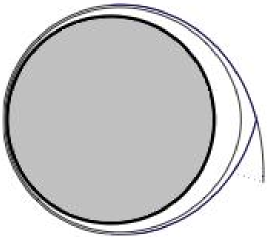

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Steady states in Pukhnachov's model

If there is no surface tension then $q_{crit} = \frac{2}{3\sqrt{\mu}} = \frac{2}{3\varepsilon}\sqrt{\frac{\omega\nu}{gR}}$ The critical flux increases as the ω increases.

Pukhnachov, [*Mathematics and Continuum Mechanics*, 2004] proved that $q_{crit} \leq 2\sqrt{3/\mu} \approx 3.464/\sqrt{\mu}$.

We improve on this:

Nonexistence of steady states

[*Chugunova, Pugh, Taranets, SIAM J. Math. Anal.*, 2010]
For positive surface tension, there is no strictly positive 2π periodic steady state with flux $q > \frac{2}{3}\sqrt{\frac{2}{\mu}} \approx 0.943/\sqrt{\mu}$. Hence $q_{crit} \leq \frac{2}{3}\sqrt{\frac{2}{\mu}}$.

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Multiple steady states

The steady state satisfies

$$u - \frac{\mu}{3} u^3 \sin(\theta) + \frac{\chi}{3} u^3 \left(u_{\theta\theta\theta} + u_\theta\right) = q.$$

For zero surface tension, given a mass, if there's a solution then it's unique.

Benilov [*J. Fluid Mech.*, 2008] did extensive numerics and asymptotics and found that for some surface tension values, there are certain masses which yield two solutions and others that yield three solutions.

We found smaller surface tension values for which a given mass can yield up to five steady states [*Badali, Chugunova, Pelinovsky, Pollack, Physics of Fluids*, 2011].

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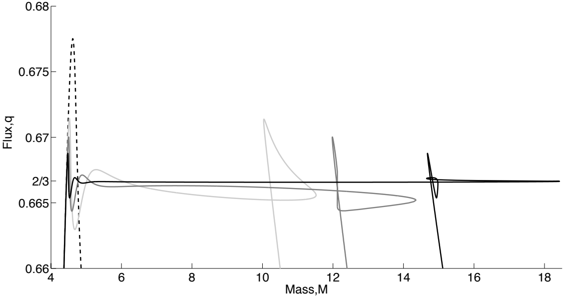
Steady states: positive surface tension, with rotation



$u - \frac{1}{3}u^3 \sin(\theta) + \frac{\chi}{3}u^3 (u_{\theta\theta\theta} + u_\theta) = q.$

$\chi = 0.005$ (dashed), 0.001 (light gray), 0.0005 (dark gray), 0.0001 (black).

Curves were generated using a custom-written turning-point algorithm and implemented in Matlab.

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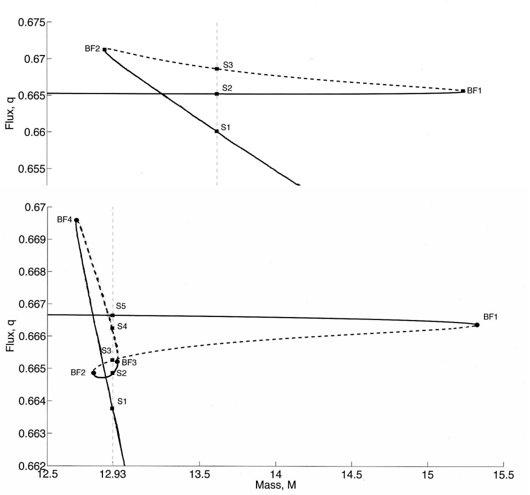
$\chi = 0.001$ and $\chi = 0.0001$ loops. Saddle–point bifurcations.



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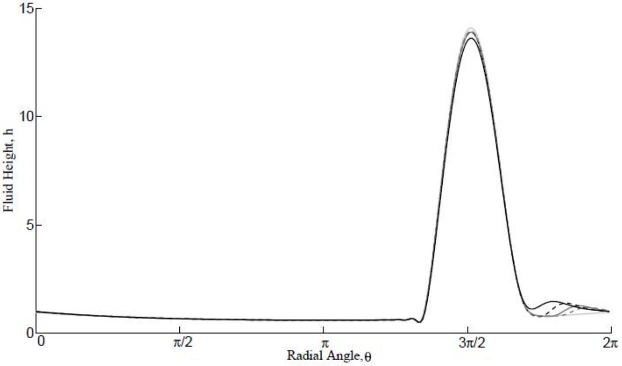

Five steady state solutions with $m = 12.93$ and $\chi = 0.00039$: solid light gray line $q = 0.6638$, solid dark gray line, $q = 0.6648$, dashed line $q = 0.6653$, dashed gray line, $q = 0.6661$, solid black line, $q = 0.6666$.

The limit of the infinitesimal thin film

The linearized equation:

[Benilov, O'Brien 2005]:

$$h_t = L[h], \quad L[h] = \varepsilon\,\partial_x(\sin x\, h_x) + h_x, \quad h(-\pi) = h(\pi).$$

- The spectrum of L consists of pure imaginary eigenvalues with infinity as the only accumulation point [*Chugunova, Volkmer, Studies in Applied Math., 2009*]
- If $|\varepsilon| < 2$ and $\varepsilon \neq 0$ then the set of eigenfunctions of L is complete in $\mathrm{L}^2(-\pi, \pi)$ but does not form a basis.
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The End

THANK YOU FOR YOUR ATTENTION !