# Determining Hilbert modular forms by the central values of Rankin-Selberg convolutions

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(Joint work with Naomi Tanabe)

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**Theorem (Luo-Ramakrishnan (1997))**

*Let $g$ and $g'$ be normalized eigenforms in $S_{2l}^{\mathrm{new}}(N)$ and $S_{2l'}^{\mathrm{new}}(N')$ respectively. Suppose that*

$$L(g \otimes \chi_d, 1/2) = L(g' \otimes \chi_d, 1/2)$$

*for almost all primitive quadratic characters $\chi_d$ of conductor prime to $NN'$. Then $g = g'$.*

### Theorem (Luo (1999))

*Let $g$ and $g'$ be two normalized eigenforms in $S_{2l}^{\mathrm{new}}(N)$ and $S_{2l'}^{\mathrm{new}}(N')$ respectively. If there exist infinitely many primes $p$ such that*

$$L(g \otimes f, 1/2) = L(g' \otimes f, 1/2)$$

*for all normalized newforms $f$ in $S_{2k}^{\mathrm{new}}(p)$, then we have $g = g'$.*

**Theorem (Ganguly-Hoffstein-Sengupta (2009))**

*Let $l$, $l'$ and $k$ denote positive integers and suppose $g$ and $g'$ are normalized eigenforms in $S_{2l}(1)$ and $S_{2l'}(1)$ respectively. If*

$$L(g \otimes f, 1/2) = L(g' \otimes f, 1/2)$$

*for all normalized eigenforms $f \in S_{2k}(1)$ for infinitely many $k$, then $g = g'$.*

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- $\mathcal{O}_F$ ring of integers in $F$, $\mathcal{D}_F$ different ideal of $F$, $d_F$ discriminant of $F$
- embeddings of $F$ $\{\sigma_1, \cdots, \sigma_n\}$. For $x \in F$ and $j \in \{1, \ldots, n\}$, we set $x_j = \sigma_j(x)$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We write $x \gg 0$ if $x_j > 0 \ \forall \ j$.

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- $X\subset F$, $X^+=\{x\in X : x \text{ totally positive}\}$
- Fix a set of representatives $\{\mathfrak{a}_i\}_{i=1}^{h_F^+}$ of the narrow class group of $F$, $Cl^+(F)$
- $\mathfrak{a}\sim\mathfrak{b} \iff \exists\ \xi\in F^{*+}$ such that $\mathfrak{a}\mathfrak{b}^{-1}=\xi\mathcal{O}_F$. We set $\xi=[\mathfrak{a}\mathfrak{b}^{-1}]$.

# Adelic Hilbert Modular Forms

By an adèlic Hilbert cusp form $\mathbf{f}$ of weight $\boldsymbol{k} \in 2\mathbb{N}^n$ and level $\mathfrak{n}$, we mean $\mathbf{f} : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ satisfying:

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1. $\mathbf{f}(\gamma z g r(\boldsymbol{\theta}) u) = \mathbf{f}(g) \exp(i\boldsymbol{k}\boldsymbol{\theta})\ \forall\ (\gamma, z, g, r(\boldsymbol{\theta}), u) \in \mathrm{GL}_2(F) \times \mathbb{A}_F^\times \times \mathrm{GL}_2(\mathbb{A}_F) \times \mathrm{SO}_2(F_\infty) \times K_0(\mathfrak{n})$.

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2. As a smooth function on $\mathrm{GL}_2^+(F_\infty)$, $\mathbf{f}$ is an eigenfunction of the Casimir element $\boldsymbol{\Delta} := (\Delta_1, \cdots, \Delta_n)$ with eigenvalue $\prod_{j=1}^{n} \frac{k_j}{2}\left(1 - \frac{k_j}{2}\right)$.

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3. $\int_{F\backslash\mathbb{A}_F} \mathbf{f}\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) dx = 0$ for all $g \in \mathrm{GL}_2(\mathbb{A}_F)$.

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Denote by $\mathcal{S}_{\boldsymbol{k}}(\mathfrak{n})$ the space of adèlic Hilbert cusp forms of weight $\boldsymbol{k}$ and level $\mathfrak{n}$.

- $\mathbf{f} = (f_1, \ldots, f_{h_F^+})$ with $f_i \in S_{\boldsymbol{k}}(\Gamma_{\mathfrak{a}_i}(\mathfrak{n}))$.
    - $f_i : \mathfrak{h}^n \to \mathbb{C}$
    - $f_i|_{\boldsymbol{k}}\gamma = f_i$ for all $\gamma \in \Gamma_{\mathfrak{a}_i}(\mathfrak{n})$
- Fourier coefficient of $\mathbf{f}$ at $\mathfrak{m} \subset \mathcal{O}_F$ : $C_{\mathbf{f}}(\mathfrak{m})$
- $\mathbf{f}$ is primitive $\Leftrightarrow$ $\mathbf{f}$ is a normalized eigenform in $\mathcal{S}_{\boldsymbol{k}}^{\text{new}}(\mathfrak{n})$.
- $\Pi_{\boldsymbol{k}}(\mathfrak{n})$ : a set of all primitive forms of weight $\boldsymbol{k}$ and level $\mathfrak{n}$.

# Rankin-Selberg Convolution

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Rankin-Selberg Convolution of $\mathbf{f}$ and $\mathbf{g}$:

$$L(\mathbf{f} \otimes \mathbf{g}, s) = \zeta_F^{\mathfrak{n}\mathfrak{q}}(2s) \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{C_{\mathbf{f}}(\mathfrak{m}) C_{\mathbf{g}}(\mathfrak{m})}{\mathrm{N}(\mathfrak{m})^s}.$$

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with

$$b_m^{\mathfrak{n}\mathfrak{q}}(\mathbf{f} \otimes \mathbf{g}) = \sum_{d^2 | m} \left( a_d^{\mathfrak{n}\mathfrak{q}} \sum_{\mathrm{N}(\mathfrak{m}) = m/d^2} C_{\mathbf{f}}(\mathfrak{m}) C_{\mathbf{g}}(\mathfrak{m}) \right).$$

Let

$$\Lambda(\mathbf{f}\otimes\mathbf{g},s)=\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})^s L_\infty(\mathbf{f}\otimes\mathbf{g},s)L(\mathbf{f}\otimes\mathbf{g},s),$$

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where

$$L_\infty(\mathbf{f}\otimes\mathbf{g},s)=\prod_{j=1}^{n}(2\pi)^{-2s}\Gamma\left(s+\frac{|k_j-l_j|}{2}\right)\Gamma\left(s-1+\frac{k_j+l_j}{2}\right).$$

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We have

$$\Lambda(\mathbf{f} \otimes \mathbf{g}, s) = \Lambda(\mathbf{f} \otimes \mathbf{g}, 1 - s).$$

# Level Aspect over Totally Real Number Field

## Theorem (H., Tanabe)

*Let* $\mathbf{g} \in \Pi_{\boldsymbol{l}}(\mathfrak{n})$ *and* $\mathbf{g}' \in \Pi_{\boldsymbol{l}'}(\mathfrak{n}')$, *with the weights* $\boldsymbol{l}$ *and* $\boldsymbol{l}'$ *being in* $2\mathbb{N}^n$. *Let* $\boldsymbol{k} \in 2\mathbb{N}^n$ *be fixed, and suppose that there exist infinitely many prime ideals* $\mathfrak{q}$ *such that*

$$L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) = L\left(\mathbf{f} \otimes \mathbf{g}', \frac{1}{2}\right)$$

*for all* $\mathbf{f} \in \Pi_{\boldsymbol{k}}(\mathfrak{q})$. *Then* $\mathbf{g} = \mathbf{g}'$.

# Weight Aspect Over Totally Real Number Field

**Theorem (H., Tanabe)**

*Let* $\mathbf{g} \in \Pi_{\boldsymbol{l}}(\mathfrak{n})$ *and* $\mathbf{g}' \in \Pi_{\boldsymbol{l}'}(\mathfrak{n}')$*, with the weights* $\boldsymbol{l}$ *and* $\boldsymbol{l}'$ *being in* $2\mathbb{N}^n$*. Let* $\mathfrak{q}$ *be a fixed prime ideal. If*

$$L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) = L\left(\mathbf{f} \otimes \mathbf{g}', \frac{1}{2}\right)$$

*for all* $\mathbf{f} \in \Pi_{\boldsymbol{k}}(\mathfrak{q})$ *for infinitely many* $\boldsymbol{k} \in 2\mathbb{N}^n$*, then* $\mathbf{g} = \mathbf{g}'$*.*

# Twisted First Moment

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$$\sum_{\mathbf{f} \in \Pi_{\boldsymbol{k}}(\mathfrak{q})} L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) C_{\mathbf{f}}(\mathfrak{p}) \omega_{\mathbf{f}},$$

where

$$w_{\mathbf{f}} = \frac{\Gamma(\boldsymbol{k} - \mathbf{1})}{(4\pi)^{\boldsymbol{k}-\mathbf{1}} |d_F|^{1/2} \langle \mathbf{f}, \mathbf{f} \rangle_{\mathcal{S}_{\boldsymbol{k}}(\mathfrak{q})}}.$$

# An Asymptotic Formula for the First Moment in the Level Aspect

## Proposition

*Consider* $\mathbf{g} \in \Pi_{\boldsymbol{l}}(\mathfrak{n})$ *and let* $\mathfrak{p}$ *be either* $\mathcal{O}_F$ *or a prime ideal. For all prime ideals* $\mathfrak{q}$ *with* $\mathrm{N}(\mathfrak{q})$ *sufficiently large, we have*

$$\sum_{\mathbf{f} \in \Pi_{\boldsymbol{k}}(\mathfrak{q})} L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) C_{\mathbf{f}}(\mathfrak{p}) \omega_{\mathbf{f}} = \frac{C_{\mathbf{g}}(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}} \gamma_{-1}(F) A_{\mathfrak{n}} \log(\mathrm{N}(\mathfrak{q})) + O(1),$$

*where* $\gamma_{-1}(F) = 2 \operatorname*{Res}_{u=0} \zeta_F(2u+1)$ *and* $A_{\mathfrak{n}} = \prod_{\substack{\mathfrak{l} \mid \mathfrak{n} \\ \mathfrak{l} : \text{prime}}} (1 - \mathrm{N}(\mathfrak{l})^{-1}).$

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$$L(\mathbf{f} \otimes \mathbf{g}, 1/2) = L(\mathbf{f} \otimes \mathbf{g}', 1/2) \qquad \forall\, \mathbf{f} \in \Pi_{\boldsymbol{k}}(\mathfrak{q}).$$

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Apply above Proposition with $\mathfrak{p} = \mathcal{O}_F$ to get $A_{\mathfrak{n}} = A_{\mathfrak{n}'}$.

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Multiplicity One Theorem $\Longrightarrow \mathbf{g} = \mathbf{g}'$.

# Approximate Functional Equation

## Proposition

*Let $G(u)$ be a holomorphic function on an open set containing the strip $|\Re(s)| \leq \frac{3}{2}$, bounded and satisfies $G(u) = G(-u)$ and $G(0) = 1$. Then we have*

$$L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) = 2 \sum_{m=1}^{\infty} \frac{b_m^{\mathfrak{n}\mathfrak{q}}(\mathbf{f} \otimes \mathbf{g})}{\sqrt{m}} V_{1/2}\left(\frac{4^n \pi^{2n} m}{\mathrm{N}(\mathfrak{D}_F^2 \mathfrak{n}\mathfrak{q})}\right),$$

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*with*

$$V_{1/2}(y) = \frac{1}{2\pi i} \int_{(3/2)} y^{-u} \frac{\Gamma\left(u + \frac{\boldsymbol{k}-\boldsymbol{l}+1}{2}\right)\Gamma\left(u + \frac{\boldsymbol{k}+\boldsymbol{l}-1}{2}\right)}{\Gamma\left(\frac{\boldsymbol{k}-\boldsymbol{l}+1}{2}\right)\Gamma\left(\frac{\boldsymbol{k}+\boldsymbol{l}-1}{2}\right)} G(u) \frac{du}{u}$$

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$$y^a V_{1/2}^{(a)}(y) \ll \left(1 + \frac{y}{\prod_{j=1}^{n} k_j^2}\right)^{-A}.$$

## Proposition (Trotabas (2011))

*Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals in $F$. For $\alpha \in \mathfrak{a}^{-1}$ and $\beta \in \mathfrak{b}^{-1}$, we have*

$$\sum_{\mathbf{f} \in H_{\boldsymbol{k}}(\mathfrak{q})} \frac{\Gamma(\boldsymbol{k}-\mathbf{1})}{(4\pi)^{\boldsymbol{k}-\mathbf{1}} |d_F|^{1/2} \langle \mathbf{f}, \mathbf{f} \rangle_{S_{\boldsymbol{k}}(\mathfrak{q})}} C_{\mathbf{f}}(\alpha\mathfrak{a}) \overline{C_{\mathbf{f}}(\beta\mathfrak{b})}$$

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$$\mathbb{1}_{\alpha\mathfrak{a}=\beta\mathfrak{b}} + C \sum_{\substack{\mathfrak{c}^2 \sim \mathfrak{a}\mathfrak{b} \\ c \in \mathfrak{c}^{-1}\mathfrak{q} \setminus \{0\} \\ \epsilon \in \mathcal{O}_F^{\times +}/\mathcal{O}_F^{\times 2}}} \frac{Kl(\epsilon\alpha, \mathfrak{a}; \beta, \mathfrak{b}; c, \mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} J_{\boldsymbol{k}-\mathbf{1}} \left( \frac{4\pi\sqrt{\boldsymbol{\epsilon\nu\xi}\left[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-\mathbf{2}}\right]}}{|\boldsymbol{c}|} \right),$$

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$$\mathbb{1}_{\alpha\mathfrak{a}=\beta\mathfrak{b}} + C \sum_{\substack{\mathfrak{c}^2 \sim \mathfrak{a}\mathfrak{b} \\ c \in \mathfrak{c}^{-1}\mathfrak{q} \setminus \{0\} \\ \epsilon \in \mathcal{O}_F^{\times +}/\mathcal{O}_F^{\times 2}}} \frac{Kl(\epsilon\alpha, \mathfrak{a}; \beta, \mathfrak{b}; c, \mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} J_{\boldsymbol{k}-\mathbf{1}} \left( \frac{4\pi \sqrt{\boldsymbol{\epsilon\nu\xi}\left[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-\mathbf{2}}\right]}}{|\boldsymbol{c}|} \right),$$

*where $C = \frac{(-1)^{\boldsymbol{k}/2}(2\pi)^n}{2|d_F|^{1/2}}$ and $H_{\boldsymbol{k}}(\mathfrak{q})$ is an orthogonal basis for the space $\mathcal{S}_{\boldsymbol{k}}(\mathfrak{q})$.*

Consider the following:

$$\sum_{\mathbf{f}\in\Pi_{\boldsymbol{k}}(\mathfrak{q})} L(\mathbf{f}\otimes\mathbf{g},1/2)C_{\mathbf{f}}(\mathfrak{p})\omega_{\mathbf{f}}$$

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$$= 2\sum_{\mathfrak{m}\subset\mathcal{O}_F}\frac{C_{\mathbf{g}}(\mathfrak{m})}{\sqrt{\mathrm{N}(\mathfrak{m})}}\sum_{d=1}^{\infty}\frac{a_d}{d}V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{m})d^2}{\mathrm{N}(\mathfrak{n}\mathfrak{q})}\right)$$

$$\times\sum_{\mathbf{f}\in\Pi_{\boldsymbol{k}}(\mathfrak{q})}\omega_{\mathbf{f}}C_{\mathbf{f}}(\mathfrak{p})C_{\mathbf{f}}(\mathfrak{m})$$

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$$\sum_{\mathbf{f}\in\Pi_{\boldsymbol{k}}(\mathfrak{q})} L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) C_{\mathbf{f}}(\mathfrak{p})\omega_{\mathbf{f}}$$

$$= 2\sum_{\{\mathfrak{a}\}} \sum_{\nu\in(\mathfrak{a}^{-1})^+/\mathcal{O}_F^{\times+}} \frac{C_{\mathbf{g}}(\nu\mathfrak{a})}{\sqrt{\mathrm{N}(\nu\mathfrak{a})}} \sum_{d=1}^{\infty} \frac{a_d^{\mathfrak{n}\mathfrak{q}}}{d} V_{\frac{1}{2}}\left(\frac{4^n\pi^{2n}\mathrm{N}(\nu\mathfrak{a})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right)$$

$$\times\left\{\mathbb{1}_{\xi\mathfrak{b}=\nu\mathfrak{a}} + C\sum_{c,\epsilon} \frac{Kl(\epsilon\nu,\mathfrak{a};\xi,\mathfrak{b};c,\mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} J_{\boldsymbol{k}-1}\left(\frac{4\pi\sqrt{\boldsymbol{\epsilon\nu\xi}\left[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}\right]}}{|\boldsymbol{c}|}\right) - \binom{\text{old}}{\text{forms}}\right\}$$

$$= \frac{C_{\mathbf{g}}(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}} M_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k},\mathfrak{q}) + CE_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k},\mathfrak{q}) - E_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k},\mathfrak{q},\text{old})$$

$$M_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k},\mathfrak{q}) = 2\frac{C_{\mathbf{g}}(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}}\sum_{d=1}^{\infty}\frac{a_d^{\mathfrak{n}\mathfrak{q}}}{d}V_{\frac{1}{2}}\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{p})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right)$$

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$$\times \sum_{\substack{\mathfrak{c}^2\sim\mathfrak{ab} \\ c\in\mathfrak{c}^{-1}\mathfrak{q}\setminus\{0\} \\ \epsilon\in\mathcal{O}_F^{\times+}/\mathcal{O}_F^{\times 2}}}\frac{Kl(\epsilon\nu,\mathfrak{a};\xi,\mathfrak{b};c,\mathfrak{c})}{\mathrm{N}(c\mathfrak{c})}J_{\boldsymbol{k}-1}\left(\frac{4\pi\sqrt{\boldsymbol{\epsilon\nu\xi}[\mathfrak{abc}^{-2}]}}{|\boldsymbol{c}|}\right)$$

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$$\times\sum_{\substack{\mathfrak{c}^2\sim\mathfrak{a}\mathfrak{b}\\ c\in\mathfrak{c}^{-1}\mathfrak{q}\setminus\{0\}\\ \epsilon\in\mathcal{O}_F^{\times+}/\mathcal{O}_F^{\times 2}}}\frac{Kl(\epsilon\nu,\mathfrak{a};\xi,\mathfrak{b};c,\mathfrak{c})}{\mathrm{N}(c\mathfrak{c})}J_{\boldsymbol{k}-1}\left(\frac{4\pi\sqrt{\boldsymbol{\epsilon\nu\xi}[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}]}}{|\boldsymbol{c}|}\right)$$

$$E_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k},\mathfrak{q},\mathrm{old}) = 2\sum_{\mathfrak{m}\subset\mathcal{O}_F}\frac{C_{\mathbf{g}}(\mathfrak{m})}{\sqrt{\mathrm{N}(\mathfrak{m})}}\sum_{d=1}^{\infty}\frac{a_d^{\mathfrak{n}\mathfrak{q}}}{d}V_{\frac{1}{2}}\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{m})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right)$$
$$\times\sum_{\mathbf{f}\in H_{\boldsymbol{k}}^{\mathrm{old}}(\mathfrak{q})}\frac{\Gamma(\boldsymbol{k}-\mathbf{1})}{(4\pi)^{\boldsymbol{k}-\mathbf{1}}|d_F|^{1/2}\langle\mathbf{f},\mathbf{f}\rangle_{S_{\boldsymbol{k}}(\mathfrak{q})}}C_{\mathbf{f}}(\mathfrak{m})\overline{C_{\mathbf{f}}(\mathfrak{p})}.$$

## Lemma

$$M_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k}, \mathfrak{q}) = \frac{C_{\mathbf{g}}(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}} \gamma_{-1}(F) \prod_{\substack{\mathfrak{l} \mid \mathfrak{n} \\ \mathfrak{l}:\,\mathrm{prime}}} (1 - \mathrm{N}(\mathfrak{l})^{-1}) \log(\mathrm{N}(\mathfrak{q})) + O(1),$$

*where* $\gamma_{-1}(F) = 2 \operatorname*{Res}_{u=0} \zeta_F(2u+1)$.

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*We have* $E_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k}, \mathfrak{q}) = O\left(\mathrm{N}(\mathfrak{q})^{-\frac{1}{2}+\epsilon}\right)$.

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Instead of

$$E_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k},\mathfrak{q}) = 2C\sum_{\{\mathfrak{a}\}}\sum_{\nu\in(\mathfrak{a}^{-1})^{+}/\mathcal{O}_F^{\times+}}\frac{C_{\mathbf{g}}(\nu\mathfrak{a})}{\sqrt{\mathrm{N}(\nu\mathfrak{a})}}\sum_{d=1}^{\infty}\frac{a_d^{\mathfrak{n}\mathfrak{q}}}{d}V_{\frac{1}{2}}\left(\frac{4^n\pi^{2n}\mathrm{N}(\nu\mathfrak{a})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right)$$

$$\times \sum_{\substack{\mathfrak{c}^2\sim\mathfrak{a}\mathfrak{b}\\ c\in\mathfrak{c}^{-1}\mathfrak{q}\backslash\{0\}\\ \epsilon\in\mathcal{O}_F^{\times+}/\mathcal{O}_F^{\times 2}}}\frac{Kl(\epsilon\nu,\mathfrak{a};\xi,\mathfrak{b};c,\mathfrak{c})}{\mathrm{N}(c\mathfrak{c})}J_{\boldsymbol{k}-1}\left(\frac{4\pi\sqrt{\boldsymbol{\epsilon\nu\xi}\left[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}\right]}}{|\boldsymbol{c}|}\right),$$

we consider

$$E_{\mathfrak{p},\mathfrak{a}}^{\mathbf{g}}(\boldsymbol{k},\mathfrak{q}) = \sum_{\nu\in(\mathfrak{a}^{-1})^{+}/\mathcal{O}_F^{\times+}}\frac{C_{\mathbf{g}}(\nu\mathfrak{a})}{\sqrt{\mathrm{N}(\nu\mathfrak{a})}}\sum_{d=1}^{\infty}\frac{a_d}{d}V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\nu\mathfrak{a})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right)$$

$$\times \sum_{\substack{c\in\mathfrak{c}^{-1}\mathfrak{q}\backslash\{0\}/\mathcal{O}_F^{\times+}\\ \eta\in\mathcal{O}_F^{\times+}}}\frac{Kl(\nu,\mathfrak{a};\xi,\mathfrak{b};c\eta,\mathfrak{c})}{|\mathrm{N}(c)|}J_{\boldsymbol{k}-1}\left(\frac{4\pi\sqrt{\boldsymbol{\nu\xi}\left[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}\right]}}{\boldsymbol{\eta}|\boldsymbol{c}|}\right).$$

# Error Term in the Level Aspect

Bound for the $J$-Bessel function: We have

$$J_u(x) \ll x^{1-\delta} \quad \text{for} \;\; 0 \leq \delta \leq 1.$$

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Crucial observation (Luo 2003): For $\lambda > 0$

$$\sum_{\eta \in \mathcal{O}_F^{\times +}} \left(\prod_{|\eta_j|>1} |\eta_j|^{-\lambda}\right) < \infty.$$

These bounds along with Weil bound for the Kloosterman sum:

$$Kl(\nu, \mathfrak{a}; \xi, \mathfrak{b}; c\eta, \mathfrak{c}) \ll \mathrm{N}\left((\nu\mathfrak{a}, \xi\mathfrak{b}, c\mathfrak{c})\right)^{\frac{1}{2}} \tau(c\mathfrak{c})\mathrm{N}(c\mathfrak{c})^{\frac{1}{2}},$$

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$$\times \sum_{\substack{c\in\mathfrak{c}^{-1}\mathfrak{q}\setminus\{0\}/\mathcal{O}_F^{\times+}\\ \eta\in\mathcal{O}_F^{\times+}}} \frac{Kl(\nu,\mathfrak{a};\xi,\mathfrak{b};c\eta,\mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} \prod_{j=1}^{n} J_{k_j-1}\left(\frac{4\pi\sqrt{\nu_j\xi_j[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}]_j}}{\eta_j|c_j|}\right)$$

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$$\ll \sum_{\eta \in \mathcal{O}_F^{\times +}} \prod_{\eta_j > 1} \eta_j^{-\delta} \sum_{\mathfrak{c} \subset \mathfrak{q}} \frac{\mathrm{N}((\nu\mathfrak{a}, \xi\mathfrak{b}, \mathfrak{c}))^{\frac{1}{2}}}{\mathrm{N}(\mathfrak{c})^{\frac{3}{2} - \delta}} \sum_{\nu} |C_{\mathbf{g}}(\nu\mathfrak{a})| \sum_{d=1}^{\infty} \frac{a_d^{\mathfrak{n}\mathfrak{q}}}{d} \left| V_{1/2}\left(\frac{4^n \pi^{2n} \mathrm{N}(\nu\mathfrak{a}) d^2}{\mathrm{N}(\mathfrak{D}_F^2 \mathfrak{n}\mathfrak{q})}\right) \right|$$

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$$\ll \sum_{\eta \in \mathcal{O}_F^{\times +}} \prod_{\eta_j > 1} \eta_j^{-\delta} \sum_{\mathfrak{c} \subset \mathfrak{q}} \frac{\mathrm{N}((\nu\mathfrak{a}, \xi\mathfrak{b}, \mathfrak{c}))^{\frac{1}{2}}}{\mathrm{N}(\mathfrak{c})^{\frac{3}{2}-\delta}} \sum_{\nu} |C_{\mathbf{g}}(\nu\mathfrak{a})| \sum_{d=1}^{\infty} \frac{a_d^{\mathfrak{n}\mathfrak{q}}}{d} \left| V_{1/2}\left(\frac{4^n \pi^{2n} \mathrm{N}(\nu\mathfrak{a}) d^2}{\mathrm{N}(\mathfrak{D}_F^2 \mathfrak{n}\mathfrak{q})}\right)\right|$$

$$\ll \mathrm{N}(\mathfrak{q})^{-\frac{3}{2}+\delta} \left( \sum_{\mathrm{N}(\nu\mathfrak{a}) \ll \mathrm{N}(\mathfrak{q})} |C_{\mathbf{g}}(\nu\mathfrak{a})| + \sum_{\mathrm{N}(\nu\mathfrak{a}) \gg \mathrm{N}(\mathfrak{q})} |C_{\mathbf{g}}(\nu\mathfrak{a})| \left(\frac{\mathrm{N}(\nu\mathfrak{a})}{\mathrm{N}(\mathfrak{q})}\right)^{-A} \right)$$

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E^{\mathbf{g}}_{\mathfrak{p},\mathfrak{a}}(\boldsymbol{k}, \mathfrak{q}) = & \sum_{\nu \in (\mathfrak{a}^{-1})^{+}/\mathcal{O}_F^{\times +}} \frac{C_{\mathbf{g}}(\nu\mathfrak{a})}{\sqrt{\mathrm{N}(\nu\mathfrak{a})}} \sum_{d=1}^{\infty} \frac{a_d^{\mathfrak{n}\mathfrak{q}}}{d} V_{\frac{1}{2}}\left(\frac{4^n \pi^{2n} \mathrm{N}(\nu\mathfrak{a}) d^2}{\mathrm{N}(\mathfrak{D}_F^2 \mathfrak{n}\mathfrak{q})}\right) \\
& \times \sum_{\substack{c \in \mathfrak{c}^{-1}\mathfrak{q} \setminus \{0\}/\mathcal{O}_F^{\times +} \\ \eta \in \mathcal{O}_F^{\times +}}} \frac{Kl(\nu, \mathfrak{a}; \xi, \mathfrak{b}; c\eta, \mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} \prod_{j=1}^{n} J_{k_j - 1}\left(\frac{4\pi \sqrt{\nu_j \xi_j [\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}]_j}}{\eta_j |c_j|}\right) \\
\ll & \sum_{\eta \in \mathcal{O}_F^{\times +}} \prod_{\eta_j > 1} \eta_j^{-\delta} \sum_{\mathfrak{c} \subset \mathfrak{q}} \frac{\mathrm{N}((\nu\mathfrak{a}, \xi\mathfrak{b}, \mathfrak{c}))^{\frac{1}{2}}}{\mathrm{N}(\mathfrak{c})^{\frac{3}{2} - \delta}} \sum_{\nu} |C_{\mathbf{g}}(\nu\mathfrak{a})| \sum_{d=1}^{\infty} \frac{a_d^{\mathfrak{n}\mathfrak{q}}}{d} \left| V_{1/2}\left(\frac{4^n \pi^{2n} \mathrm{N}(\nu\mathfrak{a}) d^2}{\mathrm{N}(\mathfrak{D}_F^2 \mathfrak{n}\mathfrak{q})}\right) \right| \\
\ll & \mathrm{N}(\mathfrak{q})^{-\frac{3}{2} + \delta} \left( \sum_{\mathrm{N}(\nu\mathfrak{a}) \ll \mathrm{N}(\mathfrak{q})} |C_{\mathbf{g}}(\nu\mathfrak{a})| + \sum_{\mathrm{N}(\nu\mathfrak{a}) \gg \mathrm{N}(\mathfrak{q})} |C_{\mathbf{g}}(\nu\mathfrak{a})| \left(\frac{\mathrm{N}(\nu\mathfrak{a})}{\mathrm{N}(\mathfrak{q})}\right)^{-A} \right) \\
\ll & \mathrm{N}(\mathfrak{q})^{-\frac{1}{2} + \delta + \epsilon}
\end{aligned}$$

Thank you.