Hypergeometric Functions and Hypergeometric Abelian Varieties

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Classical hypergeometric functions are well-understood. They are related to

- periods of algebraic varieties
- triangle groups, modular forms on arithmetic triangle groups
- Ramanujan type identities, combinatorial identities, physical applications...

Hypergeometric functions over finite fields are developed theoretically by Evans, Greene, Katz, McCarthy, Ono,... They are related to

- L-functions of algebraic varieties
- character sum identities
- supercongruences (Apéry or Ramanujan type) ....

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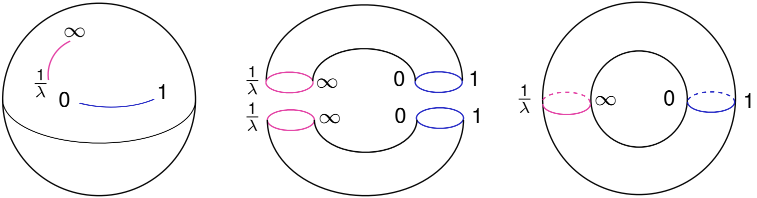
For $\lambda \neq 0, 1$, let $E_\lambda : y^2 = x(1-x)(1-\lambda x)$ be the elliptic curve in Legendre normal form.

- 

- A period of E_λ is

$$\Omega(E_\lambda) = \int_0^1 \frac{dx}{y} = \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}}.$$

and

$$\frac{\Omega(E_\lambda)}{\pi} = {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right] := \sum_{n=0}^{\infty} \binom{\frac{1}{2}+n-1}{n}^2 \lambda^n.$$

For $\lambda \in \mathbb{Q}$ and $\lambda \neq 0, 1 \mod p$,

$$\#\widetilde{E_\lambda}(\mathbb{F}_p) = p + 1 - a_p(\lambda),$$

where

$$a_p(\lambda) = \sum_{x \in \mathbb{F}_p} \left(\frac{x(1-x)(1-\lambda x)}{p} \right).$$

The value $a_p(\lambda)$ is

- the trace of Frobenius map;
- the p-th Fourier coefficient of certain modular form.

$a_p(\lambda)$ can be thought as a finite field analogue of the period

$$\Omega(E_\lambda) = \int_0^1 (x(1-x)(1-\lambda x))^{-1/2}\, dx.$$

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${}_2F_1$-hypergeometric Function

Let $a, b, c \in \mathbb{Q}$. The hypergeometric function ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right]$ is defined by

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, $$

where $(a)_n = a(a+1)\ldots(a+n-1)$ is the Pochhammer symbol.

For fixed a, b, c and argument z, the function ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right]$

- can be viewed as a quotient of periods on certain algebraic varieties.
- satisfies a hypergeometric differential equation, whose monodromy group is a triangle group.

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Hypergeometric Differential Equation

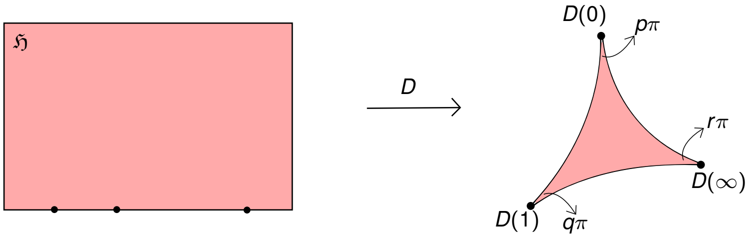
For fixed a, b, c and argument z, the function ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right]$ satisfies the hypergeometric differential equation

$$HDE(a,b,c;z):\ F'' + \frac{(a+b+1)z-c}{z(1-z)}F' + \frac{ab}{z(1-z)}F = 0,$$

with 3 regular singularities at 0, 1, and ∞.

Theorem (Schwarz)

Let f, g be two independent solutions to $HDE(a, b; c; \lambda)$ at a point $z \in \mathfrak{H}$, and let $p = |1 - c|$, $q = |c - a - b|$, and $r = |a - b|$. Then the Schwarz map $D = f/g$ gives a bijection from $\mathfrak{H} \cup \mathbb{R}$ onto a curvilinear triangle with vertices $D(0), D(1), D(\infty)$, and corresponding angles $p\pi, q\pi, r\pi$.



The universal cover of $\Delta(p, q, r)$ is one of the followings:

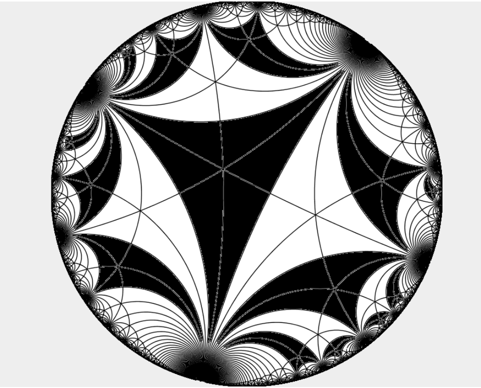
- the unit sphere $(p + q + r > 1)$;
- the Euclidean plane $(p + q + r = 1)$;
- the hyperbolic plane $(p + q + r < 1)$.

When p, q, r are rational numbers in the lowest form with $0 = \frac{1}{\infty}$, let e_i be the denominators of p, q, r arranged in the non-decreasing order, the monodromy group is isomorphic to the triangle group (e_1, e_2, e_3), where

$$(e_1, e_2, e_3) := \langle x, y \mid x^{e_1} = y^{e_2} = (xy)^{e_3} = id \rangle.$$

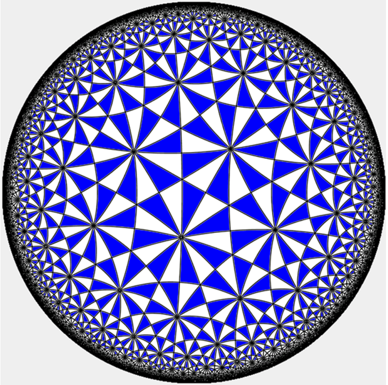
Example

For ${}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix} ; \lambda\right]$, the triangle $\Delta(p, q, r) = \Delta\,(0, 0, 0)$ is a hyperbolic triangle. The corresponding monodromy group is the arithmetic triangle group $(\infty, \infty, \infty) \simeq \Gamma(2)$.



Example

For ${}_2F_1\left[\begin{matrix}\frac{1}{84} & \frac{13}{84} \\ & \frac{1}{2}\end{matrix} ; \lambda\right]$, the triangle $\Delta(p,q,r) = \Delta\left(1/2, 1/3, 1/7\right)$ is a hyperbolic triangle. The corresponding monodromy group is the arithmetic triangle group $(2,3,7)$.



- Euler's integral representation of the $_2F_1$ with $c > b > 0$

$$_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right] = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a}dx$$
$$= {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right] B(b, c-b),$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the Beta function.

- Following Wolfart, we can realize $_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right]$ as a *period* of

$$C_\lambda^{[N;i,j,k]} : y^N = x^i(1-x)^j(1-\lambda x)^k,$$

where $N = lcd(a,b,c)$, $i = N(1-b)$, $j = N(1+b-c)$, $k = Na$.

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Examples

The function $B\left(\frac{12}{84}, \frac{29}{84}\right)\, {}_2F_1\left[\begin{matrix} \frac{1}{84} & \frac{13}{84} \\ & \frac{1}{2} \end{matrix} ; \lambda\right]$ is a period of the curve

$$C_\lambda^{[84;71,55,1]} : \; y^{84} = x^{71}(1-x)^{55}(1-\lambda x).$$

For the curve $C_\lambda^{[6;4,3,1]} : \; y^6 = x^4(1-x)^3(1-\lambda x)$,

- $B\left(\frac{1}{3}, \frac{1}{2}\right)\, {}_2F_1\left[\begin{matrix} \frac{1}{6} & \frac{1}{3} \\ & \frac{5}{6} \end{matrix} ; \lambda\right]$ is a period.
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Motivation

Study the arithmetic of

- generalized Legendre curves

$$y^N = x^i(1-x)^j(1-zx)^k,$$

which are parameterized by Shimura curves;

- general hypergeometric varieties

$$y^N = x_1^{i_1}\cdots x_n^{i_n}(1-x_1)^{j_1}\cdots(1-x_n)^{j_n}(1-zx_1x_2x_3\cdots x_n)^k.$$

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Period Functions over Finite Fields

Let p be a prime, and $q = p^s$.

- Let $\widehat{\mathbb{F}_q^\times}$ denote the group of multiplicative characters on $\mathbb{F}_q^\times$.
- Extend $\chi \in \widehat{\mathbb{F}_q^\times}$ to $\mathbb{F}_q$ by setting $\chi(0) = 0$.

Definition

Let $\lambda \in \mathbb{F}_q$, and $A, B, C \in \widehat{\mathbb{F}_q^\times}$. Define

$$ {}_2\mathcal{P}_1 \left(\begin{matrix} A & B \\ & C \end{matrix} \middle| \lambda \right) = \sum_{x \in \mathbb{F}_q} B(x)\overline{B}C(1-x)\overline{A}(1-\lambda x). $$

This is a finite field analogue of

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Let $X_\lambda^{[N;i,j,k]}$ be the smooth model of

$$C_\lambda^{[N;i,j,k]}: \; y^N = x^i(1-x)^j(1-\lambda x)^k.$$

Let q be an odd prime power, and let i, j, k be natural numbers with $1 \leq i, j, k < N$. Further, let $\eta_N \in \widehat{\mathbb{F}_q^\times}$ be a character of order N. Then for $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$,

$$\#\widetilde{X}_\lambda^{[N;i,j,k]} \text{``}=\text{''}\, 1 + q + \sum_{m=1}^{N-1} \sum_{x \in \mathbb{F}_q} \eta_N^m(x^i(1-x)^j(1-\lambda x)^k).$$

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- For each $d \mid N$, there is a natural covering from $C_\lambda^{[N;i,j,k]}$ to $C_\lambda^{[d;i,j,k]}$.
- For a given curve $C_\lambda^{[N;i,j,k]}$, we let J_λ^{new} be the subvariety of $\mathrm{Jac}(X_\lambda^{[N;i,j,k]})$ which is not induced from $C_\lambda^{[d;i,j,k]}$ for all $d \mid N$, $d < N$.
- $\dim J_\lambda^{new} = \varphi(N)$.
- Let K be the Galois closure of $\mathbb{Q}(\lambda)$. For any fixed prime ℓ, one can construct a compatible family of degree-$2\varphi(N)$ representations

$$\rho_\ell^{\text{new}}(\lambda): G_K := Gal(\overline{K}/K) \to GL_{2\varphi(N)}(\mathbb{Z}_\ell)$$

via the Tate module of J_λ^{new}.

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For any good prime $\mathfrak{p} \in \mathcal{O}_K$ with residue field $\mathbb{F}_q$,

$$\mathrm{Tr}\rho_\ell^{\text{new}}(\lambda)(\text{Frob}_\mathfrak{p}) = -\sum_{m \in (Z/N\mathbb{Z})^\times} {}_2\mathcal{P}_1\left(\begin{matrix}\eta_N^{-km} & \eta_N^{im} \\ & \eta_N^{(i+j)m}\end{matrix}\middle|\lambda\right)$$

Let ζ be a primitive Nth root of unity. The map $A_\zeta : (x, y) \mapsto (x, \zeta^{-1}y)$ induces an action on the ρ_ℓ. Consequently,

$$\rho_\ell^{\text{new}}(\lambda)|_{G_{K(\zeta)}} = \bigoplus_{\gcd(m,N)=1} \sigma_{m,\ell}(\lambda).$$

Here $\sigma_{m,\ell}(\lambda)$ is 2-dimensional when $(n, N) = 1$.

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Theorem (Fuselier, Long, Ramakrishna, Swisher, T.)

If $\gcd(m, N) = 1$, *then*

$$-\mathrm{Tr}\sigma_{m,\ell}(Frob_q) \quad \text{and} \quad {}_2\mathcal{P}_1\left(\begin{matrix} \eta_N^{-km} & \eta_N^{im} \\ & \eta_N^{m(i+j)} \end{matrix} \middle| \lambda\right)$$

agree up to different embeddings of $\mathbb{Q}(\zeta_N)$ *in* $\mathbb{C}$

Theorem (Deines, Fuselier, Long, Swisher, T.)

Let $N = 3, 4, 6$, *and* $N \nmid i + j + k$. *Then for each* $\lambda \in \overline{\mathbb{Q}}$, *the endomorphism algebra of* J_λ^{new} *contains a quaternion algebra over* $\mathbb{Q}$ *if and only if*

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For a given λ, let $S_m := {}_2\mathcal{P}_1 \left(\begin{matrix} \eta_N^{-km} & \eta_N^{im} \\ & \eta_N^{(i+j)m} \end{matrix} \middle| \lambda \right)$.

- Atkin-Li-Liu-Long: If End(J^{new}) contains a quaternion algebra, then the function

$$F(\eta_N) := S_1/S_{N-1} = J(\eta_N^i, \eta_N^j)/J(\eta_N^{-k}, \eta_N^{i+j+k})$$

is a character.

- Yamamoto: The quotient

$$B\left(\frac{i}{N}, \frac{j}{N}\right) / B\left(-\frac{k}{N}, \frac{i+j+k}{N}\right)$$

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${}_2\mathbb{F}_1$-hypergeometric Functions over Finite Fields

$$\begin{aligned} \Gamma(a) &\quad\leftrightarrow\quad g(A) \\ B(a,b) &\quad\leftrightarrow\quad J(A,B) \end{aligned}$$

Define

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$$a = \frac{i}{N} \quad \leftrightarrow \quad A = \eta_N^i$$

$$\Gamma(a) \quad \leftrightarrow \quad g(A)$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \leftrightarrow \quad J(A,B) = \frac{g(A)g(B)}{g(AB)} \text{ if } A \neq \overline{A}$$

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi}, \quad a \notin \mathbb{Z} \quad \leftrightarrow \quad g(A)g(\overline{A}) = A(-1)q, \quad A \neq \varepsilon$$

Gauss' multiplication formula

$$\Gamma(ma)(2\pi)^{(m-1)/2} = m^{ma-\frac{1}{2}}\Gamma(a)\Gamma\left(a+\frac{1}{m}\right)\cdots\Gamma\left(a+\frac{m-1}{m}\right)$$

$$\updownarrow$$

$$\prod_{\substack{\chi\in\widehat{\mathbb{F}_q^\times} \\ \chi^m=\varepsilon}} g(\chi\psi) = -g(\psi^m)\psi(m^{-m})\prod_{\substack{\chi\in\widehat{\mathbb{F}_q^\times} \\ \chi^m=\varepsilon}} g(\chi)$$

Hasse-Davenport relation

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\begin{array}{ccc}
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Theorem (Fuselier-Long-Ramakrishna-Swisher-T.)

Let q be an odd prime power, $z \neq 1 \in \mathbb{F}_q^\times$, ϕ be the quadratic character, and $A \in \widehat{\mathbb{F}_q^\times}$ of order larger than 2. Then

$$
{}_2\mathbb{F}_1\left[\begin{matrix} A & \phi A \\ & \phi \end{matrix} ; \; z\right] = \begin{cases} 0, & \text{if } z \text{ is not a square}, \\ \overline{A}^2(1+\sqrt{z}) + \overline{A}^2(1-\sqrt{z}), & \text{if } z \text{ is a square.} \end{cases}
$$

This is the analogue to the classical result

$$
{}_2F_1\left[\begin{matrix} a & a+\frac{1}{2} \\ & \frac{1}{2} \end{matrix} ; \; z\right] = \frac{1}{2}\left((1+\sqrt{z})^{-2a} + (1-\sqrt{z})^{-2a}\right).
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Theorem (T.-Yang)

Let p be a prime congruent to 1 modulo 4, $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[i]$ lying above p. Let $\psi_{\mathfrak{p}}$ be the quartic multiplicative character on $\mathbb{F}_p^\times$ satisfying $\psi_{\mathfrak{p}}(x) \equiv x^{(p-1)/4} \mod \mathfrak{p}$, for every $x \in \mathbb{Z}[i]$. Then, for $a \neq 0, 1$, if one of a and $1-a$ is not a square in $\mathbb{F}_p^\times$, we have

$$ {}_2\mathcal{P}_1 \left(\begin{matrix} \phi\psi_{\mathfrak{p}} & \psi_{\mathfrak{p}} \\ & \phi \end{matrix} \middle| a \right) = 0, $$

and

$$ {}_2\mathcal{P}_1 \left(\begin{matrix} \phi\psi_{\mathfrak{p}} & \psi_{\mathfrak{p}} \\ & \phi \end{matrix} \middle| a \right) = 2\psi_{\mathfrak{p}}(-1)\phi\,(1+b)\,\chi(\mathfrak{p}), $$

if $a = b^2$ for some $b \in \mathbb{F}_p^\times$, where χ is the Hecke character associated to the elliptic curve $E:\ y^2 = x^3 - x$ satisfying $\chi(\mathfrak{p}) \in \mathfrak{p}$ for all primes $\mathfrak{p}$ of $Z[i]$.

The curve $C_4: \quad y^4 = x(x-1)(x-a)$ has genus 3 and it is a 2-fold cover of the following 3 elliptic curves

$$\begin{aligned} C_2 &: \quad y^2 = x(x-1)(x-a), \\ E_+ &: \quad y^2 = x^3 + (1+b)^2 x, \\ E_- &: \quad y^2 = x^3 + (1-b)^2 x. \end{aligned}$$

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Quadratic Formula

Theorem (Fuselier, Long, Ramakrishna, Swisher, and T.)

Let $B, D \in \widehat{\mathbb{F}_q^\times}$, and set $C = D^2$. When $D \neq \phi$, $B \neq D$ and $x \neq \pm 1$, we have

$$\overline{C}(1-x)\ {}_2\mathbb{F}_1\left[\begin{matrix} D\phi\overline{B} & D \\ & C\overline{B} \end{matrix}; \frac{-4x}{(1-x)^2}\right] = {}_2\mathbb{F}_1\left[\begin{matrix} B & C \\ & C\overline{B} \end{matrix}; x\right].$$

This is the analogue to the classical result

$$(1-z)^{-c}\,{}_2F_1\left[\begin{matrix} \frac{1+c}{2}-b & \frac{c}{2} \\ & c-b+1 \end{matrix}; \frac{-4z}{(1-z)^2}\right] = {}_2F_1\left[\begin{matrix} b & c \\ & c-b+1 \end{matrix}; z\right].$$

Analogue to the classical result

$$ {}_2F_1\left[\begin{matrix} a & a-\frac{1}{2} \\ & 2a \end{matrix} ; z\right] = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a}. $$

Let q be an odd prime power, $z \in \mathbb{F}_q^\times$, ϕ be the quadratic character, and $A \in \widehat{\mathbb{F}_q^\times}$ of order larger than 2. Then

$$ {}_2\mathbb{F}_1\left[\begin{matrix} A & \phi A \\ & A^2 \end{matrix} ; z\right] = \begin{cases} 0, & \text{if } \phi(1-z) = -1, \\ \overline{A}^2\left(\frac{1+\sqrt{1-z}}{2}\right) + \overline{A}^2\left(\frac{1-\sqrt{1-z}}{2}\right), & \text{if } \phi(1-z) = 1. \end{cases} $$

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Let $X_\lambda^{[6;4,3,1]}$ and $X_\lambda^{[12;9,5,1]}$ be the smooth models of

$$y^6 = x^4(1-x)^3(1-\lambda), \quad \text{and } \; y^{12} = x^9(1-x)^5(1-\lambda),$$

respectively.

Theorem (Fuselier, Long, Ramakrishna, Swisher, and T.)

Let $\lambda \in \overline{\mathbb{Q}}$ such that $\lambda \neq 0, \pm 1$. Let $J_{\lambda,1}^{new}$ (resp. $J_{\frac{-4\lambda}{(1-\lambda)^2},2}^{new}$) be the primitive part of the Jacobian variety of $X_\lambda^{[6;4,3,1]}$ (resp. $X_\lambda^{[12;9,5,1]}$). Then

$$J_{\frac{-4\lambda}{(1-\lambda)^2},2}^{new} \sim J_{\lambda,1}^{new} \oplus J_{\lambda,1}^{new}$$

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As an analogue of the classical hypergeometric series, we inductively define

$$ {}_{n+1}\mathcal{P}_n\left[\begin{matrix} A_0 & A_1 & \dots & A_n \\ & B_1 & \dots & B_n \end{matrix};\lambda\right] := \sum_{y\in\mathbb{F}_q} A_n(y)\overline{A}_nB_n(1-y)\cdot {}_n\mathcal{P}_{n-1}\left[\begin{matrix} A_0 & A_1 & \dots & A_{n-1} \\ & B_1 & \dots & B_{n-1} \end{matrix};\lambda y\right] $$

$$ {}_{n+1}\mathbb{F}_n\left[\begin{matrix} A_0 & A_1 & \cdots & A_n \\ & B_1 & \cdots & B_n \end{matrix};\ \lambda\right] := \frac{1}{\prod_{i=1}^n J(A_i, B_i\overline{A_i})}\,{}_{n+1}\mathcal{P}_n\left[\begin{matrix} A_0 & A_1 & \dots & A_n \\ & B_1 & \dots & B_n \end{matrix};\lambda\right], $$

where A_i, $B_j \in \widehat{\mathbb{F}_q^\times}$, and $\lambda \in \mathbb{F}_q$.

Example: Consider the higher dimensional analogue of the legendre curve:

$$C_{n,\lambda}: \quad y^n = (x_1 x_2 \cdots x_{n-1})^{n-1}(1-x_1)\cdots(1-x_{n-1})(1-\lambda x_1 x_2 x_3 \cdots x_{n-1})$$

- $C_{2,\lambda}$ are known as Legendre curves.
- Up to a scalar multiple, ${}_nF_{n-1}\left[\begin{matrix} \frac{j}{n} & \frac{j}{n} & \cdots & \frac{j}{n} \\ & 1 & \cdots & 1 \end{matrix}; \lambda\right]$ for any $1 \le j \le n-1$, when convergent, can be realized as a period of $C_{n+1,\lambda}$.
- Let $q = p^e \equiv 1 \pmod n$ be a prime power. Let η_n be a primitive order n character and ε the trivial multiplicative character in $\widehat{\mathbb{F}_q^\times}$. Then

$$\#C_{n,\lambda}(\mathbb{F}_q) = 1 + q^{n-1} + \sum_{i=1}^{n-1} {}_n\mathcal{P}_{n-1}\left(\begin{matrix} \eta_n^i & \eta_n^i & \cdots & \eta_n^i \\ & \varepsilon & \cdots & \varepsilon \end{matrix}; \lambda\right).$$

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Local L-functions of $C_{3,1}$ and $C_{4,1}$

Theorem (Deines, Long, Fuselier, Swisher, T.)

Let η_3, *and* η_4 *denote characters of order* 3, *or* 4, *respectively, in* $\widehat{\mathbb{F}_q^\times}$.

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(Greene's transformation formula)

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$$C_{4,1}: y^4 = (x_1x_2x_3)^3(1-x_1)(1-x_2)(1-x_3)(x_1-x_2x_3)$$

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where $a(p)$ is the pth coefficient of the weight-4 Hecke eigenform $\eta(2z)^4\eta(4z)^4$, with $\eta(z)$ being the Dedekind eta function.

The factor of the zeta function $Z_{C_{4,1}}(T,p)$ corresponding to

$$y^2 = (x_1x_2x_3)^3(1-x_1)(1-x_2)(1-x_3)(x_1-x_2x_3)$$

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- Hasse-Davenport relation
  Let $\mathbb{F}$ be a finite field and $\mathbb{F}_s$ an extension field over $\mathbb{F}$ of degree s. If $\chi \neq \varepsilon \in \widehat{\mathbb{F}^\times}$ and $\chi_s = \chi \circ N_{\mathbb{F}_s/\mathbb{F}}$ a character of $\mathbb{F}_s$. Then

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Batyrev-Van Straten: Calabi-Yau manifolds whose Picard Fuchs equations are hypergeometric functions of the form

$$ {}_4F_3\left[\begin{matrix} d_1 & 1-d_1 & d_2 & 1-d_2 \\ & 1 & 1 & 1 \end{matrix}\; ;\; z\right], \quad d_1, d_2 \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}. $$

Conjectures: $z = 1$

- [Cohen]

$$ {}_4\mathbb{F}_3\left[\begin{matrix} \eta_3 & \overline{\eta}_3 & \eta_4 & \overline{\eta}_4 \\ & \varepsilon & \varepsilon & \varepsilon \end{matrix}\; ;\; 1\right] = -J(\eta_3, \eta_3)^3 - J(\overline{\eta}_3, \overline{\eta}_3)^3 + \eta_{12}(-1)p $$

- [Long] Numerically, Long finds the weight 4 cuspidal Hecke forms corresponding to $d_1 = \frac{1}{2}$ and $d_2 = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$.

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