Free and Bi-free Extremes

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Classical

f, g independent r.v.

$f \vee g$ (i.e. $\max(f, g)$)

$F_f(t) = \mu_f((-\infty, t]) = \Pr(f \leq t)$

distribution function

$$F_{f \vee g} = F_f \cdot F_g$$

limit distributions of

$$\frac{f_1 \vee \cdots\cdots \vee f_n - B_n}{A_n}, \quad n \longrightarrow \infty$$

$f_1, f_2, \ldots$ i.i.d.

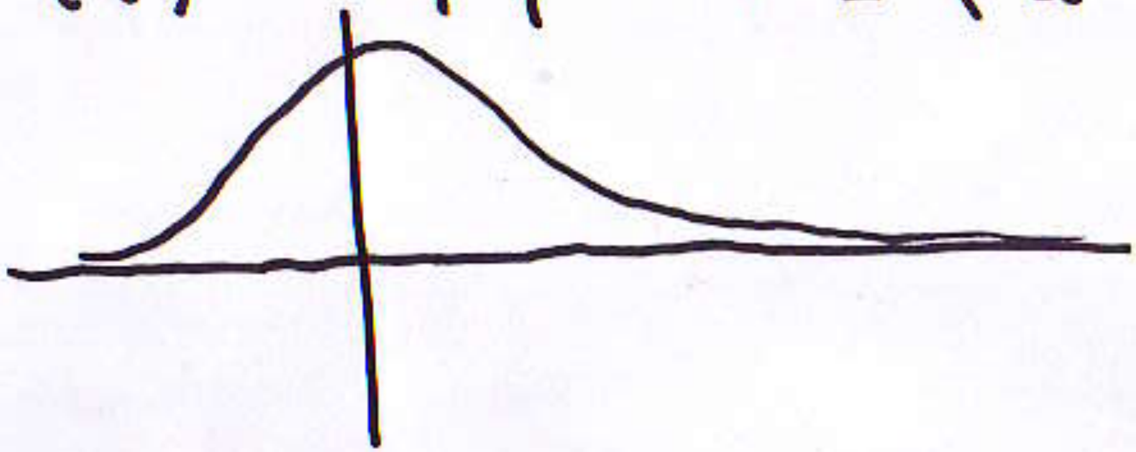
Max-stable distributions

(Insurance, Withstanding Floodings, Earthquakes . . .)

Classification

1° Gumbel

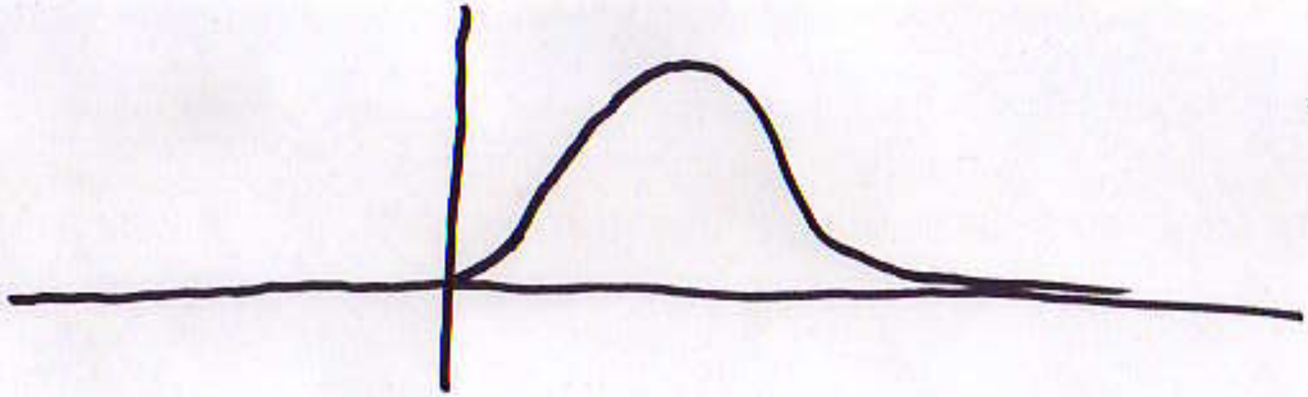
$$F(t) = \exp\left\{-\exp\left[-\left(\frac{t-b}{a}\right)\right]\right\} \qquad \begin{array}{l} -\infty < t < \infty \\ a > 0 \\ b \in \mathbb{R} \end{array}$$



2°. Frechet

$$F(t) = \begin{cases} 0 & t \le b \\ \exp\left\{-\left(\frac{t-b}{a}\right)^{-\alpha}\right\}, & t > b \end{cases}$$

$a > 0,\ b \in \mathbb{R},\ \alpha > 0$

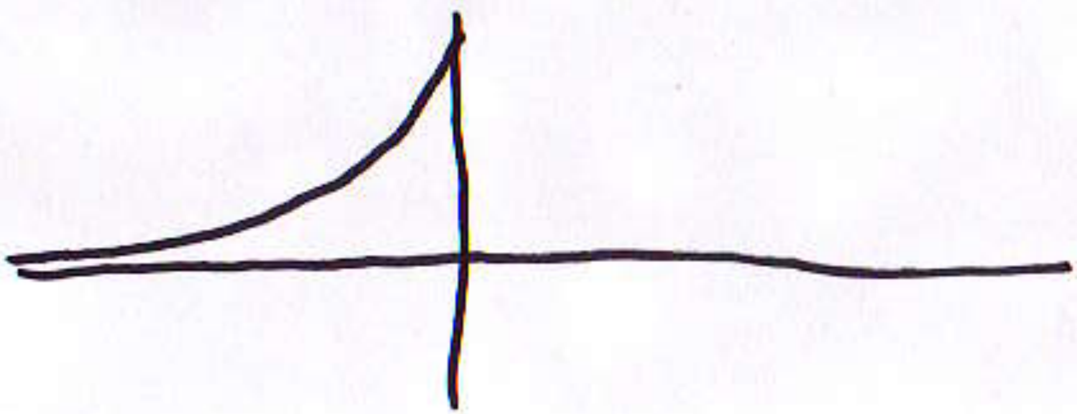


Frechet

3° Weibull

$$F(t) = \begin{cases} \exp\left\{-\left[-\left(\frac{t-b}{a}\right)^{\alpha}\right]\right\} & t < b \\ 1 & t \geq b \end{cases}$$

$a > 0,\ b \in \mathbb{R},\ \alpha > 0$



Free (Ben Arous – V.)

$(\mathcal{A}, \varphi)$ v. Neumann alg. with normal state

P, Q hermitian projections in $\mathcal{A}$

$P \wedge Q$ = proj. onto $P\mathcal{H} \cap Q\mathcal{H}$

$P \vee Q$ = proj. onto $\overline{P\mathcal{H} + Q\mathcal{H}}$

P, Q free $\Longrightarrow$ $\varphi(P \wedge Q) = (\varphi(P) + \varphi(Q) - 1)_+$

$\varphi(P \vee Q) = \min(\varphi(P) + \varphi(Q), 1)$

$X = X^*$, $Y = Y^*$ in (A, φ)

$$X \prec Y \overset{\text{def}}{\Longleftrightarrow} E(X; (-\infty, a]) \geq E(Y; (-\infty, a])$$

all $a \in \mathbb{R}$

Spectral Order (Ando)

max w.r.t. spectral order

$$E(X \vee Y; (-\infty, a]) = E(X; (-\infty, a]) \wedge E(Y; (-\infty, a])$$

Distribution of X is

$$\mu_X(\omega) = \varphi(E(X;\omega)), \ \omega \subset \mathbb{R} \text{ Borel}$$

μ_X probability measure on $\mathbb{R}$

Free Max-convolution X, Y free

$$\mu_X \boxslash \mu_Y = \mu_{X \vee Y}$$

If $F_\mu(t) = \mu((-\infty, t])$

$$F \boxslash G(t) = (F(t) + G(t) - 1)_+$$

operation on distribution functions.

Free Max-stable Laws

equivalent

$$\underbrace{\mu \boxed{\vee} \cdots \boxed{\vee} \mu}_{n} = T_* \mu$$

$T(t) = a_n t + b_n, \quad a_n > 0$

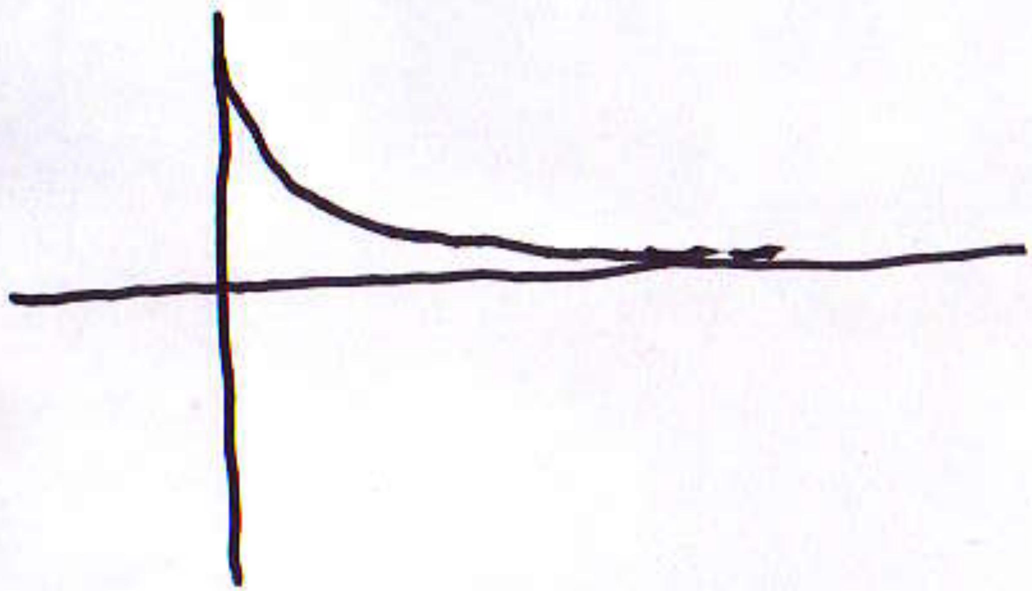
Classification analogous to classification of classical (with some differences ...)

so-called generalized Pareto distributions

The free Max-stable Laws are affine transforms $(at+b,\ a>0)$ of :

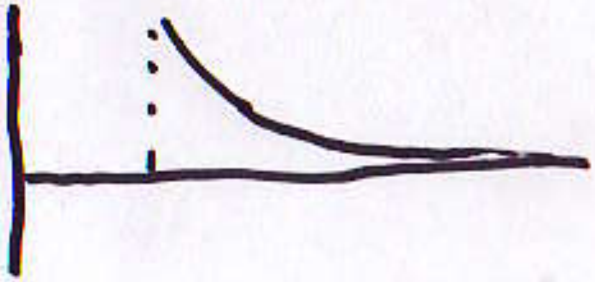
1°. exponential distribution

$$F(t) = (1 - e^{-t})_+$$



2°. Pareto distribution

$$F(t) = (1 - t^{-\alpha})_+ , \quad \alpha > 0$$

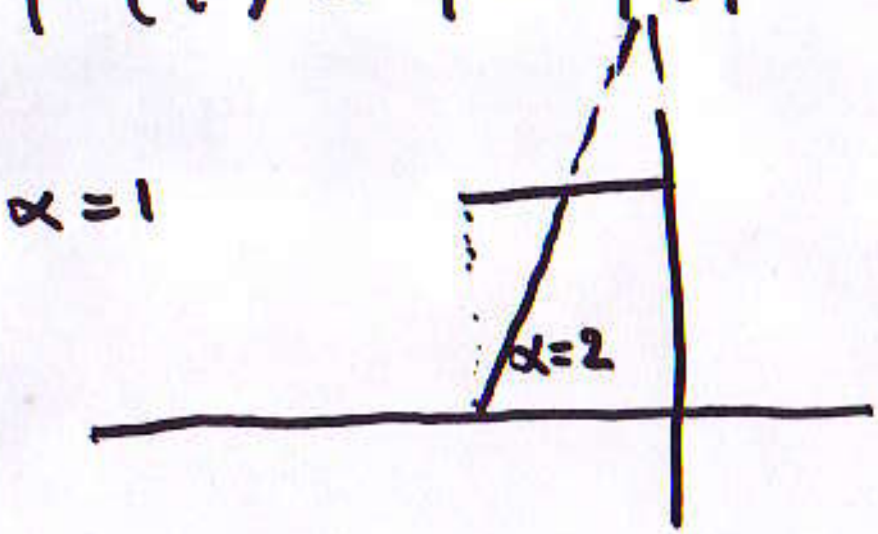


$$\alpha t^{\alpha - 1}, \quad t \geq 1$$

3°. Beta law

$$F(t) = 1 - |t|^{\alpha} \text{ if } -1 \leq t \leq 0$$

$$\alpha > 0$$



$$\alpha |t|^{\alpha - 1}$$

Classical and Free Max-domains of attraction coincide (Max-analogue of Bercovici-Pata)

<u>Free</u> Free Max-stable laws

$\parallel$

<u>Classical</u> limit laws in Peaks over Thresholds

de Haan and Balkema
"Residual lifetime at great age"

X r.v., u threshold $\Pr(X > u) > 0$

$(X - u \mid X \geq u)$ limit of affine transforms of distribution as $u \uparrow$

$$F^{[u]}(x) = \frac{F(u+x) - F(u)}{1 - F(u)}, \quad x > 0$$

limit of

$F^{[u]}(a_u x + b_u)$ as $F(u) \uparrow 1$.

Free Extremal Projection valued Process over a Set (particular case)

(M, τ) v. Neumann algebra with normal faithful tracial state

(Ω, Σ, μ) probability measure on Ω

1°. $\Sigma \ni \omega \longrightarrow P(\omega) \in \mathrm{Proj}(M)$

$\omega_1, \dots, \omega_n$ disjoint $\Rightarrow P(\omega_1), \dots, P(\omega_n)$ free

2°. $\omega = \bigcup_{1 \le j \le n} \omega_j \Rightarrow P(\omega) = \bigvee_{1 \le j \le n} P(\omega_j)$

3°. $\tau(P(\omega)) = \mu(\omega)$.

Realization

(M, τ) containing $L^\infty(\Omega, \Sigma, \mu)$

so that $\tau|L^\infty$ is $\int \cdot \, d\mu$

$C \in M$ circular element

$\{C, C^*\}$, $L^\infty(\Omega, \Sigma, \mu)$ freely indep.

$P(\omega) =$ range projection of $C \chi_\omega C^*$

$\omega \longrightarrow P(\omega)$

has properties $1^\circ - 3^\circ$.

<u>Remark</u> $\Pi(\omega) = C \chi_\omega C^*$, $\omega \in \Sigma$

free Poisson process over Ω ...

Bi-free

Systems of Left and Right variables

$(\mathcal{A}, \varphi)$, $((z_i)_{i \in I}, (z_j)_{j \in J}) \subset \mathcal{A}$

left var. right var.

Simplest case: two-faced pair (a, b)

left right

bi-partite $[a, b] = 0$

$a = a^*$, $b = b^*$ distribution of (a, b)

probability measure with compact-supp on $\mathbb{R}^2$.

bi-free independence (a.k.a. bi-freeness)

Like defining classical independence from tensor products of Hilbert spaces, using instead free products of Hilbert spaces with state vectors and the fact that on free products of Hilbert spaces there are left and right factorizations, hence left and right operators . . . .

$\left((z'_i)_{i \in I'}, (z'_j)_{j \in J'} \right)$ and $\left((z''_i)_{i \in I''}, (z''_j)_{j \in J''} \right)$
bi-free

$\Rightarrow$ $(z'_i)_{i \in I'}$ and $(z''_i)_{i \in I''}$ free
$(z'_j)_{j \in J'}$ and $(z''_i)_{i \in I''}$ classically indep

. . . – – . – – – –

(a, b) and (a', b') bifree, bi-partite
$a = a^*$, $b = b^*$, $a' = a'^*$, $b' = b'^*$,
realization with
$\{a, a'\}$ commuting with $\{b, b'\}$.

(P,Q), (P',Q') bi-free, bi-partite, projections

$$\varphi(P\wedge P') = (\varphi(P)+\varphi(P')-1)_+$$
$$\varphi(Q\wedge Q') = (\varphi(Q)+\varphi(Q')-1)_+$$

– if $\varphi(P\wedge P')>0$, $\varphi(Q\wedge Q')>0$, $\varphi(PQ)>0$, $\varphi(P'Q')>0$

then

$$\frac{\varphi(P\wedge P')\varphi(Q\wedge Q')}{\varphi((P\wedge P')(Q\wedge Q'))} = \frac{\varphi(P)\varphi(Q)}{\varphi(PQ)} + \frac{\varphi(P')\varphi(Q')}{\varphi(P'Q')} - 1$$

– otherwise $\varphi((P\wedge P')(Q\wedge Q'))=0$

operation on probability measures on $\mathbb{R}^2$

bi-free max-convolution

$$\mu_{(a,b)} \boxtimes\!\!\boxtimes\, \mu_{(a',b')} = \mu_{(a \vee a',\, b \vee b')}$$

operation on bi-variate distribution functions

$$F_\mu(s,t) = \mu((-\infty,s] \times (-\infty,t])$$

$$F_\mu \boxtimes\!\!\boxtimes\, F_\nu = F_{\mu \boxtimes\!\boxtimes \nu}$$

F bi-variate, F_1 and F_2 marginals

$$H = F \boxed{\nabla}\boxed{\nabla} G$$

$$H_j = (F_j + G_j - 1)_+ \quad , \; j = 1,2$$

$$\frac{H_1(s)H_2(t)}{H(s,t)} - 1 =$$

$$= \left(\frac{F_1(s)F_2(t)}{F(s,t)} - 1\right) + \left(\frac{G_1(s)G_2(t)}{G(s,t)} - 1\right)$$

if $F(s,t)>0, G(s,t)>0, H_1(s)>0, H_2(t)>0$

and otherwise $H(s,t)=0$.

Problems

1°. Bi-free Max-stable laws?
Bi-free Max-∞-divisible laws?

2°. Are there classical problems which give rise to bi-free Max-laws?
(like Peaks over Thresholds for free Max-stable)

3°. Applications? Free Floodings?
Bi-free Floodings??

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