The Painlevé Equations and Discrete Asymptotics

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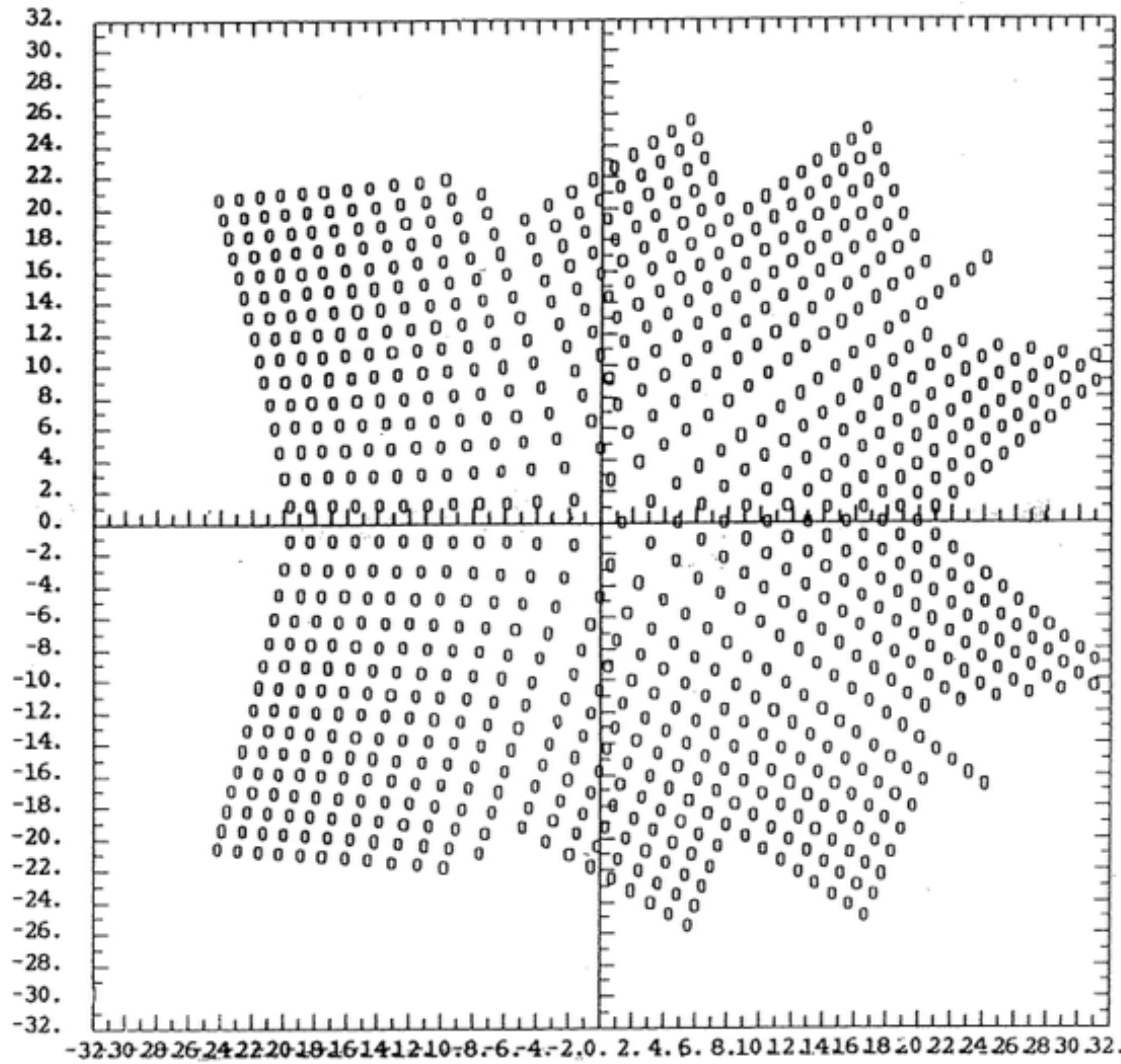
@monsoon0

P_I



$$w_{tt} = 6w^2 + t,\ t \in \mathbb{C}$$

Questions

- Do we know all possible solutions?
- What are their global properties?
- Is the solution space connected?

Okamoto’s Space of Initial Values

- Okamoto (1979) showed that the space of initial values of the Painlevé equations can be compactified and regularised after exactly nine blow-ups.

- Sakai (2001) classified all equations (differential and discrete) with this property, thereby providing a set of all possible Painlevé systems.

- We study Okamoto’s space of initial values in the asymptotic limit

Boutroux's Coordinates

Boutroux (1913) showed that

$$w(t) = t^{1/2}u(z), z = 4t^{5/4}/5$$

transforms P_I to

$$\ddot{u} = 6u^2 + 1 - \frac{\dot{u}}{z} + \frac{4u}{25z^2}$$

or in system form

$$\begin{cases} \dot{u}_1 = u_2 - \dfrac{2u_1}{5z} \\ \dot{u}_2 = 6u_1^2 + 1 - \dfrac{3u_2}{5z} \end{cases}$$

where

$$u_2 = \dot{u}_1 + 2\,u_1/(5z)$$

Perturbation of Elliptic Pencil

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6u_1^2 + 1 \end{pmatrix} - \frac{1}{5z} \begin{pmatrix} 2u_1 \\ 3u_2 \end{pmatrix}$$

A perturbation of an elliptic curve:

$$u_2^2 = 4u_1^3 + u_1 + 2E$$

$$\Rightarrow \quad \frac{dE}{dz} = \frac{1}{5z}(6E + 4u_1)$$

Perturbation of Elliptic Pencil

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6u_1^2 + 1 \end{pmatrix} + \frac{1}{5z} \begin{pmatrix} 2u_1 \\ 3u_2 \end{pmatrix}$$

A perturbation of an elliptic curve:

$$u_2^2 = 4u_1^3 + u_1 + 2E$$

$$\Rightarrow \quad \frac{dE}{dz} = \frac{1}{5z}(6E + 4u_1)$$

In the projective plane

$$\overbrace{[1 : \frac{u_{011}}{u_{010}} : \frac{u_{012}}{u_{010}}]}^{Affine\ coordinates} \;\Leftrightarrow\; \overbrace{[u_{010} : u_{011} : u_{012}]}^{Homogeneous\ coordinates}$$

$$u_{010} = 0 \;\Leftrightarrow\; \mathcal{L}_0$$

First chart:

$$[u_1^{-1} : 1 : u_1^{-1}\, u_2] = [u_{021} : 1 : u_{022}]$$

Second chart:

$$[u_2^{-1} : u_1\, u_2^{-1} : 1] = [u_{031} : u_{032} : 1]$$

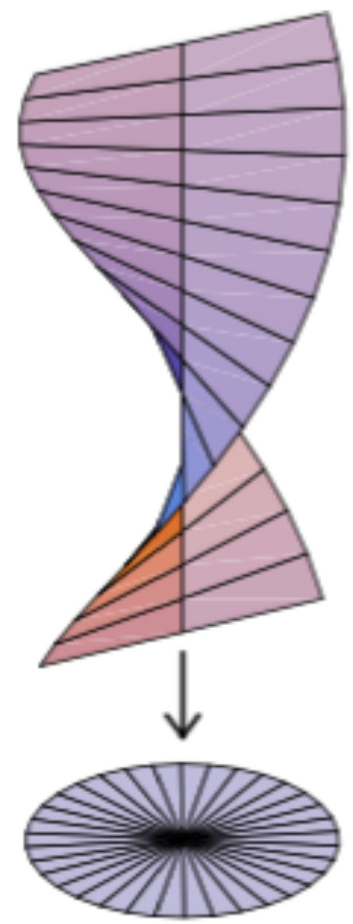
Transformed P_I

$$\begin{cases} \dot{u}_{021} = -u_{021}u_{022} + 2(5z)^{-1}u_{021} \\ \dot{u}_{022} = u_{021} + 6u_{021}^{-1} - u_{022}^{2} - (5z)^{-1}u_{022} \end{cases}$$

$$\begin{cases} \dot{u}_{031} = -u_{031}^{2} - 6u_{032}^{2} + 3(5z)^{-1}u_{031} \\ \dot{u}_{032} = -u_{031}u_{032} - 6u_{031}^{-1}u_{032}^{3} + 1 + (5z)^{-1}u_{032} \end{cases}$$

Base point: $b_0 : u_{031} = 0, u_{032} = 0$

Blowing up



from JJ Duistermaat, QRT maps and elliptic surfaces, Springer 2010

First blow-up

$$[1 : u_{111} : u_{112}] = [1 : u_{031}/u_{032} : u_{032}]$$

$$[1 : u_{121} : u_{122}] = [1 : u_{031} : u_{032}/u_{031}]$$

$$\begin{cases} \dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111} \\ \dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112} \end{cases}$$

$$\begin{cases} \dot{u}_{121} = u_{121}^{2}\left(-6u_{122}^{2} - 1\right) + 3\,(5z)^{-1}\,u_{121} \\ \dot{u}_{122} = u_{121}^{-1} - 2\,(5z)^{-1}\,u_{122} \end{cases}$$

Base point: $b_1 : u_{111} = 0, u_{112} = 0$

$$L_1 : u_{112} = 0,\ \ L_0^{(1)} : u_{111} = 0$$

First blow-up

$$[1 : u_{111} : u_{112}] = [1 : u_{031}/u_{032} : u_{032}]$$

$$[1 : u_{121} : u_{122}] = [1 : u_{031} : u_{032}/u_{031}]$$

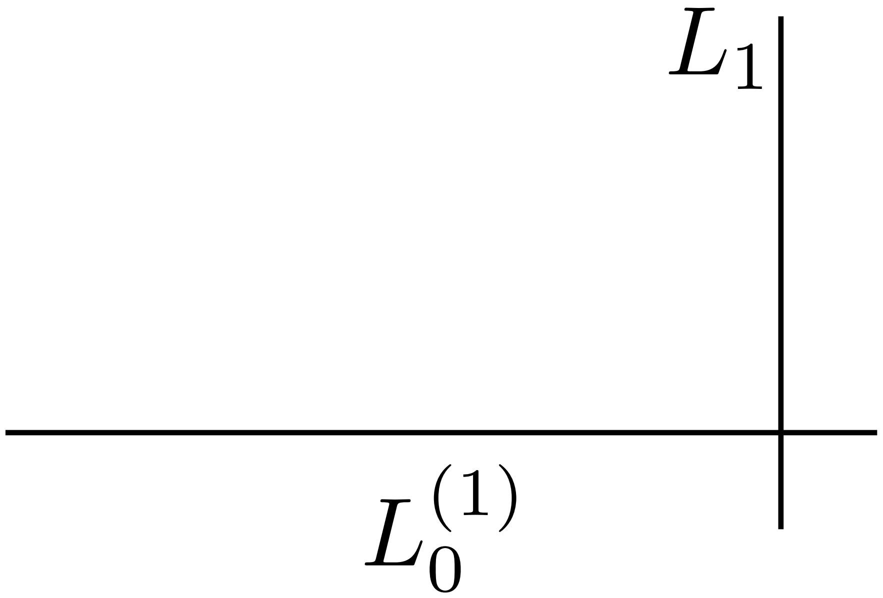
$$\begin{cases} \dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111} \\ \dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112} \end{cases}$$

$$\begin{cases} \dot{u}_{121} = u_{121}^{2}\left(-6u_{122}^{2} - 1\right) + 3\,(5z)^{-1}\,u_{121} \\ \dot{u}_{122} = u_{121}^{-1} - 2\,(5z)^{-1}\,u_{122} \end{cases}$$

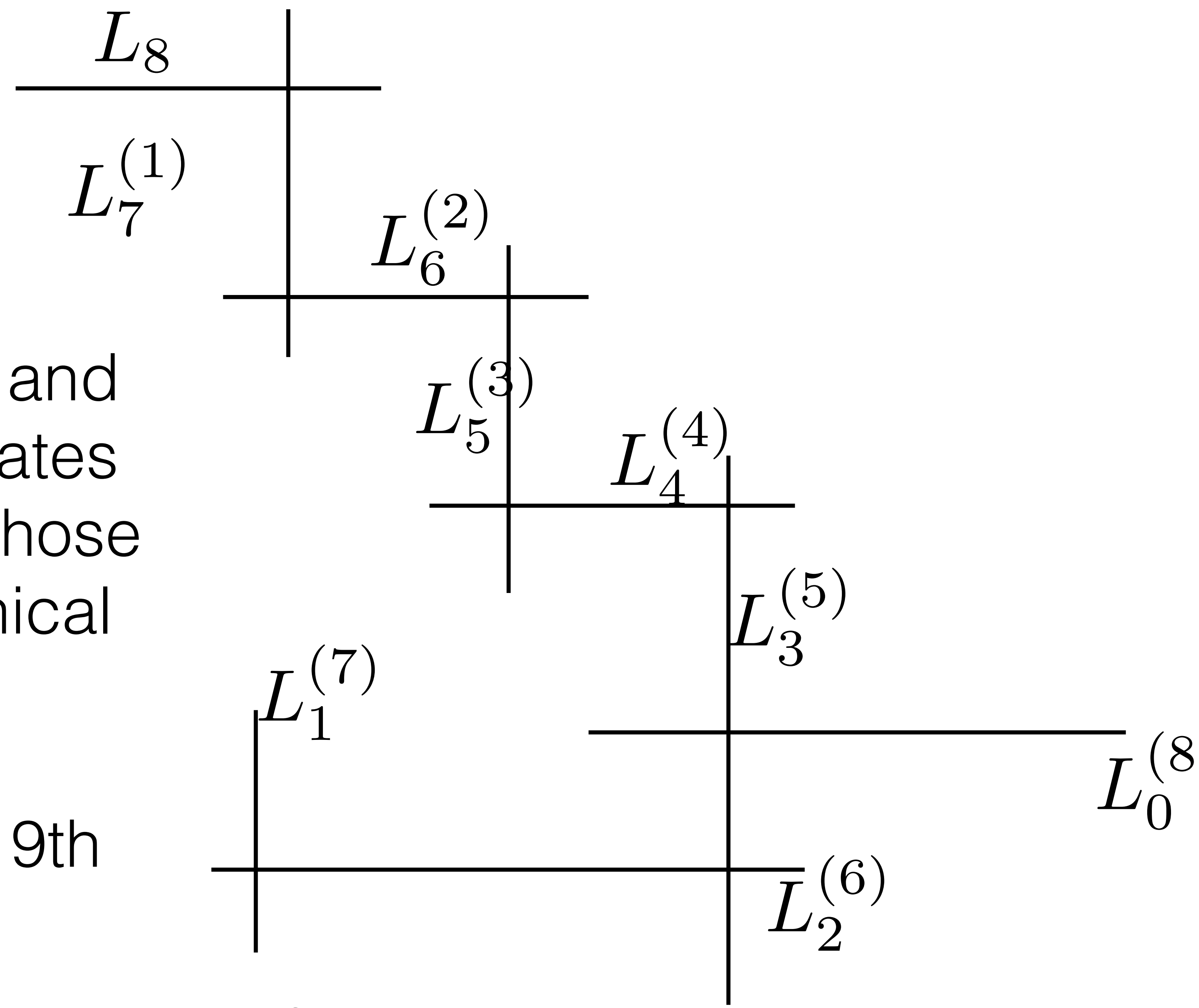
Base point: $b_1 : u_{111} = 0, u_{112} = 0$

$$L_1 : u_{112} = 0,\ \ L_0^{(1)} : u_{111} = 0$$

Exceptional lines



Between first and eight blow-ups



The base points and blow-up coordinates as the same as those of the anti-canonical pencil (the autonomous system). But the 9th base pt is

$$b_8 : u_{811} = -2^8/(5\,z), u_{812} = 0$$

Ninth blow-up I

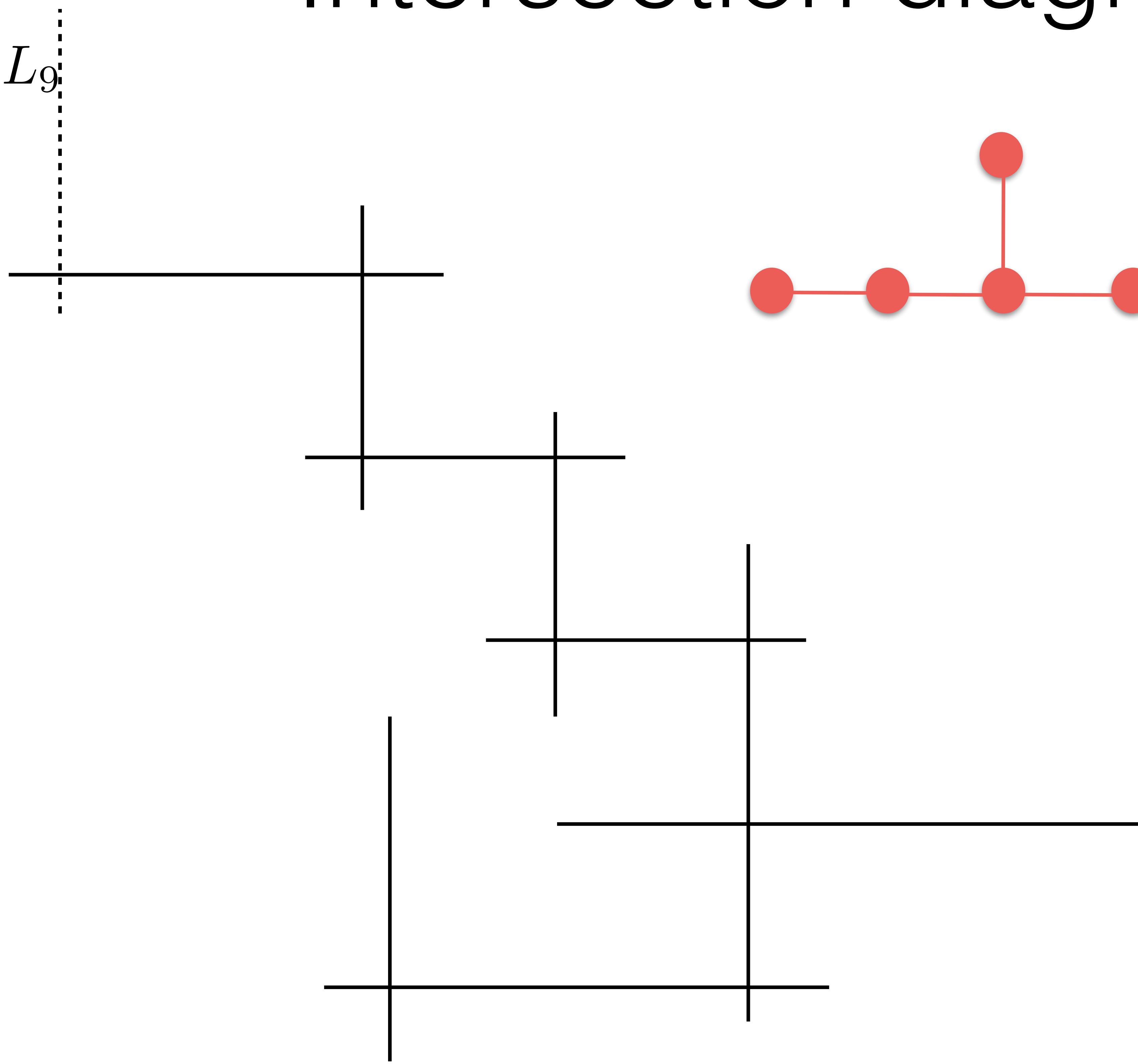
$$
\begin{aligned}
\dot{u}_{911} = & \left(4 + 32u_{912}^4 + u_{911}u_{912}^6 - \frac{2^8}{(5z)}u_{912}^5\right)^{-1} \\
& \times \Big[u_{912}\Big(-2^{11} - 2^6 \cdot 5\, u_{911}u_{912}^2 + 2^{13} \cdot 7u_{912}^4 \\
& \qquad - 3^2u_{911}^2u_{912}^4 + 2^{12}u_{911}u_{912}^6 + 2^{16} \cdot 3u_{912}^8 + 2^3 \cdot 3^2u_{911}^2u_{912}^8 \\
& \qquad + 2^{12} \cdot 5u_{911}u_{912}^{10} + 2^6 \cdot 11u_{911}^2u_{912}^{12} + 2^3u_{911}^3u_{912}^{14}\Big) \\
& - \frac{2}{(5z)}\Big(2^2 \cdot 3u_{911} - 2^{12} \cdot 3^2u_{912}^2 - 2^5 \cdot 3^2 \cdot 7u_{911}u_{912}^4 \\
& \qquad + 2^{15} \cdot 3 \cdot 5u_{912}^6 + 3u_{911}^2u_{912}^6 + 2^{10} \cdot 17u_{911}u_{912}^8 \\
& \qquad + 2^{17} \cdot 19u_{912}^{10} + 2^{13} \cdot 3 \cdot 7u_{911}u_{912}^{12} + 2^7 \cdot 23u_{911}^2u_{912}^{14}\Big) \\
& + 2^9\,(5z)^{-2}u_{912}^3\Big(-2^6 \cdot 3 \cdot 5 + 3u_{911}u_{912}^2 + 2^{13}u_{912}^4 \\
& \qquad + 2^{14} \cdot 5u_{912}^8 + 2^8 \cdot 11u_{911}u_{912}^{10}\Big) - 2^{24} \cdot 7(5z)^{-3}u_{912}^{12}\Big]
\end{aligned}
$$

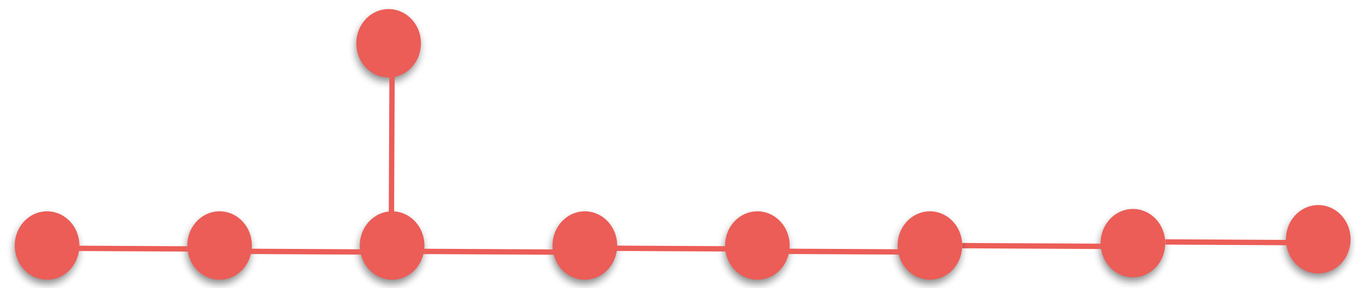
Ninth blow-up II

$$\dot{u}_{912} = -\left(4 + 32u_{912}^4 + u_{911}u_{912}^6 - \frac{2^8}{(5z)}u_{912}^5\right)^{-1}$$
$$\times \Big[2 - 2^4u_{912}^4 - u_{911}u_{912}^6 + 2^8u_{912}^8$$
$$+ 2^3u_{911}u_{912}^{10} + 2^{10}u_{912}^{12} + 2^6u_{911}u_{912}^{14} + u_{911}^2u_{912}^{16}$$
$$- (5z)^{-1}u_{912}\big(2^2 - 2^5\cdot 7u_{912}^4 - u_{911}u_{912}^6 + 2^{11}u_{912}^8$$
$$+ 2^{14}u_{912}^{12} + 2^9u_{911}u_{912}^{14}\big) + 2^8(5z)^{-2}u_{912}^6\left(1 + 2^8u_{912}^8\right)\Big]$$

- There are no further base points.
- The zero set of the denominator gives $L_0^{(9)}$
- The Painlevé vector field is regular and transversal to L_9

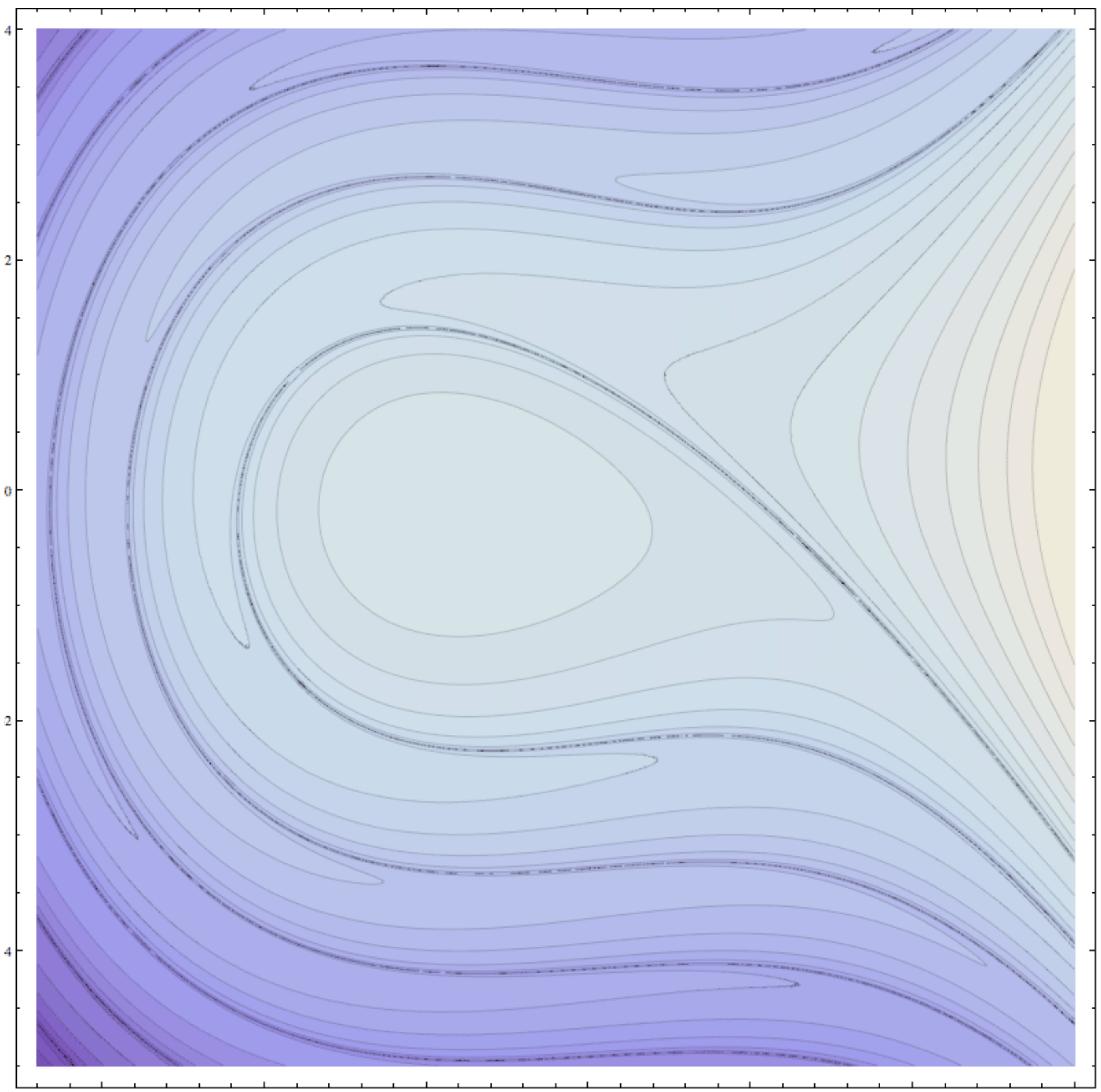
Intersection diagram



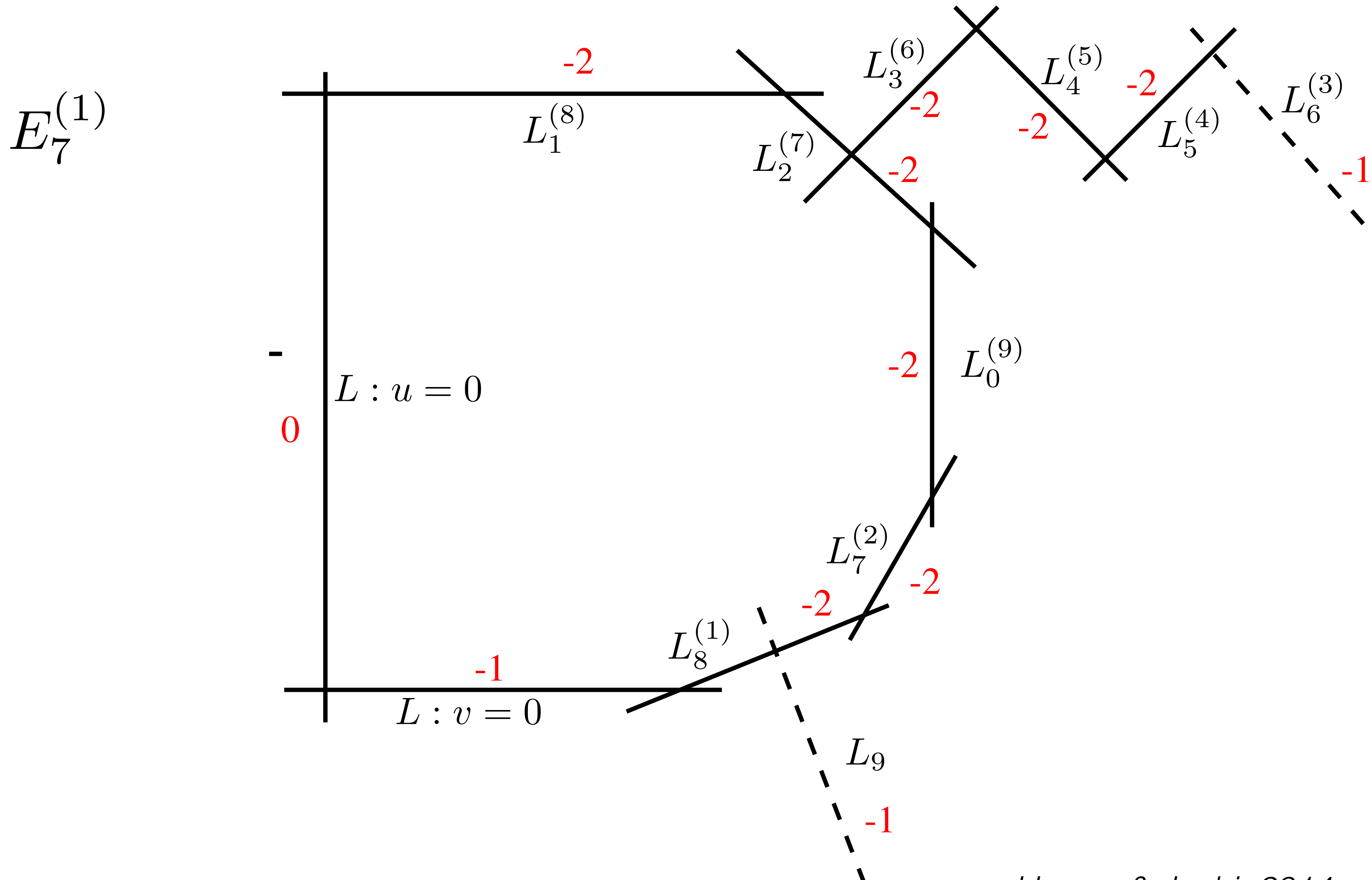


$\Leftrightarrow E_8^{(1)}$

A snapshot

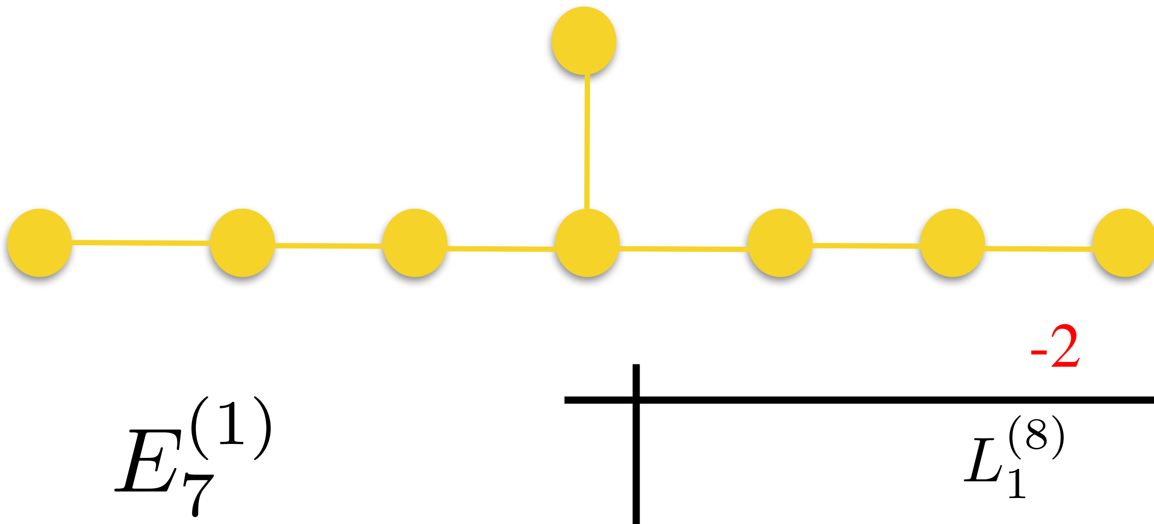


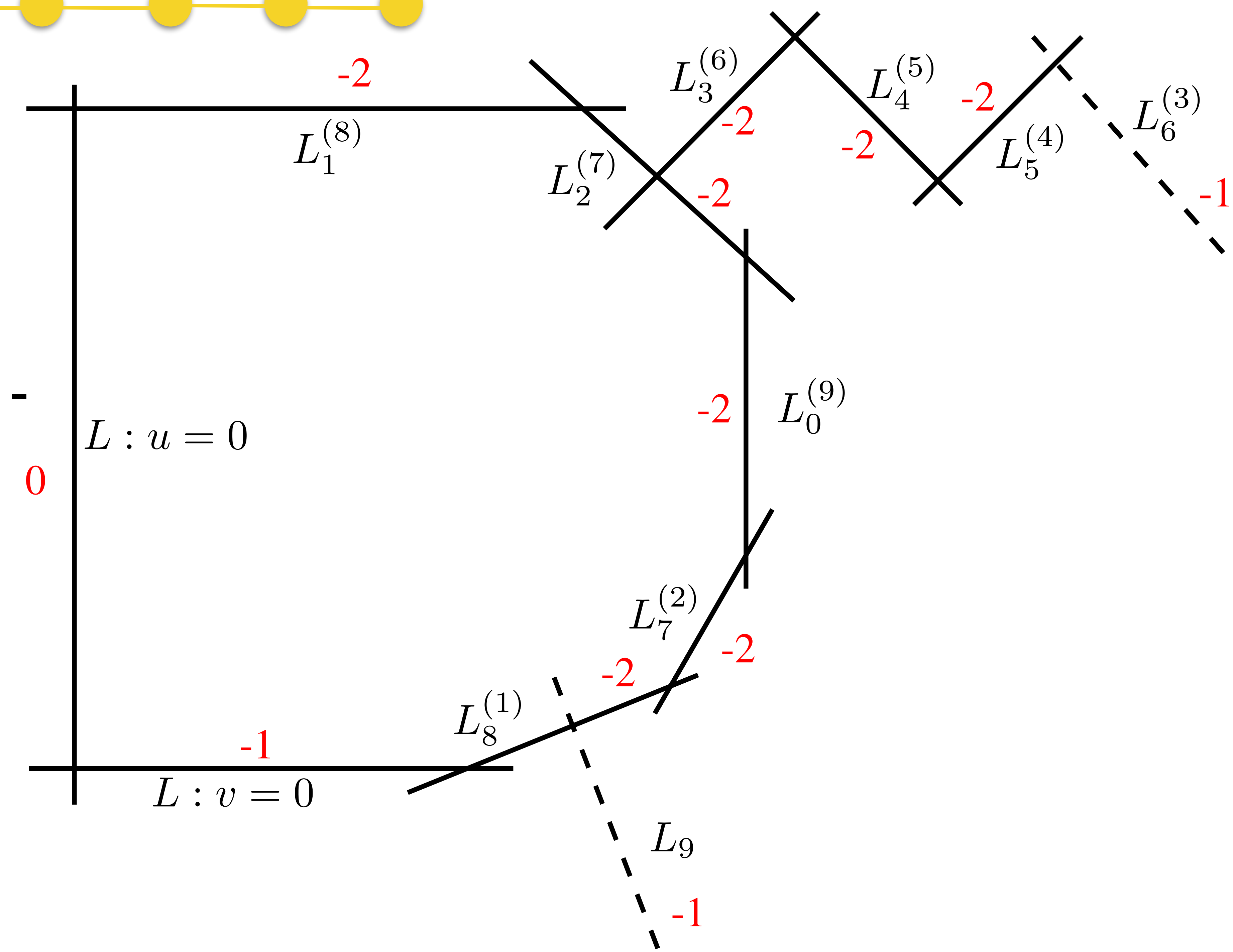
$\mathrm{P_{II}}$



Howes & Joshi, 2014

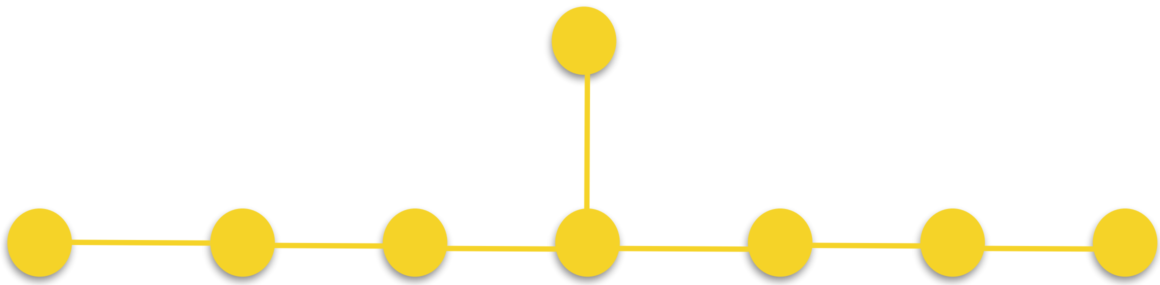
$\mathrm{P_{II}}$

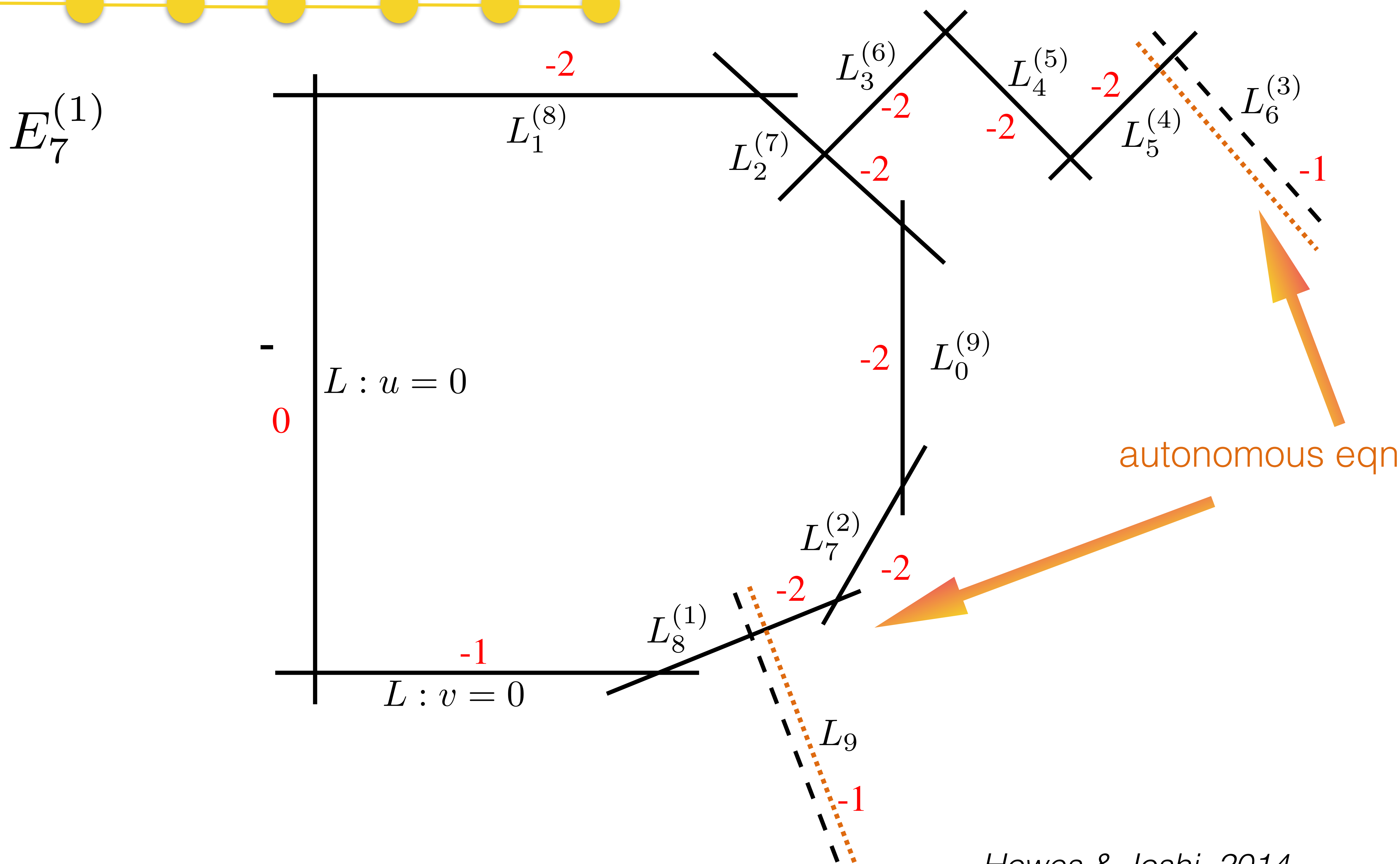




Howes & Joshi, 2014

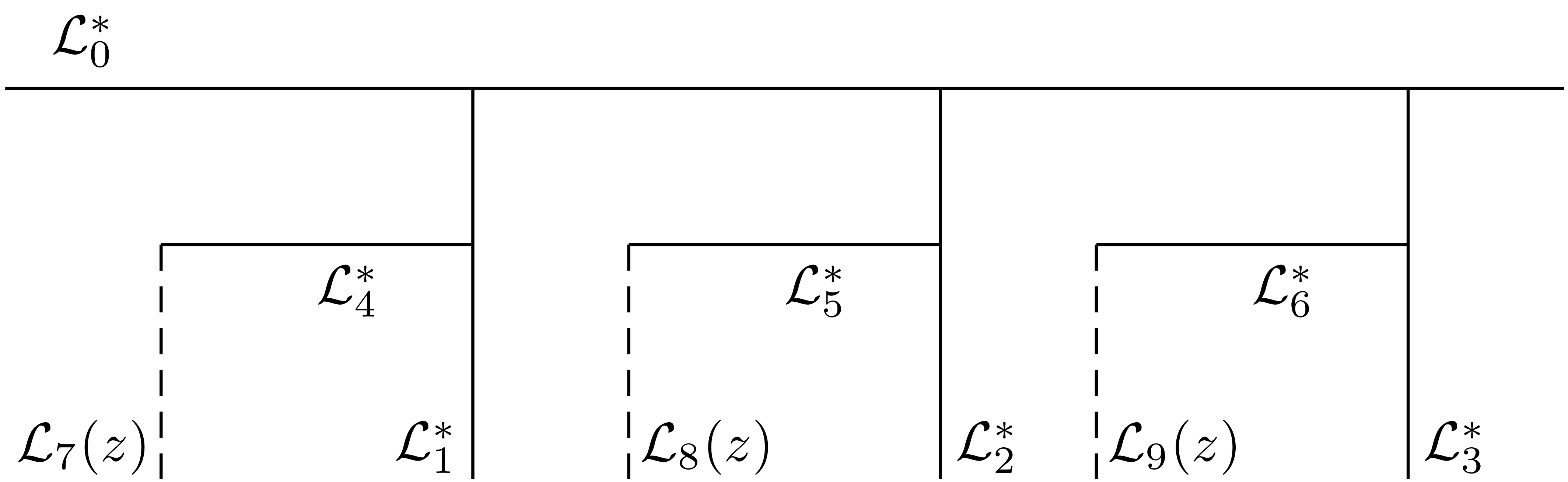
$\mathrm{P_{II}}$





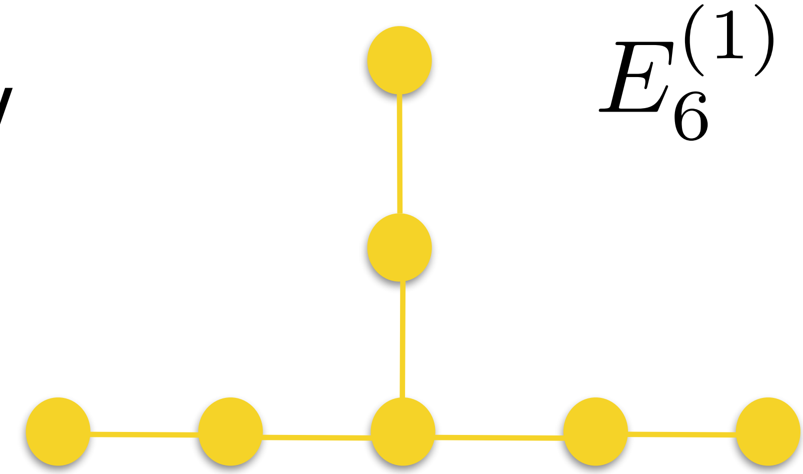
Howes & Joshi, 2014

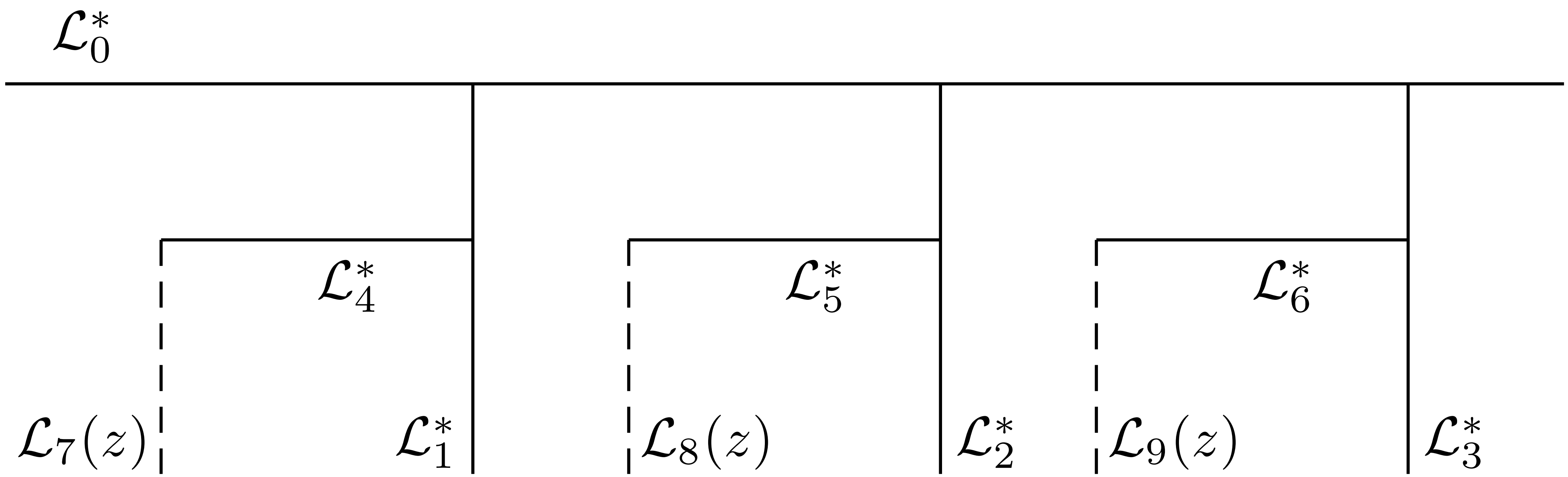
P_{IV} $E_6^{(1)}$



Joshi & Radnovic, 2016

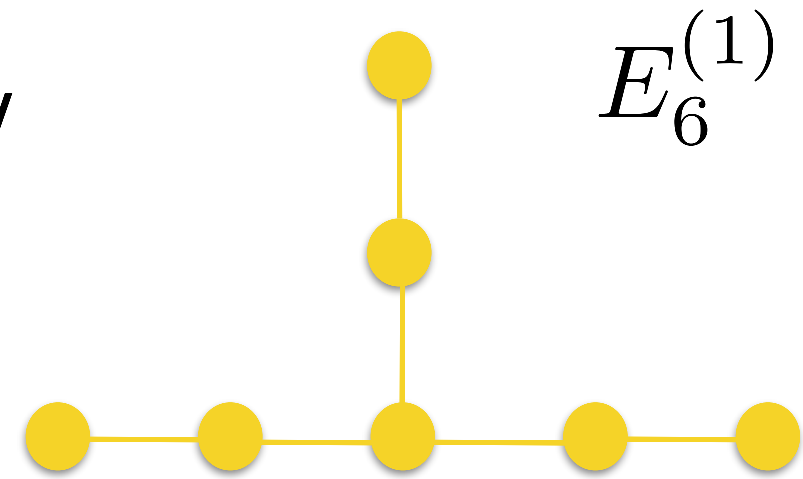
P_{IV}

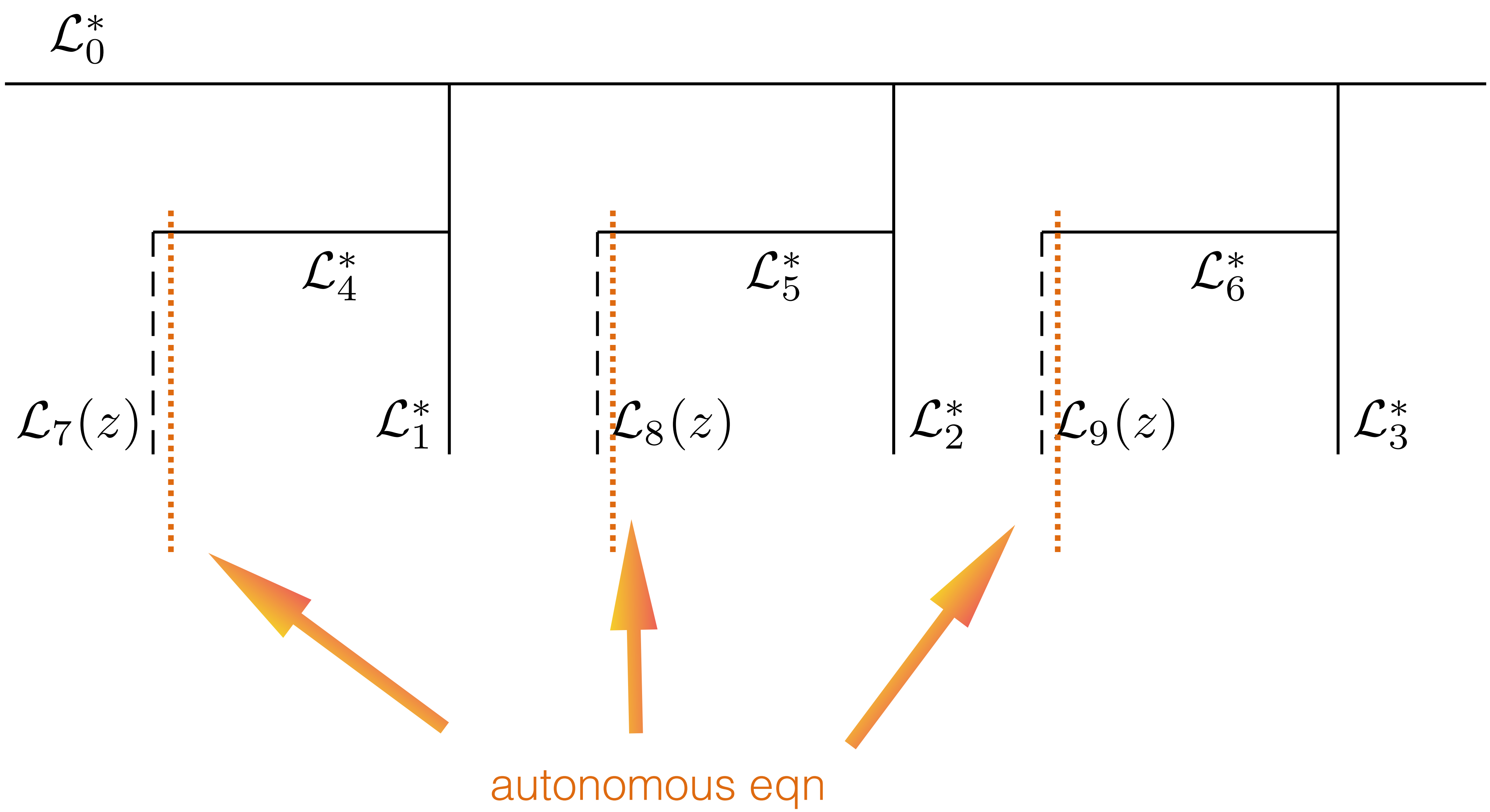




Joshi & Radnovic, 2016

$\mathrm{P_{IV}}$





Joshi & Radnovic, 2016

The infinity set and the limit set

For $z \in \mathbb{C} \backslash \{0\}$

- Let $\mathcal{S} = \bigcup S_9(z)$
- The infinity set is $I(z) := \bigcup_{i=0}^{8} L_i^{(9-i)}(z)$
- For each solution $U(z) \in S_9(z) \backslash I(z)$, let Ω_U denote the set of

$$s \in S_9(\infty) \backslash I(\infty) \text{ s.t. } \exists z_j \in \mathbb{C} \text{ with } z_j \to \infty$$

$$\text{and } U(z_j) \to s \text{ as } j \to \infty$$

Global results for P_I , P_{II} , $P_{IV,}$ P_V

- The union of exceptional lines is a repeller for the flow.
- There exists a complex limit set, which is non-empty, connected and compact.
- Every solution of P_I , every solution of P_{II} whose limit set is not {0}, and every non-rational solution of P_{IV} and P_V intersects the last exceptional line(s) infinitely many times => infinite number of movable poles and movable zeroes in every neighbourhood of the limit point.

Duistermaat & J, Arch Rational Mech. Anal 2011

Howes & J, Constr. Approx 2014

J & Radnovic, Constr. Approx 2016   J & Radnovic, 2016

Special Solutions

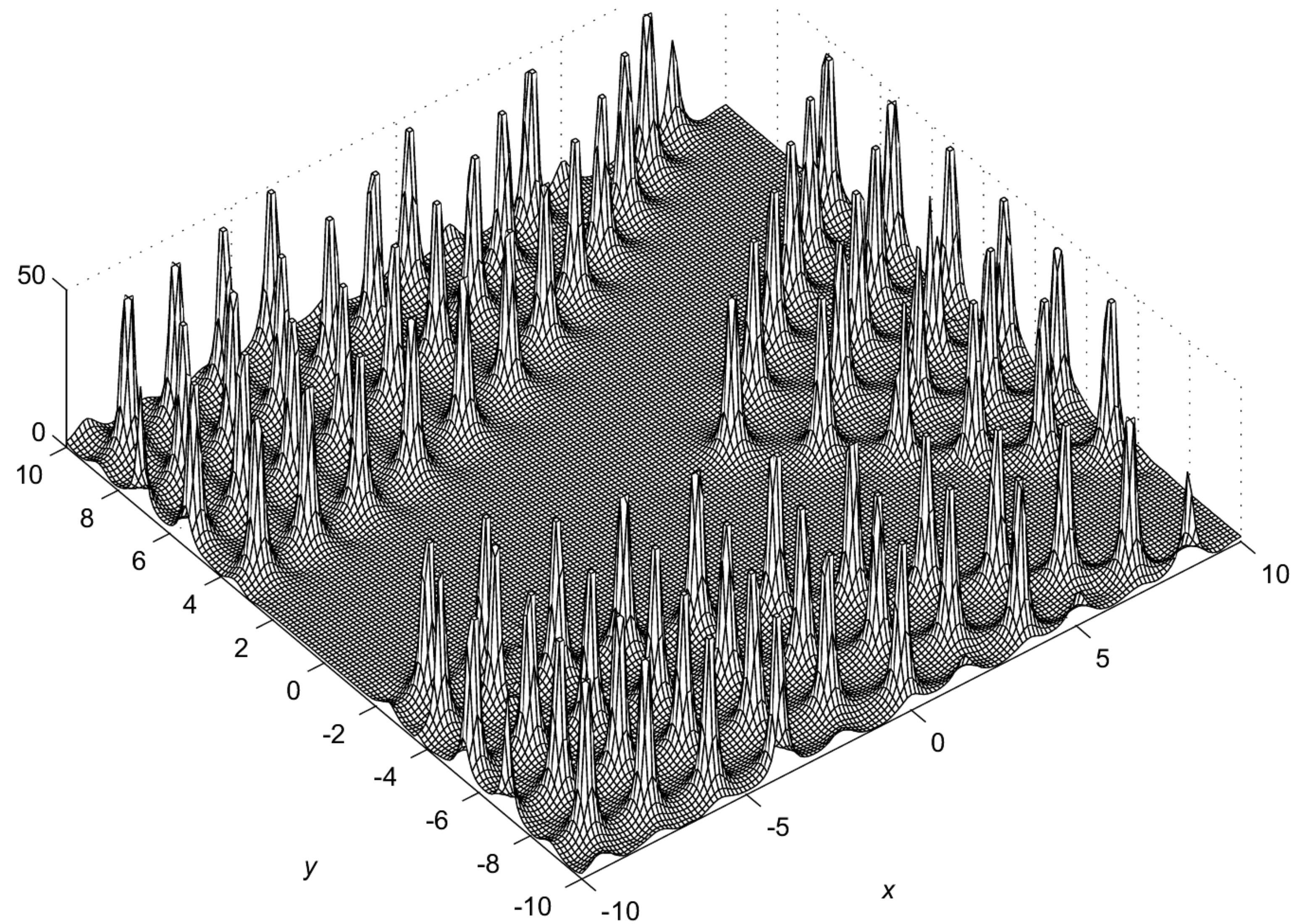


FIG. 3.1. *Magnitude of the solution $u(z)$ to the P_I equation in case of ICs $u(0) = -0.1875$, $u'(0) = 0.3049$, displayed over the domain $z = x + iy$, $-10 \leq x \leq 10$, $-10 \leq y \leq 10$.*

Fornberg and Weideman 2009

Sectors

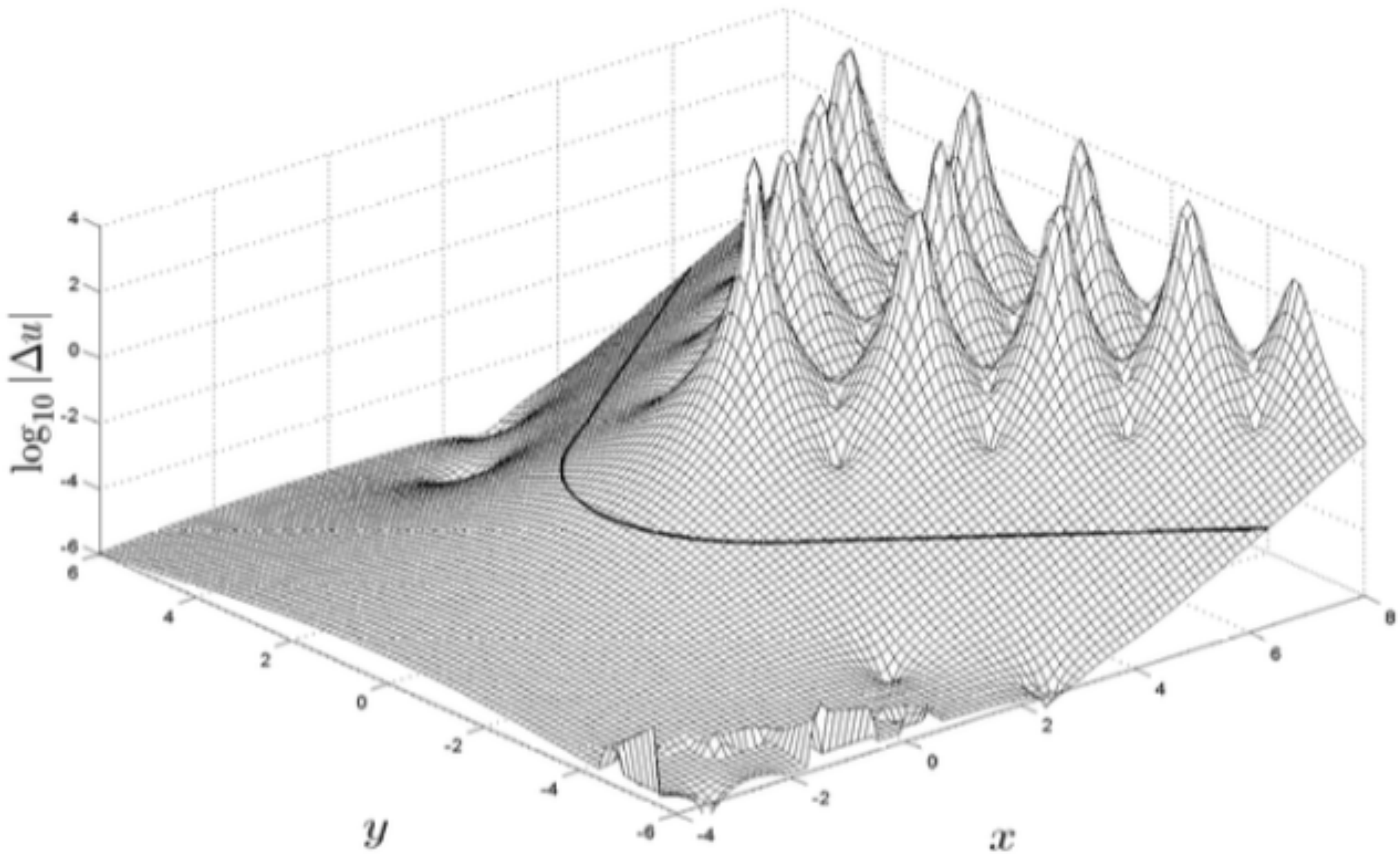
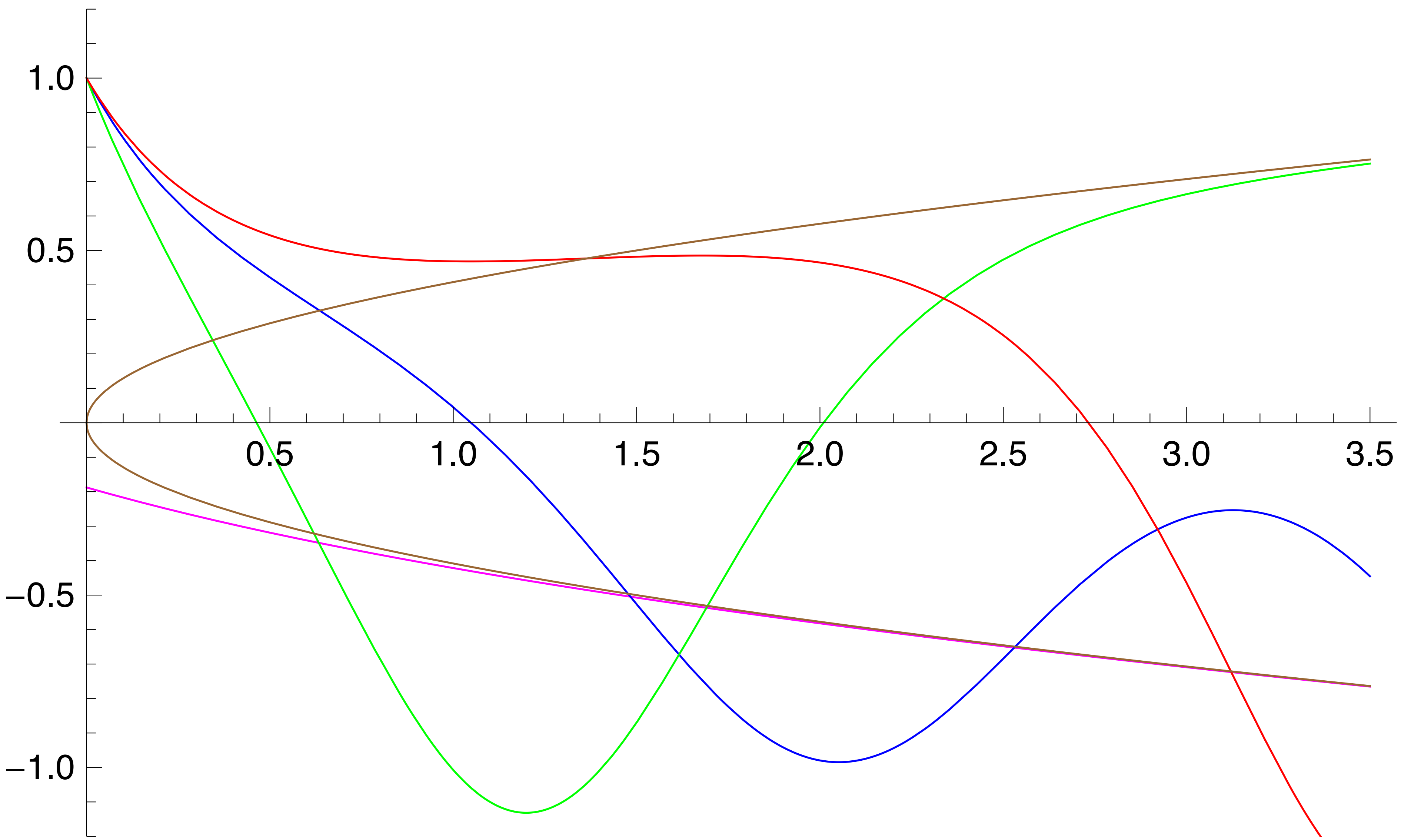


FIG. 3.3. *The approximation (3.3) applied to a numerical near-tritronquée solution in the vicinity of the origin (cf. Figures 1.1c and 3.1). The level curve* 0.001 *(solid line) is an example of a suitable pole field edge description.*

Fornberg and Weideman 2009

Consider P_I

$$w_{tt} = 6w^2 - t \qquad w(t), t \in \mathbb{R}$$



Hidden Solutions of P_I

- Solutions asymptotic to

$$\Pi_{\pm} = \left\{(x, y) \mid x > 0, y = \pm\sqrt{x/6}\right\}$$

have formal expansions

$$y_f = \frac{x^{1/2}}{\sqrt{6}} \sum_{k=0}^{\infty} \frac{a_k}{(x^{1/2})^{5k}}$$

$$a_k \underset{k\to\infty}{=} -2\,c((k-1)!)^2 \left(25/(8\sqrt{6})\right)^k$$

The coeffts a_k are important in 2D quantum gravity (Di Francesco, Ginsparg, Zinn-Justin 1994).

The Real Tritronquée

- Theorem: ∃ unique solution Y(x) of PI which has asymptotic expansion

$$y_f = -\sqrt{\frac{x}{6}}\sum_{k=0}^{\infty}\frac{a_k}{(x^{1/2})^{5k}}, \text{ in } |\arg(x)| \le 4\pi/5$$

  and

  - *Y(x)* is real for real *x*
  - Its interval of existence *I* contains $\mathbb{R}$
  - *Y(x)* lies below *Π*-
  - It is monotonically decaying in *I*.

From Joshi & Kitaev Studies in Appl Math (2001)

What about discrete Painlevé equations?

Symmetric dP1

$$w_{n+1} + w_n + w_{n-1} = \frac{\alpha\, n + \beta}{w_n} + \gamma$$

Fokas, Its & Kitaev 1991

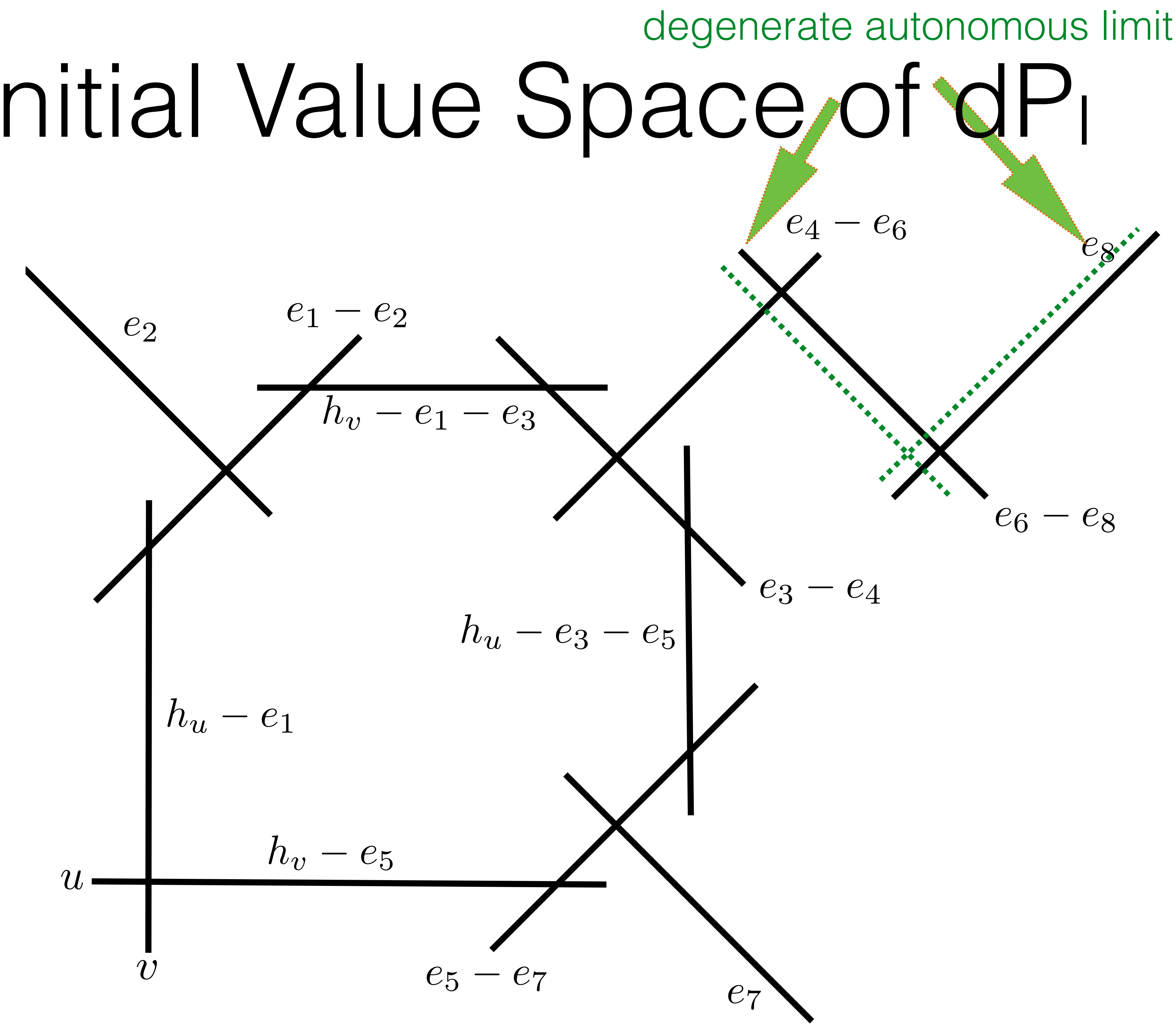
- Consider $n \to \infty$ by taking local iterates in a nhd of infinity: $n = \eta^2 + m, \eta = \epsilon^{-1/2} \to \infty$

J. 1997

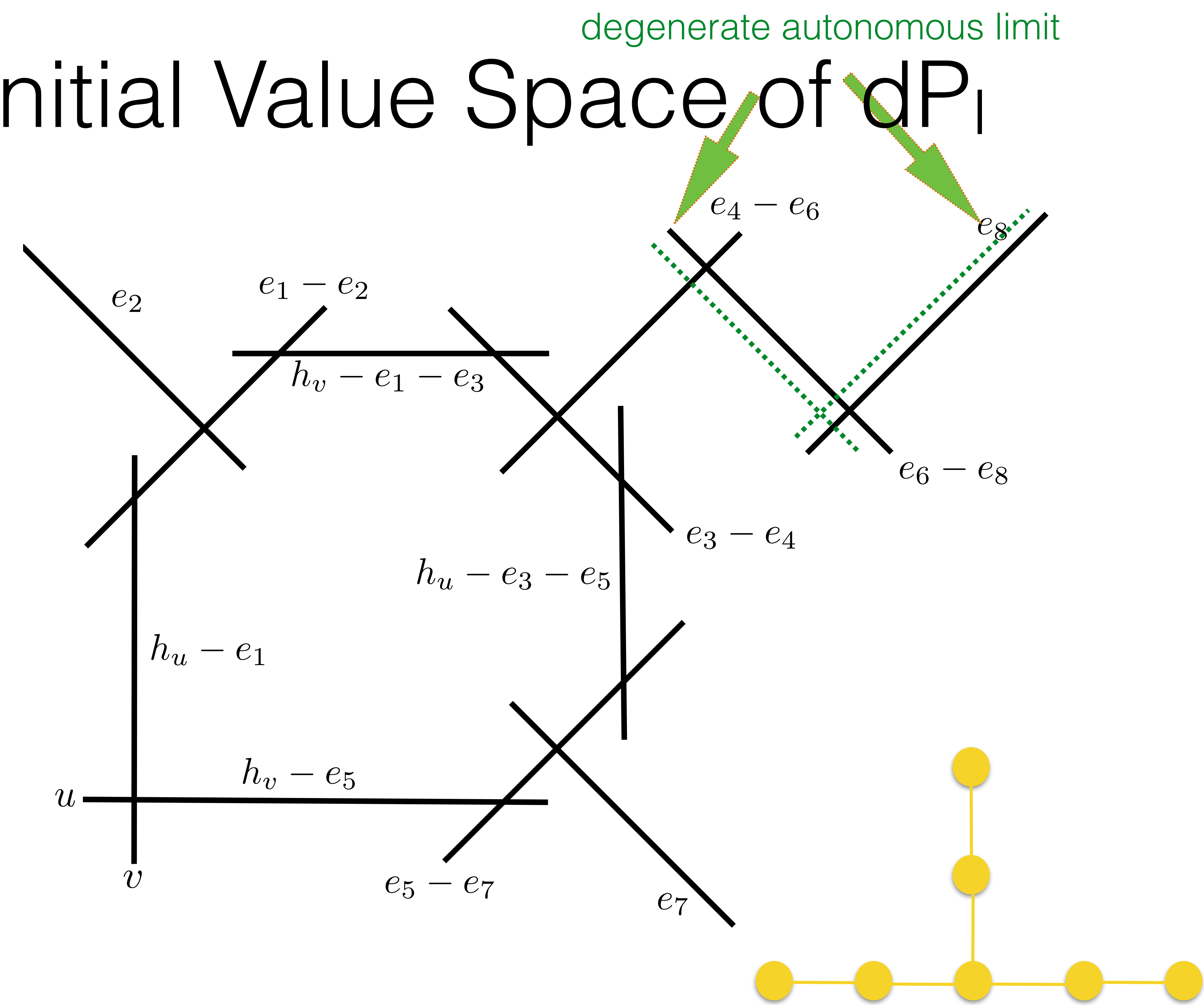
Vereschagin 1995

J. & Takei 2016

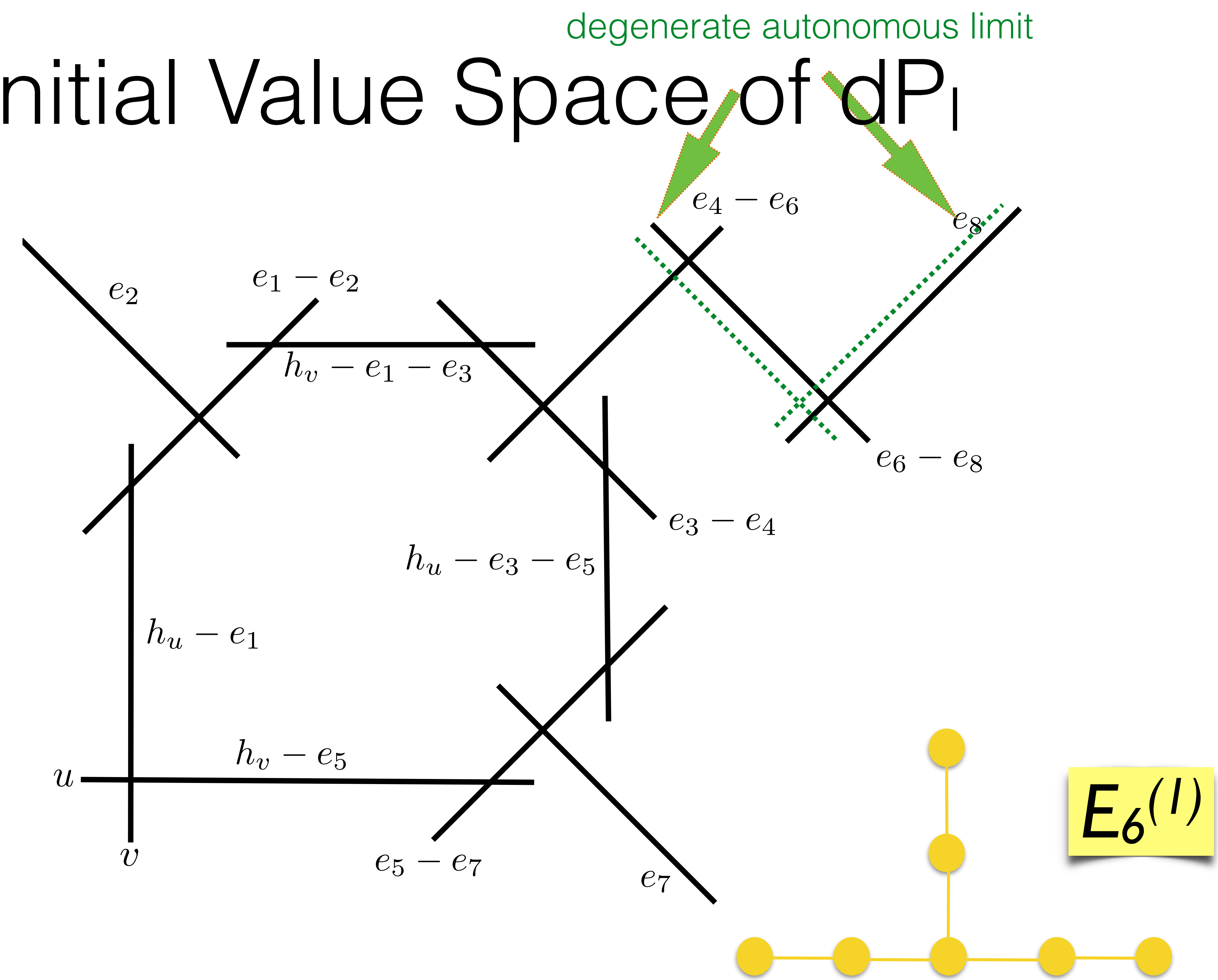
Initial Value Space of dP$_{\mathrm{I}}$



Initial Value Space of dP_I



Initial Value Space of dP$_{\text{I}}$



Late Terms

For an asymptotic series

$$f(z) \sim \sum_{n=0}^{\infty} \epsilon^n f_n(z), \qquad \epsilon \to +0$$

with factorially growing coefficients, we write

$$f_n(z) \sim \frac{F(z)\Gamma(n+\gamma)}{\chi(z)^{n+\gamma}}$$

Remainder

It follows that

$$f(z) = \sum_{n=0}^{N} \epsilon^n f_n(z) + R_N(z)$$

implies

$$R_N(z) \sim \mathcal{S} F(z)\, e^{-\chi(z)/\epsilon},$$

with Stokes lines following

$$\Im\big(\chi(z)\big) = 0$$

See e.g. Olde Daalhuis et al 1995

Scaling

$$\begin{cases} w_{2k} & = \frac{u(s)}{\epsilon^{1/2}} \\ w_{2k-1} & = \frac{v(s)}{\epsilon^{1/2}} \end{cases} \qquad s = \epsilon\, n$$

- dPI becomes

$$\big(v(s+\epsilon) + u(s) + v(s-\epsilon)\big)u(s) = \alpha\, s + \epsilon\, \beta + \epsilon^{1/2}\, \gamma\, u(s)$$
$$\big(u(s+\epsilon) + v(s) + u(s-\epsilon)\big)v(s) = \alpha\, s + \epsilon\, \beta + \epsilon^{1/2}\, \gamma\, u(s)$$

- Series expansions as $\epsilon \to 0$

$$u(s) \sim \sum_{m=0}^{\infty} \epsilon^{m/2}\, u_m(s)$$
$$v(s) \sim \sum_{m=0}^{\infty} \epsilon^{m/2}\, v_m(s)$$

J. & Lustri 2015

Types of solutions

- Type A

$$u \sim \pm\sqrt{-\alpha s} + \frac{\gamma\epsilon^{1/2}}{2} \mp \frac{(4\beta - \gamma^2)\epsilon}{8\sqrt{-\alpha s}} + \dots$$

$$v \sim \mp\sqrt{-\alpha s} + \frac{\gamma\epsilon^{1/2}}{2} \pm \frac{(4\beta - \gamma^2)\epsilon}{8\sqrt{-\alpha s}} + \dots$$

- Type B

$$u = v \sim \pm\sqrt{\frac{\alpha s}{3}} + \frac{\gamma\epsilon^{1/2}}{6} \mp \pm \frac{\sqrt{3}(12\beta + \gamma^2)\epsilon}{72\sqrt{\alpha s}} + \dots$$

Late-order terms: Type A

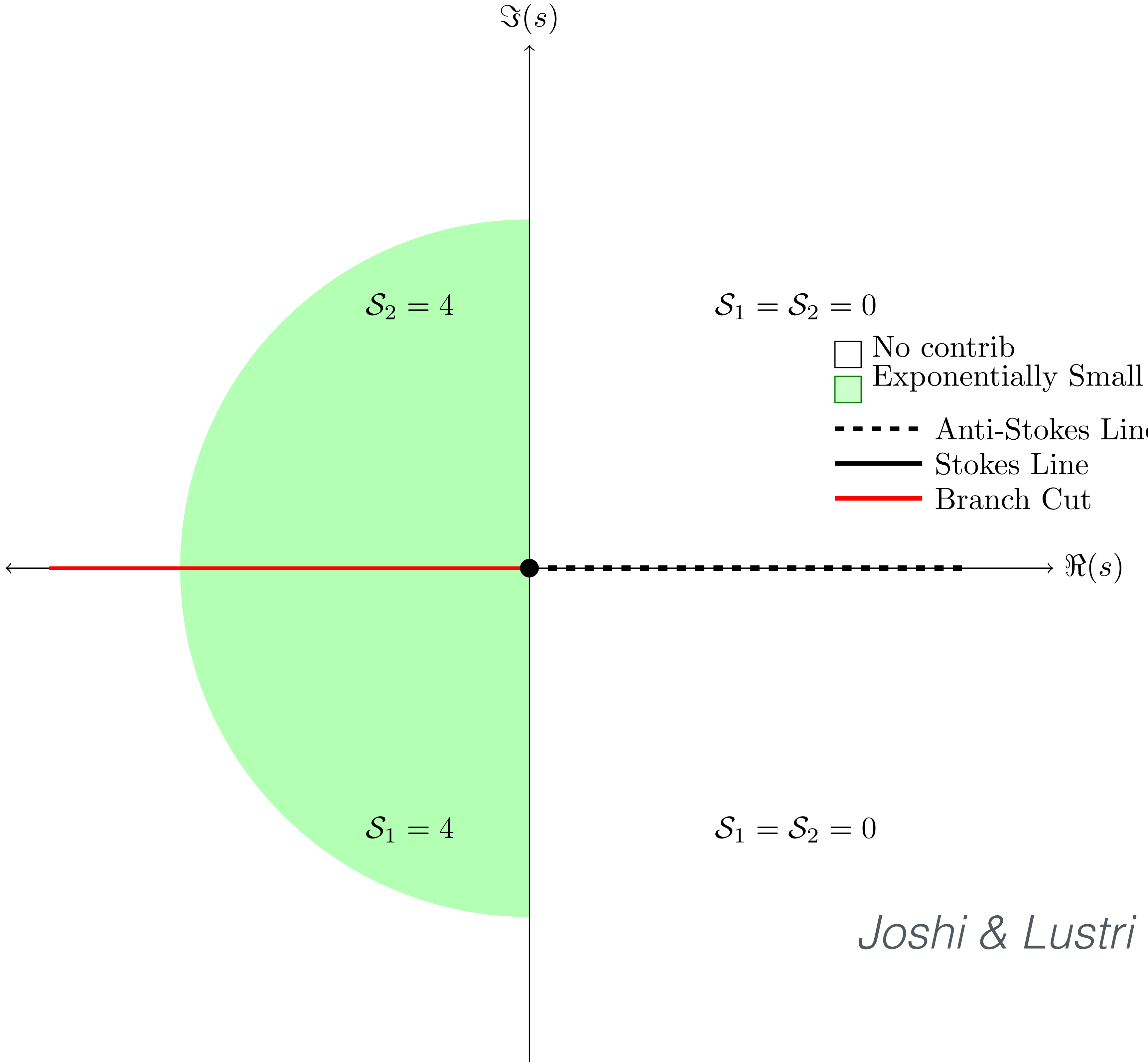
$$u_m \sim \frac{\Lambda_1\,\Gamma(\frac{m-1}{2})}{(i\pi\,s/2)^{\frac{m-1}{2}}} + \frac{\Lambda_2\,\Gamma(\frac{m-1}{2})}{(-i\pi\,s/2)^{\frac{m-1}{2}}}$$

$$v_m \sim \frac{\Lambda_3\,\Gamma(\frac{m-1}{2})}{(i\pi\,s/2)^{\frac{m-1}{2}}} + \frac{\Lambda_4\,\Gamma(\frac{m-1}{2})}{(-i\pi\,s/2)^{\frac{m-1}{2}}}$$

- Optimal truncation

$$u(s) \sim \sum_{m=0}^{N_o} \epsilon^{m/2} u_m(s) + S_1\,\Lambda_1 (-i)^{s/\epsilon} + S_2\,\Lambda_2\, i^{s/\epsilon}$$

Stokes Sectors: Type A



Joshi & Lustri 2015

Symmetric dP2

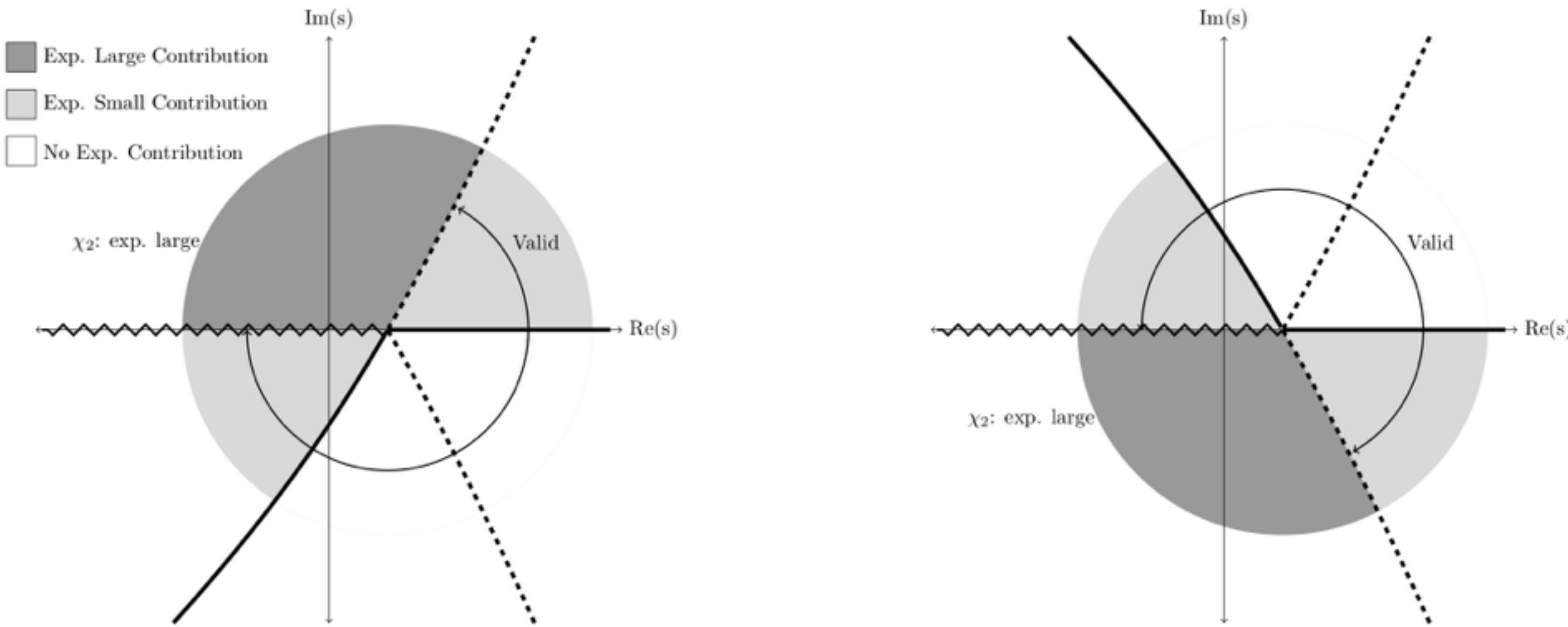
$$w_{n+1} + w_{n-1} = \frac{(\alpha\, n + \beta)\, w_n + \gamma}{1 - w_n^2}$$

There are two scalings:

$$\text{(i) } w_n = \epsilon f_n$$

$$\text{(ii)} w_n = \frac{g_n}{\epsilon}$$

Stokes Sectors Type (i)

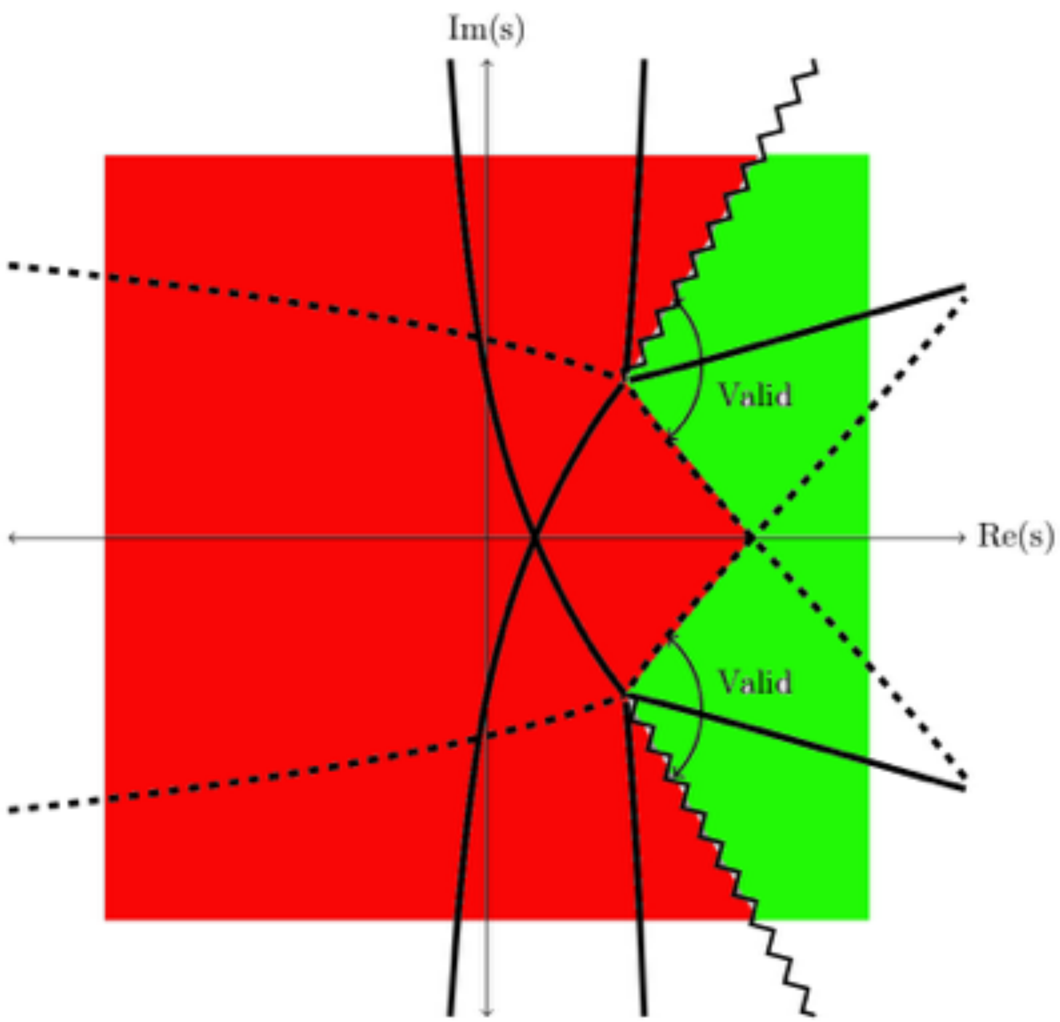


(a) First special asymptotic solution.

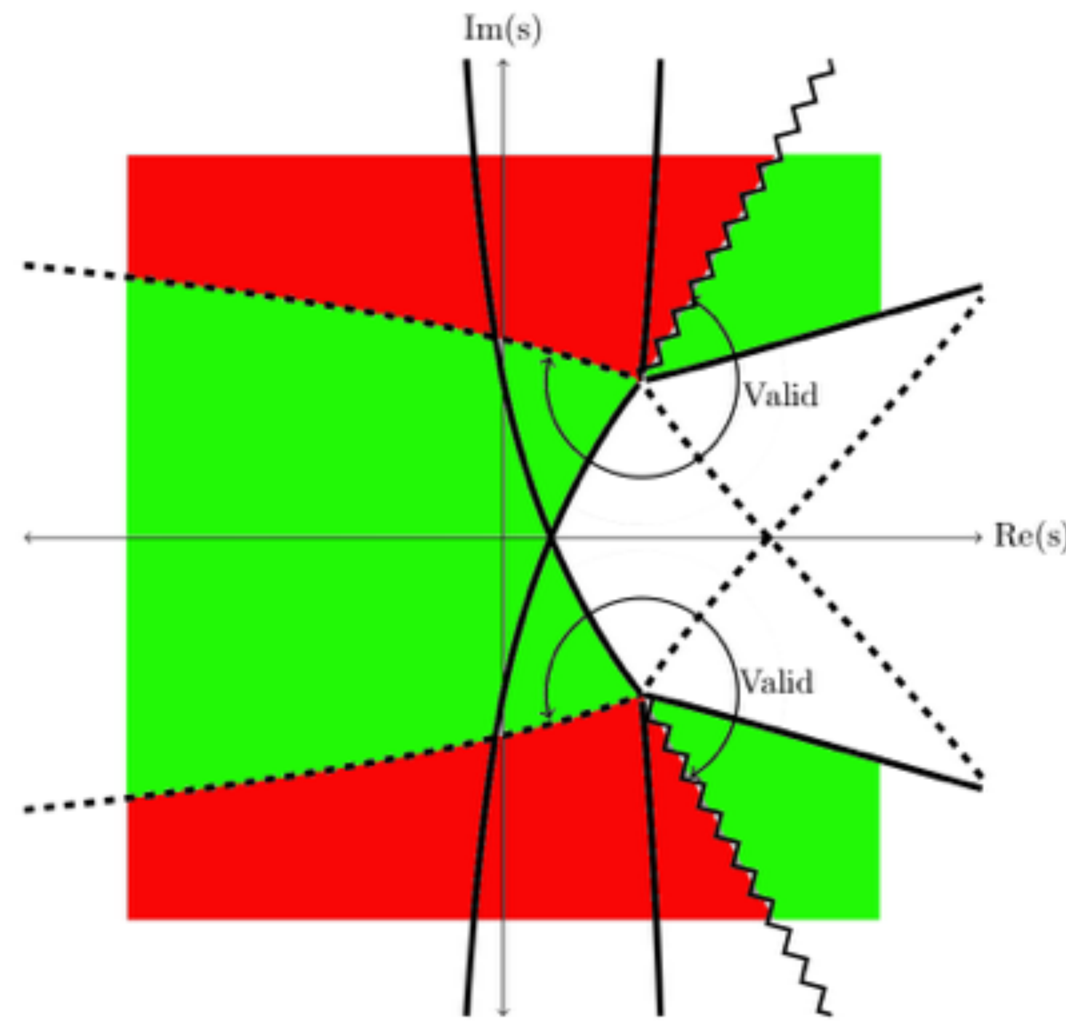
(b) Second special asymptotic solution.

Joshi, Lustri & Luu 2016

Stokes Sectors Type (ii)



(a) Composite general asymptotic solution.



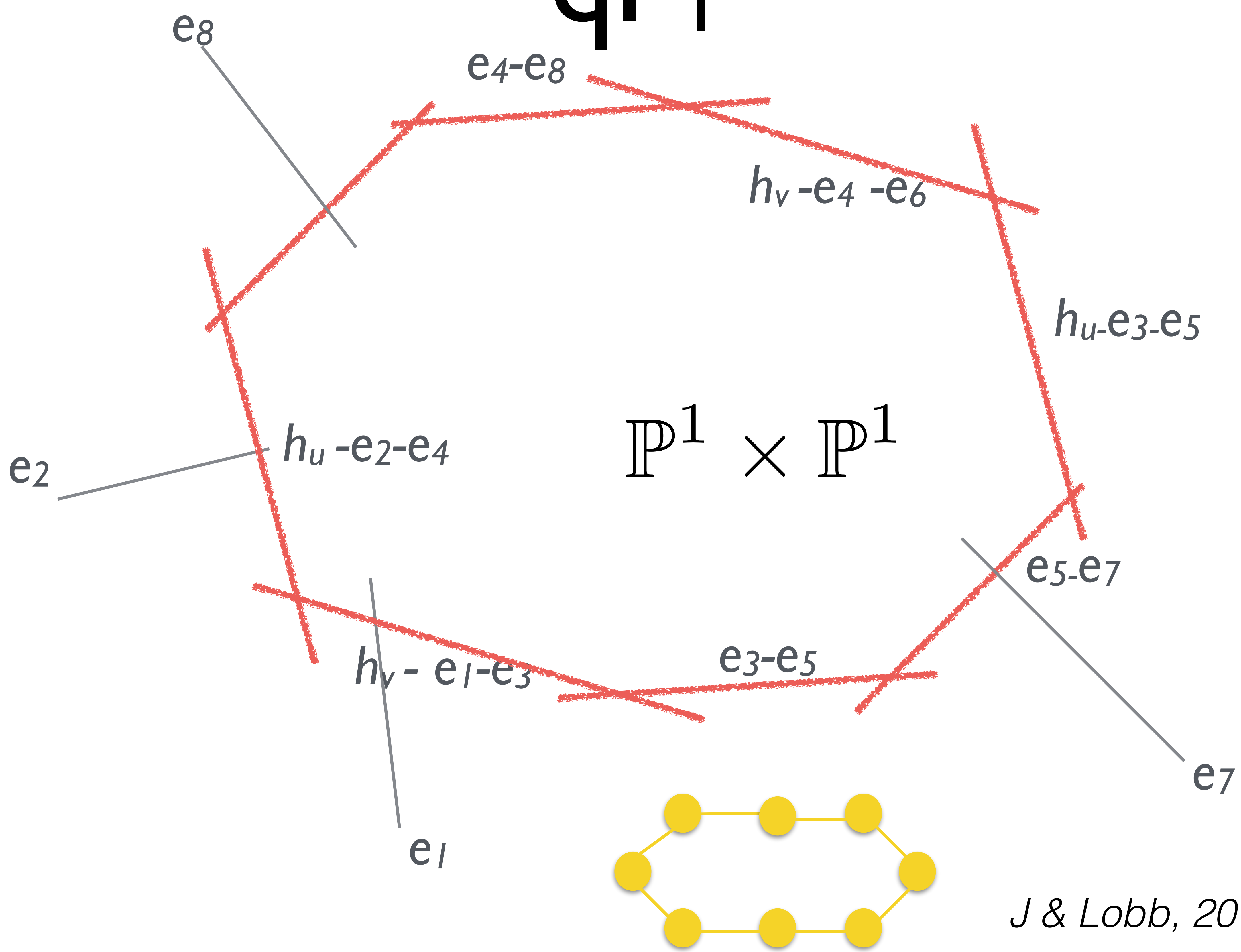
(b) Composite special asymptotic solution.

Joshi, Lustri & Luu 2016

q-discrete Painlevé equations

qP$_I$

$A_7^{(1)}$



J & Lobb, 2016

qP1

$$\Rightarrow \quad \overline{w}\,\underline{w} = \frac{1}{w} - \frac{1}{\xi\, w^2} \qquad (\mathrm{qP_I})$$

$$\overline{w} = w(q\,\xi), w = w(\xi), \underline{w} = w(\xi/q)$$

$\mapsto$ **PI:** $\quad y'' = 6\,y^2 - t \quad$ in continuum limit.

Behaviours near fixed points

$$\overline{w} \sim w, \quad \underline{w} \sim w, \quad |\xi| \to \infty$$

$$\Rightarrow \quad w^4 = w + \mathcal{O}(1/\xi)$$

$$\Rightarrow \quad w = \begin{cases} \omega + \mathcal{O}(1/\xi) \\ \mathcal{O}(1/\xi) \end{cases} \qquad \omega^3 = 1$$

- qP_I is invariant under rotation by argument 2π/3, so ω can be replaced by unity.
- The second case lies in neighbourhood of a merger of two base points: *(1/ξ ,0), (q/ξ , 0)*.

Near zero

- Near $w = 1/\xi,\ \underline{w} = q/\xi,\ \exists$ a formal series solution

$$w(\xi) = \sum_{n=1}^{\infty} \frac{b_n}{\xi^n}$$

where

$$b_1 = 1,\ b_2 = 0,\ b_3 = 0$$

$$b_n = \sum_{r=2}^{n-2} \sum_{k=1}^{r-1} \sum_{m=1}^{n-r-1} b_k\, b_{r-k}\, b_m\, b_{n-r-m} q^{(r-2\,k)},\ n \geq 4$$

Divergence

The coefficients of the asymptotic series grow very fast:

$$b_{3\,p+1} \underset{p\to\infty}{=} \mathcal{O}\left(|q|^{3\,p\,(p-1)/2} \prod_{k=0}^{p-1}\left(1+q^{-3k}\right)^2\right), |q| > 1$$

$$b_{3\,p+2} = 0,\ b_{3\,p+3} = 0,\ \forall p \geq 0$$

so the series diverges for all ξ, except $1/\xi = 0$

Quicksilver solution

- The vanishing solution occurs in a neighbourhood of two *merging* base points.
- Although the series expansion is divergent, we can prove a true solution exists with this behaviour.
- We gave it a new name: *quicksilver* solution
- It is unstable in initial-value space.

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Summary

- Global dynamics of solutions of non-linear equations, whether they are differential or discrete, can be found through geometry.
- Questions about global dynamics of discrete problems remain open.
- Special transcendental solutions give rise to Stokes phenomena in asymptotic limits.
- Connection problems for q-discrete Painlevé equations remain unsolved.
- Tantalising questions about finite properties of solutions remain open.