Tensor Triangular Geometry with Applications to Classical Lie Superalgebras

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Overview of the Talk

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I) Algebra (Tensor Triangulated Categories, Finite Group Schemes, Lie Algebras/Superalgebras, Commutative Rings)

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II) Geometry (Zariski Spaces, Spec of a Ring, N-Spec, Support Varieties)

III) Combinatorics (atypicality, Symmetric Groups, verification of classifying support data,)

$\boxed{\text{Tensor Triangulated Category}}$

Structure of Tensor Categories and Topological Spaces

Tensor Triangulated Category

Zariski Space

Structure of Tensor Categories and Topological Spaces

$$\begin{tikzcd}
\boxed{\text{Tensor Triangulated Category}} \arrow[drr, "\boxed{\text{Support Data}}"] & & \\
& & \boxed{\text{Zariski Space}}
\end{tikzcd}$$

Tensor Triangulated Categories

Definition

A *tensor triangulated category* (TTC) is a triple $(\mathbf{K}, \otimes, \mathbf{1})$ such that

(i) $\mathbf{K}$ is a triangulated category,

(ii) $\mathbf{K}$ has a symmetric monodial tensor product $\otimes : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ which is exact in each variable with unit object $\mathbf{1}$.

Finite Group Schemes

Example

Let G be a finite group scheme (equivalently a finite-dimensional cocommutative Hopf algebra). Let

(i) $\mathbf{S}^c = \mathrm{mod}(G)$ finite-dimensional modules for G

(ii) $\mathbf{S} = \mathrm{Mod}(G)$.

The categories $\mathbf{S}^c$ and $\mathbf{S}$ are symmetric monoidal tensor categories. Now consider

(iii) $\mathbf{K}^c = \mathrm{stmod}(G)$ stable module category of finite-dimensional modules for G

(iv) $\mathbf{K} = \mathrm{StMod}(G)$ stable module category for $\mathrm{Mod}(G)$.

Then $\mathbf{K}^c$ and $\mathbf{K}$ are tensor triangulated categories (with shift $\Sigma = \Omega^{-1}$).

Perfect Complexes

Example

Let R be a commutative Noetherian ring. Let

(i) $\mathbf{K}^c = D^b_{perf}(R)$ bounded derived category of finitely generated projective R-modules

(ii) $\mathbf{K} = D(R)$ derived category of R-modules.

Then $\mathbf{K}^c$ and $\mathbf{K}$ are tensor triangulated categories.

Ideals in a TTC

Definition

(a) A *(tensor) ideal* in $\mathbf{K}$ is a triangulated subcategory $\mathbf{I}$ of $\mathbf{K}$ such that $M \otimes N \in \mathbf{I}$ for all $M \in \mathbf{I}$ and $N \in \mathbf{K}$.

(b) An ideal $\mathbf{I}$ is *thick* if $M_1 \oplus M_2 \in \mathbf{I}$ then $M_j \in \mathbf{I}$ for $j = 1, 2$.

(c) A *prime ideal* $\mathbf{P}$ of $\mathbf{K}$ is a proper thick tensor ideal such that if $M \otimes N \in \mathbf{P}$ then either $M \in \mathbf{P}$ or $N \in \mathbf{P}$.

Balmer Spectrum

Definition

The *Balmer spectrum* is defined as

$$\mathrm{Spc}(\mathbf{K}) = \{\mathbf{P} \subset \mathbf{K} \mid \mathbf{P} \text{ is a prime ideal}\}.$$

The topology on $\mathrm{Spc}(\mathbf{K})$ is given by closed sets of the form

$$Z(\mathcal{C}) = \{\mathbf{P} \in \mathrm{Spc}(\mathbf{K}) \mid \mathcal{C} \cap \mathbf{P} = \varnothing\}$$

where $\mathcal{C}$ is a family of objects in $\mathbf{K}$.

Zariski Spaces

Definition

Assume throughout that X is a Noetherian topological space. In this case any closed set in X is the union of finitely many irreducible closed sets. We say that X is a *Zariski space* if in addition any irreducible closed set Y of X has a unique generic point (i.e., $y \in Y$ such that $Y = \overline{\{y\}}$).

Example

Let $X = \text{Spec}(R)$ where R is a commutative Noetherian ring. Then X is a Zariski space. If R is graded one can also consider $X = \text{Proj}(\text{Spec}(R))$.

Zariski Spaces: Notation

(i) $\mathcal{X}$ be the collection of closed subsets of X.

(ii) $\mathcal{X}_{irr}$ be the set of irreducible closed sets.

(iii) A subset W in X is *specialization closed* if and only if $W = \cup_{j \in J} W_j$ where W_j are closed sets.

(iv) $\mathcal{X}_{sp}$ be the collection of all specialization closed subsets of X.

Support Data

Definition

A *support data* is an assignment $V : \mathbf{K}^c \to \mathcal{X}_{sp}$ which satisfies the following six properties (for $M, M_i, N, Q \in \mathbf{K}$):

(S1) $V(0) = \varnothing$, $V(\mathbf{1}) = X$;

(S2) $V(\oplus_{i \in I} M_i) = \bigcup_{i \in I} V(M_i)$ whenever $\oplus_{i \in I} M_i$ is an object of $\mathbf{K}$;

(S3) $V(\Sigma M) = V(M)$;

(S4) for any distinguished triangle $M \to N \to Q \to \Sigma M$ we have

$$V(N) \subseteq V(M) \cup V(Q);$$

(S5) $V(M \otimes N) = V(M) \cap V(N)$;

(S6) $V(M) = V(M^*)$ for $M \in \mathbf{K}^c$ [the compact objects have a duality].

We will be interested in support data which satisfy an additional two properties:

(S7) $V(M) = \varnothing$ if and only if $M = 0$;

(S8) for any $W \in \mathcal{X}$ there exists an $M \in \mathbf{K}^c$ such that $V(M) = W$ (Realization Property).

Definition

We say that $\mathcal{V} : \mathbf{K} \to \mathcal{X}_{sp}$ *extends* $V : \mathbf{K}^c \to \mathcal{X}$ if

(i) $\mathcal{V}$ satisfies properties (S1)–(S5) for objects in $\mathbf{K}$;

(ii) $\mathcal{V}(M) = V(M)$ for all $M \in \mathbf{K}^c$; and

(iii) if V satisfies (S7) then $\mathcal{V}$ satisfies (S7).

Classifying Thick Tensor Ideals in a TTC

Theorem (BKN, Dell'Ambrogio)

Let $\mathbf{K}$ *be a compactly generated TTC. Let* X *be a Zariski space and let* $V : \mathbf{K}^c \to \mathcal{X}$ *be a support data defined on* $\mathbf{K}^c$ *satisfying the additional conditions (S7) and (S8). Moreover, assume* $\mathcal{V} : \mathbf{K} \to \mathcal{X}_{sp}$ *extends* V*.*
Given the above setup there is a pair of mutually inverse maps

$$\{\textit{thick tensor ideals of } \mathbf{K}^c\} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{X}_{sp},$$

given by

$$\Gamma(\mathbf{I}) = \bigcup_{M \in \mathbf{I}} V(M), \qquad \Theta(W) = \mathbf{I}_W,$$

where $\mathbf{I}_W = \{M \in \mathbf{K}^c \mid V(M) \subseteq W\}$*.*

Computing the Balmer Spectrum

Theorem

Let $\mathbf{K}$ *be a compactly generated TTC and let* X *be a Zariski space. Assume that* $V : \mathbf{K}^c \to \mathcal{X}$ *is a support data defined on* $\mathbf{K}^c$ *satisfying the additional conditions (S7) and (S8). Further assume that we have a support data* $\mathcal{V} : \mathbf{K} \to \mathcal{X}_{sp}$ *which extends* V*. Then there is a homeomorphism*

$$f : X \to \mathrm{Spc}(\mathbf{K}^c).$$

Finite Group Schemes

Example

Let G be a finite group scheme, $A := \mathrm{H}^\bullet(G,k) = \mathrm{Ext}^\bullet_G(k,k)$ be the cohomology ring. Set $\mathbf{K}^c = \mathrm{stmod}(G)$ and $X = \mathrm{Proj}(\mathrm{Spec}(A))$.

(i) $\{$thick $\otimes$-ideals of $\mathbf{K}^c\}$ are in one-to-correspondence with $\mathcal{X}_{sp}$.

(ii) $\mathrm{Spc}(\mathbf{K}^c) \cong \mathrm{Proj}(\mathrm{Spec}(A))$.

The (classifying) support data is given by

$$V(M) = \{P \in \mathrm{Proj}(\mathrm{Spec}(A)) : \mathrm{Ext}^\bullet_G(M,M)_P \neq 0\}.$$

Restricted Lie Algebras

Example

Let G be a reductive algebraic group over k an algebraically closed field of characteristic $p > 0$. Let $\mathfrak{g} = \text{Lie } G$, $u(\mathfrak{g})$ be the restricted universal enveloping algebra (f.d. Hopf algebra), and $\mathcal{N}$ be the nilpotent cone for $\mathfrak{g}$.

Theorem (Friedlander-Parshall, Andersen-Jantzen, 1984)

Let $p > h$

(a) $\mathrm{H}^{2\bullet}(u(\mathfrak{g}), k) \cong k[\mathcal{N}]$

(b) $\mathrm{H}^{2\bullet+1}(u(\mathfrak{g}), k) = 0$

Then the Balmer Spectrum "realizes the nilpotent cone":

$$\mathrm{Spc}(\mathrm{stmod}(u(\mathfrak{g})) \cong \mathrm{Proj}(\mathrm{Spec}(k[\mathcal{N}])).$$

Perfect Complexes

Example

Let R be a commutative Noetherian ring, $\mathbf{K}^c = D^b_{perf}(R)$ and $X = \mathrm{Spec}(R)$. Then

(i) $\{$thick $\otimes$-ideals of $\mathbf{K}^c\}$ are in one-to-correspondence with $\mathcal{X}_{sp}$.

(ii) $\mathrm{Spc}(\mathbf{K}^c) \cong \mathrm{Spec}(R)$.

The support data which gives this classification is

$$V(C^\bullet) = \{P \in \mathrm{Spec}(R) : \mathrm{H}^*(C^\bullet)_P \neq 0\}.$$

Lie Superalgebras

Throughout let $k = \mathbb{C}$. Let $\mathfrak{g}$ be a *Lie superalgebra* which is a $\mathbb{Z}_2$-graded vector space

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with a bracket operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which preserves the $\mathbb{Z}_2$-grading and satisfies graded versions of the usual Lie bracket axioms.

Definition

A finite dimensional Lie superalgebra $\mathfrak{g}$ is called *classical* if there is a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$ and an action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ which differentiates to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.

$\mathfrak{g}l(m|n)$

Example

The underlying vector space for $\mathfrak{g} = \mathfrak{gl}(m|n)$ is the set of $(m+n) \times (m+n)$ matrices over $\mathbb{C}$. We have $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, where $\mathfrak{g}_{\bar{0}}$ consists of matrices of the form:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Moreover, $\mathfrak{g}_{\bar{1}}$ consists of matrices

$$\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}.$$

The supercommutator is given by

$$[E_{i,j}, E_{k,l}] = E_{i,j}E_{k,l} - (-1)^{\bar{E}_{i,j}\bar{E}_{k,l}} E_{k,l}E_{i,j}.$$

The Stable Module Category

Consider the following categories:

(i) $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$: finite-dimensional $\mathfrak{g}$-modules for a classical Lie superalgebra (i.e., $\mathfrak{g} = \mathfrak{gl}(m|n)$) which are completely reducible over $\mathfrak{g}_{\bar{0}}$

(ii) $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$: $\mathfrak{g}$-modules for a classical Lie superalgebra (i.e., $\mathfrak{g} = \mathfrak{gl}(m|n)$) which are completely reducible over $\mathfrak{g}_{\bar{0}}$

(iii) $\mathbf{K}^c = \mathrm{stmod}(\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})})$

(iv) $\mathbf{K} = \mathrm{StMod}(\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})})$

The category $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ is self-injective which makes $\mathbf{K}$ and $\mathbf{K}^c$ into triangulated categories. In fact, both $\mathbf{K}$ and $\mathbf{K}^c$ are tensor triangulated categories.

Support Data Arising from the Cohomology of $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$

Let $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, M)$ be the relative Lie algebra cohomology of the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ which is obtained from the complex

$$C^\bullet = \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(\Lambda^\bullet_{super}(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M).$$

Theorem (BKN)

Let $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ be as above. Then

$$\mathrm{Ext}^\bullet_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}}(\mathbb{C}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \cong (\Lambda^\bullet_{super}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})^*)^{G_{\bar{0}}} \cong S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}.$$

Note that the cohomology ring is finitely generated because $G_{\bar{0}}$ is reductive.

Using the finite generation of cohomology we can define the following support data for modules in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$:

$$V_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \{P \in \mathrm{Proj}(\mathrm{Spec}(\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}},\mathbb{C}))) : \mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M,M)_P \neq 0\}.$$

WARNING: This support data will not classify thick tensor ideals because $V_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(-)$ does not detect projectivity. In fact $V_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda)) = \varnothing$ for any Kac module $K(\lambda)$.

Connections with Combinatorial Invariants

Theorem (BKN)

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, *and* $L(\lambda)$ *be a simple module in* $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$. *Then*

$$\dim V_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L(\lambda)) = \mathrm{atyp}(\lambda).$$

where $\mathrm{atyp}(\lambda)$, *is the maximal number of linearly independent mutually orthogonal, positive isotropic roots* $\alpha \in \Delta^+$ *such that* $(\lambda + \rho, \alpha) = 0$.

Cohomological Interpretation of Analogs of the Chevalley Restriction Theorem

Theorem (BKN)

Let $\mathfrak{g}$ be a classical Lie superalgebra.

(a) *If $\mathfrak{g}$ admits a stable action then there exists a subalgebra $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$ isomorphic to $\oplus \mathfrak{s}l(1|1)$ such that the restriction map*

$$\mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \to \mathrm{H}^{\bullet}(\mathfrak{f}, \mathfrak{f}_{\bar{0}}; \mathbb{C})^{N/N_0}.$$

is an isomorphism.

(b) *If $\mathfrak{g}$ admits a polar action then there exists a subalgebra $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ isomorphic to $\oplus \mathfrak{q}(1)$ such that the restriction map*

$$\mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \to \mathrm{H}^{\bullet}(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^{W}.$$

is an isomorphism.

Detecting Subalgebras for $\mathfrak{gl}(m|n)$

Set $r = \min(m, n)$ and consider the following $m \times n$ matrix (generic matrix of rank r)

$$X[t_1, t_2, \ldots, t_r] = \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & t_2 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & t_r & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Detecting Subalgebras for $\mathfrak{gl}(m|n)$, con't

Let

$$\mathfrak{f}_{\bar{1}} = \{ \begin{pmatrix} 0 & X[t_1, \dots, t_r] \\ X[s_1, \dots, s_r] & 0 \end{pmatrix} : \ t_i, s_j \in \mathbb{C} \text{ for all } 1 \leq i, j \leq r \}$$

with $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$ and $\mathfrak{f} = \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}$.

Let

$$\mathfrak{e}_{\bar{1}} = \{ \begin{pmatrix} 0 & X[t_1, \dots, t_r] \\ X[t_1, \dots, t_r] & 0 \end{pmatrix} : \ t_i \in \mathbb{C} \text{ for all } 1 \leq i \leq r \}$$

with $\mathfrak{e}_{\bar{0}} = [\mathfrak{e}_{\bar{1}}, \mathfrak{e}_{\bar{1}}]$ and $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$.

N-Spec

We have an algebraic group N acting rationally on a graded commutative ring R by automorphisms which preserve the grading. This action induces an action of N on $X = \text{Proj}(\text{Spec}(R))$.

Consider $X_N = \text{Proj}(N\text{-Spec}(R))$ which is the set of homogeneous N-prime ideals of R. There exists a canonical map $\rho : X \twoheadrightarrow X_N$ with $\rho(P) = \cap_{g \in N}\ gP =: \cap_g\ gP$. The topology on X_N is given by declaring $W \subseteq X_N$ closed if and only if $\rho^{-1}(N)$ is closed in X.

An important property is that $\cap_g\ gP_1 = \cap_g\ gP_2$ for $P_1, P_2 \in X$ if and only if $\overline{N \cdot P_1} = \overline{N \cdot P_2}$ in X

Classification of Thick Tensor Ideals

Theorem

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, let $\mathfrak{f}$ be the detecting subalgebra of $\mathfrak{g}$, and let $N = \mathrm{Norm}_{G_{\bar{0}}}(\mathfrak{f}_{\bar{1}})$.

Then there is a bijection between the set of thick tensor ideals of $\mathrm{Stab}(\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})})$ and the set of specialization closed subsets of $\mathrm{Proj}(N\text{-}\mathrm{Spec}(S^\bullet(\mathfrak{f}_{\bar{1}}^*)))$.

Determining the Balmer Spectrum

Theorem

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$*, let* $\mathfrak{f}$ *be the detecting subalgebra of* $\mathfrak{g}$*, and let* $N = \mathrm{Norm}_{G_{\bar{0}}}(\mathfrak{f}_{\bar{1}})$.

Then there is a homeomorphism between $\mathrm{Spc}(\mathbf{K}^c)$ *and* $\mathrm{Proj}(N\text{-}\mathrm{Spec}(S^\bullet(\mathfrak{f}^*_{\bar{1}})))$.

In particular, there is a bijection between the set of prime thick tensor ideals of $\mathrm{Stab}(\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})})$ *and the collection of irreducible N-stable closed subsets of* $\mathrm{Proj}(\mathrm{Spec}(S^\bullet(\mathfrak{f}^*_{\bar{1}})))$.

Key Property that Needs Verification for the Support Data V

One proves the preceding theorems by constructing a support data V by first constructing a support data on modules for $\mathfrak{f}$ (detecting subalgebra). To apply the results of the Main Theorem it only remains to verify that V satisfies (S1)-(S8). The key property to check is (S8) [Realization Property].

Using the rank variety description of V, we can reduce this problem to showing that for any N-invariant closed subvariety W in $\mathrm{Proj}(\mathfrak{f}_{\bar{1}})$ *there exists a module M in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$* such that

$$\begin{aligned} W &= V_{\max}(M) \\ &\cong V^r_{\mathfrak{f}_{\bar{1}}}(M) \\ &:= \mathrm{Proj}(\{x \in \mathfrak{f}_{\bar{1}} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\}). \end{aligned}$$

Realization of supports

1) There exists a torus T such that Lie $\mathrm{T} = \mathfrak{f}_{\bar{0}}$ with $N = \Sigma_r \ltimes T$. In order to finish the proof one needs to realize closed subvarieties of $\mathfrak{f}_{\bar{1}}$ of the form $\Sigma_r \cdot W$ where W is a closed T-invariant subvariety.

2) The computation of the $\mathfrak{f}_{\bar{1}}$-supports for the Kac modules and the simple modules for $\mathfrak{gl}(m|n)$ are used to realize some of the cases. For the other cases one needs to consider simple modules over parabolic superalgebra and apply the geometric induction functor studied by Penkov and Serganova.

3) Another interesting facet of these computations involves using spectral sequence techniques that were employed by Nakano, Parshall and Vella in their proof of the Jantzen Conjecture for support varieties of Weyl modules (for the first Frobenius kernel).

Future Directions: Developing a General Theory

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra.

1) BKN showed that $c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}}(M) \leq \dim \mathfrak{g}_1$. Let $L = \oplus L_\lambda^{\dim P_\lambda}$ (direct sum of simple modules). Does there exist a ring homomorphism from

$$S(\mathfrak{g}_{\bar{1}}^*) \to \operatorname{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L, L)?$$

2) Is there some support theory on $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ such that the dimension of the support of a object equals the complexity?

Future Directions: Developing a General Theory

3) Let K be a extension of $\mathbb{C}$ of transcendence degree greater than or equal to $\dim \mathfrak{g}_{\bar{1}}$. Define

$$V_{\mathfrak{g}_{\bar{1}}}(M) = \{x \in K \otimes \mathfrak{g}_1 : \ (K \otimes M)|_{\langle x \rangle} \text{ is not projective}\} \cup \{0\}.$$

Does $V_{\mathfrak{g}_{\bar{1}}}(M)$ detect projectivity?

4) What is the relationship between the thick tensor ideals in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ and specialization closed sets of $\operatorname{Proj}(G_{\bar{0}}\text{-}\operatorname{Spec}(S^\bullet(\mathfrak{g}_{\bar{1}}^*)))$?

Thank you for your attention.