Mutation via Hovey twin cotorsion pairs and model structures in extriangulated category

(joint work with Yann Palu)

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Introduction: Cotorsion pair

$\mathscr{E}$: exact category

Definition

In this talk, any 'subcategory' is always assumed to be full, additive, closed by isomorphisms.

$(\mathcal{U}, \mathcal{V})$: cotorsion pair

$\underset{\text{def}}{\Longleftrightarrow}$ (0) $\mathcal{U}, \mathcal{V} \subseteq \mathscr{E}$: subcategories closed by direct summands

(1) $\mathrm{Ext}^1_{\mathscr{E}}(\mathcal{U}, \mathcal{V}) = 0$

(2) ${}^{\forall} X \in \mathscr{E}$, ${}^{\exists}$exact sequences

$$\begin{matrix} 0 \to X \to V_X \to U_X \to 0 \\ 0 \to V^X \to U^X \to X \to 0 \end{matrix} \qquad \left(\begin{matrix} U_X, U^X \in \mathcal{U}, \\ V_X, V^X \in \mathcal{V} \end{matrix} \right)$$

Example

- $\mathscr{E}$ has enough projectives, $(\mathcal{U}, \mathcal{V}) = (\mathrm{Proj}(\mathscr{E}), \mathscr{E})$
- $\mathscr{E}$ has enough injectives, $(\mathcal{U}, \mathcal{V}) = (\mathscr{E}, \mathrm{Inj}(\mathscr{E}))$

Introduction: Cotorsion pair

$\mathscr{T}$: triangulated category

Definition

$(\mathcal{U}, \mathcal{V})$: cotorsion pair

$\underset{\text{def}}{\Longleftrightarrow}$ (0) $\mathcal{U}, \mathcal{V} \subseteq \mathscr{T}$: subcategories closed by direct summands

(1) $\mathrm{Ext}^1_{\mathscr{T}}(\mathcal{U}, \mathcal{V}) \, (= \mathscr{T}(\mathcal{U}, \mathcal{V}[1])) = 0$

(2) ${}^\forall X \in \mathscr{T}$, ${}^\exists$distinguished triangle

$$X \to V_X \to U_X \to X[1] \qquad (U_X \in \mathcal{U},\ V_X \in \mathcal{V})$$

Remark

$(\mathcal{U}, \mathcal{V})$:cotorsion pair $\Leftrightarrow$ $(\mathcal{U}, \mathcal{V}[1])$: torsion pair

in the sense of [Iyama & Yoshino '08]

$\mathscr{T}$: triangulated category

Definition

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Example

1. $\mathcal{U}[1] \subseteq \mathcal{U} \iff (\mathcal{U}[-1], \mathcal{V}[1]) = (t^{\leq 0}, t^{\geq 0})$: t-structure [Beĭlinson-Bernstein-Deligne '82]
2. $\mathcal{U} = \mathcal{V} \iff \mathcal{U}(= \mathcal{V}) \subseteq \mathscr{C}$: cluster-tilting subcat. [Keller-Reiten '07], [Koenig-Zhu '08]
3. $\mathcal{U}[-1] \subseteq \mathcal{U} \iff (\mathcal{U}, \mathcal{V})$: co-t-structure [Pauksztello '08]

$\mathscr{E}$: exact category

Definition

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$\underset{\text{def}}{\Longleftrightarrow}$

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$$0 \to X \to V_X \to U_X \to 0$$
$$0 \to V^X \to U^X \to X \to 0$$
$$\begin{pmatrix} U_X, U^X \in \mathcal{U}, \\ V_X, V^X \in \mathcal{V} \end{pmatrix}$$

$\mathscr{T}$: triangulated category

Definition

$(\mathcal{U}, \mathcal{V})$: cotorsion pair

$\underset{\text{def}}{\Longleftrightarrow}$

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(1) $\mathrm{Ext}^1_{\mathscr{T}}(\mathcal{U}, \mathcal{V}) = 0$

(2) ${}^{\forall} X \in \mathscr{T}$, ${}^{\exists}$dist. $\triangle$

$$X \to V_X \to U_X \to X[1]$$

$$(U_X \in \mathcal{U},\ V_X \in \mathcal{V})$$

Introduction : Use of Ext^1

- Cotorsion pair (CP)

$$\cdots \text{ defined on } \begin{cases} \text{exact} \\ \text{triangulated} \end{cases} \text{ categories}$$

by using Ext^1

Question 1 Can we formalize CPs on an (additive category + "Ext-functor" + α) ?

Introduction : Use of Ext^1

Question 1 Can we formalize CPs on
an (additive category + "Ext-functor" + α) ?

- In exact category $\mathscr{E}$,

$$\delta \in \mathrm{Ext}^1_{\mathscr{E}}(C, A) \quad \cdots$$

equivalence class of
exact sequence (= conflation)

$$[A \xrightarrow{x} B \xrightarrow{y} C]$$

$$\begin{tikzcd} & B \arrow[dr, "y"] \arrow[dd, "{}^{\exists} b"', "\cong"] & \\ A \arrow[ur, "x"] \arrow[dr, "x'"'] \arrow[r, phantom, "\circlearrowright"] & & C \arrow[l, phantom, "\circlearrowright"] \\ & B' \arrow[ur, "y'"'] & \end{tikzcd}$$

In triangulated category $\mathscr{T}$,

$$\delta \in \mathrm{Ext}^1_{\mathscr{T}}(C, A)$$
$$= \mathscr{T}(C, A[1])$$

completed to a dist.$\triangle$

$$\boxed{A \xrightarrow{x} B \xrightarrow{y} C} \overset{\delta}{\dashrightarrow}$$

$$[A \xrightarrow{x} B \xrightarrow{y} C]$$

- Use of Ext^1-functor is also important in *relative homological algebra.*

Question 2 Can we start from "Ext-functor",
instead of sequence/triangle ?

Introduction : Use of Ext^1

Question 1 Can we formalize CPs on an (additive category + "Ext-functor" + α) ?

Question 2 Can we start from "Ext-functor", instead of sequence/triangle ?

Our viewpoint :

- Ext^1 is the "body" of exact structure or triangulation
- equivalence classes of sequences = "*realization*" of Ext^1

$\rightsquigarrow$ Formulation by

$\mathscr{C}$	$\mathbb{E}$	$\mathfrak{s}$
additive category	biadditive functor	realization of $\mathbb{E}$

(later)

Introduction : Use of Ext^1

Question 1 Can we formalize CPs on an (additive category + "Ext-functor" + α) ?

Question 2 Can we start from "Ext-functor", instead of sequence/triangle ?

Formulation by

$\mathscr{C}$	$\mathbb{E}$	$\mathfrak{s}$
additive category	biadditive functor	realization of $\mathbb{E}$

$\Longrightarrow$

- Both $\left\{\begin{array}{l}\text{exact}\\ \text{triangulated}\end{array}\right.$ categories can be dealt with.
- CPs can be defined.
- Closed by taking $\left\{\begin{array}{l}\text{`extension-closed' sub.}\\ \text{some ideal quotient}\end{array}\right.$

Today

- *Extriangulated* category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$
- Cotorsion pairs
- Pairs of CPs $\quad\longleftrightarrow\quad$ model structures
- Reduction & mutation of CPs

Extriangulated category

$$(\mathscr{C}, \mathbb{E}, \mathfrak{s})$$

Extriangulated category

= additive category + "Ext^1-functor" + realization

Extriangulated

Example

- exact category $\left(\begin{array}{c}\text{with small}\\ \mathrm{Ext}^1\text{-groups}\end{array}\right)$
- triangulated category
- extension-closed subcategory of a triangulated category

Extriangulated category

= additive category + "Ext^1-functor" + realization

Externally triangulated

Example

- exact category $\left(\begin{array}{c}\text{with small} \\ \mathrm{Ext}^1\text{-groups}\end{array}\right)$
- triangulated category
- extension-closed subcategory of a triangulated category

Extriangulated category

= additive category + "Ext^1-functor" + realization

Exact + *triangulated*

Example

- exact category $\left(\begin{array}{c}\text{with small}\\ \mathrm{Ext}^1\text{-groups}\end{array}\right)$
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Extriangulated category

= additive category + "Ext^1-functor" + realization

Ext^1- triangulated

Example

- exact category $\left(\begin{array}{c}\text{with small}\\ \mathrm{Ext}^1\text{-groups}\end{array}\right)$
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Extriangulated category

= additive category + "Ext^1-functor" + realization

Ext^1- triangulated

Definition

$(\mathscr{C}, \mathbb{E}, \mathfrak{s})$: *extriangulated* category

$\underset{\text{def}}{\Longleftrightarrow}$ $\mathscr{C}$: additive category,

with the following (ET1),(ET2),(ET3),(ET4),

(ET3)$^{\mathrm{op}}$,(ET4)$^{\mathrm{op}}$

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

$\underset{\text{def}}{\Longleftrightarrow}$ $\mathscr{C}$: additive category with (ET1),(ET2),(ET3),(ET4),(ET3)$^{\text{op}}$,(ET4)$^{\text{op}}$

(ET1) $\mathbb{E}\colon \mathscr{C}^{\text{op}} \times \mathscr{C} \to Ab$ is a biadditive functor

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

$\mathscr{C}$: additive category

(ET1) $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is a biadditive functor

Definition $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension.*

(Also denoted by ${}_A\delta_C$)

Definition $(a, c)\colon {}_A\delta_C \to {}_{A'}\delta'_{C'}$ is a morphism of $\mathbb{E}$-extensions

$$\underset{\text{def}}{\Longleftrightarrow} \quad \mathbb{E}(C, a)(\delta) = \mathbb{E}(c, A')(\delta')$$

$$\begin{array}{ccccc} \mathbb{E}(C, A) & \xrightarrow{\mathbb{E}(C,a)} & \mathbb{E}(C, A') & \xleftarrow{\mathbb{E}(c,A')} & \mathbb{E}(C', A') \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ \delta & & & & \delta' \end{array}$$

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

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$$\underset{\text{def}}{\Longleftrightarrow} \qquad a_*\delta = c^*\delta'$$

$$\begin{array}{ccccc} \mathbb{E}(C, A) & \xrightarrow{a_*} & \mathbb{E}(C, A') & \xleftarrow{c^*} & \mathbb{E}(C', A') \\ \Uparrow\!\!\!\!\in & & & & \Uparrow\!\!\!\!\in \\ \delta & & & & \delta' \end{array}$$

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

$\mathscr{C}$: additive category

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Definition $A, C \in \mathscr{C}$

Sequences $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ are equivalent

$$\underset{\text{def}}{\Longleftrightarrow} \quad \begin{tikzcd} & B \arrow[rd, "y"] \arrow[dd, "b", "\cong"'] & \\ A \arrow[ru, "x"] \arrow[rd, "x'"'] \arrow[r, phantom, "\circlearrowright"] & {} \arrow[r, phantom, "\circlearrowright"] & C \\ & B' \arrow[ru, "y'"'] & \end{tikzcd}$$

Equivalence class is denoted by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

Definition

$\mathfrak{s}$: to each $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$,

associates $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

$\mathscr{C}$: additive category

(ET1) $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is a biadditive functor

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$

Definition $\mathfrak{S}$ is a *realization* of $\mathbb{E}$

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}$morphism $(a, c)\colon {}_A\delta_C \to {}_{A'}\delta'_{C'}$,

$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$,

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "y"] \arrow[d, "{}^{\exists}b"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

Definition

$\mathfrak{S}$: to each $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$,

associates $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

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Definition $\mathfrak{s}$ is *additive*

$\underset{\text{def}}{\Longleftrightarrow}$

- $\mathfrak{s}({}_A0_C) = [A \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} A \oplus C \xrightarrow{[0\ 1]} C]$
- $\mathfrak{s}(\delta \oplus \delta') = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C']$

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

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(ET3) $\cdots$ analog of (TR3)

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C],\ \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'],$$

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "{}^{\exists}c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

such that $(a, c)\colon \delta \to \delta'$ is a morphism.

(ET3)$^{\mathrm{op}}$ Dually,

$$\begin{tikzcd} A \arrow[r, "x"] & B \arrow[r, "y"] \arrow[d, "b"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "{}^{\exists}a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

such that $(a, c)\colon \delta \to \delta'$ is a morphism.

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

$\mathscr{C}$: additive category

(ET1) $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is a biadditive functor

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$

(ET3),(ET3)$^{\mathrm{op}}$ $\cdots$ analog of (TR3) and its dual

Proposition If $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1) and (ET2),

TFAE:

(1) $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET3) and (ET3)$^{\mathrm{op}}$

(2) For any $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$,

obtained by Yoneda Lemma

$$\mathscr{C}(-,A) \xrightarrow{x\circ -} \mathscr{C}(-,B) \xrightarrow{y\circ -} \mathscr{C}(-,C) \xrightarrow{\delta_\sharp} \mathbb{E}(-,A) \xrightarrow{x_*} \mathbb{E}(-,B)$$

$$\mathscr{C}(C,-) \xrightarrow{-\circ y} \mathscr{C}(B,-) \xrightarrow{-\circ x} \mathscr{C}(A,-) \xrightarrow{\delta^\sharp} \mathbb{E}(C,-) \xrightarrow{y^*} \mathbb{E}(B,-)$$

are exact.

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(ET3),(ET3)$^{\mathrm{op}}$ $\cdots$ analog of (TR3) and its dual

(ET4) $\cdots$ analog of (TR4)

$$\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{f'} D],\ \mathfrak{s}(\delta') = [B \xrightarrow{g} C \xrightarrow{g'} F]$$

$$\Rightarrow \quad {}^{\exists}E \in \mathscr{C},\ {}^{\exists}{}_{A}\delta''_{E},\ \mathfrak{s}(\delta'') = [A \xrightarrow{g\circ f} C \xrightarrow{{}^{\exists}h'} E]$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[d, "{}^{\exists}d"] \\
A \arrow[r, "g\circ f=h"'] & C \arrow[r, "h'"'] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "{}^{\exists}e"] \\
& F \arrow[r, equal] & F
\end{tikzcd}$$

satisfying

(i) $\mathfrak{s}(f'_*\delta') = [D \xrightarrow{d} E \xrightarrow{e} F]$

(ii) $d^*\delta'' = \delta$

(iii) $f_*\delta'' = e^*\delta'$

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

(ET4) $\cdots$ analog of (TR4)

$$\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{f'} D],\ \mathfrak{s}(\delta') = [B \xrightarrow{g} C \xrightarrow{g'} F]$$

$$\Longrightarrow \quad {}^{\exists}E \in \mathscr{C},\ {}^{\exists}{}_{A}\delta''_{E},\ \mathfrak{s}(\delta'') = [A \xrightarrow{g \circ f} C \xrightarrow{{}^{\exists}h'} E]$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[r, dashed, "\delta"] \arrow[d, "{}^{\exists}d"] \arrow[dr, phantom, "\circlearrowright"] & A[1] \arrow[d, equal] \\
A \arrow[r, "g \circ f = h"'] & C \arrow[r, "h'"'] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, dashed, "\delta''"] \arrow[d, "{}^{\exists}e"] \arrow[dr, phantom, "\circlearrowright"] & A[1] \arrow[d, "f[1]"] \\
& F \arrow[r, equal] \arrow[d, dashed, "\delta'"'] \arrow[dr, phantom, "\circlearrowright"] & F \arrow[r, dashed, "\delta'"'] \arrow[d, dashed, "f'_{*}\delta'"] & B[1] \\
& B[1] \arrow[r, "f'[1]"'] & D[1] &
\end{tikzcd}$$

satisfying

(i) $\mathfrak{s}(f'_{*}\delta') = [D \xrightarrow{d} E \xrightarrow{e} F]$

(ii) $d^{*}\delta'' = \delta$

(iii) $f_{*}\delta'' = e^{*}\delta'$

Correspond to $\cdots$

(When $\mathscr{C}$ is triangulated)

(ET4)$^{\mathrm{op}}$ = dual of (ET4)

Extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$

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(ET3),(ET3)$^{\mathrm{op}}$ $\cdots$ analog of (TR3) and its dual

(ET4) $\cdots$ analog of (TR4)

$$\mathfrak{s}(\delta)=[A\xrightarrow{f}B\xrightarrow{f'}D],\ \mathfrak{s}(\delta')=[B\xrightarrow{g}C\xrightarrow{g'}F]$$

$$\Rightarrow\quad {}^{\exists}E\in\mathscr{C},\ {}^{\exists}{}_{A}\delta''_{E},\ \mathfrak{s}(\delta'')=[A\xrightarrow{g\circ f}C\xrightarrow{{}^{\exists}h'}E]$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "f'"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[d, "{}^{\exists}d"] \\
A \arrow[r, "g\circ f=h"'] & C \arrow[r, "h'"'] \arrow[d, "g'"] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "{}^{\exists}e"] \\
& F \arrow[r, equal] & F
\end{tikzcd}$$

satisfying

(i) $\mathfrak{s}(f'_{*}\delta')=[D\xrightarrow{d}E\xrightarrow{e}F]$

(ii) $d^{*}\delta''=\delta$

(iii) $f_{*}\delta''=e^{*}\delta'$

(ET4)$^{\mathrm{op}}$ $=$ dual of (ET4)

Extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$

Definition

$\mathscr{C}$: additive category

$(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an *extriangulated category*

$\underset{\text{def}}{\Leftrightarrow}$ (ET1) $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is a biadditive functor

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$

(ET3),(ET3)$^{\mathrm{op}}$ $\cdots$ analog of (TR3) and its dual

(ET4),(ET4)$^{\mathrm{op}}$ $\cdots$ analog of (TR4) and its dual

Example

- exact category $\left(\begin{array}{c}\text{with small}\\ \mathrm{Ext}^1\text{-groups}\end{array}\right)$
- triangulated category

We import terminology from exact/triangulated categories.

Terminology in an ex-$\triangle$ category

We import terminology from exact/triangulated categories.

$(\mathscr{C}, \mathbb{E}, \mathfrak{s})$: ex-$\triangle$ category

Definition A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is a *conflation*

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}{}_A\delta_C$ s.t. $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ Cone(x)

We call: *inflation* *deflation*

CoCone(y)

Definition A full subcategory $\mathcal{D} \subseteq \mathscr{C}$ (closed by isomorphisms)

is *extension-closed*

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}$conflation $A \xrightarrow{x} B \xrightarrow{y} C$, $A, C \in \mathcal{D}$ implies $B \in \mathcal{D}$

Definition $P \in \mathscr{C}$ is *projective* $\underset{\text{def}}{\Longleftrightarrow}$ $\mathbb{E}(P, -) = 0$ $\mathrm{Proj}(\mathscr{C}) \subseteq \mathscr{C}$

$\in$

$\mathscr{C}$ has *enough projectives* $\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}C \in \mathscr{C}$, ${}^{\exists}$ deflation $P \to C$

Dually for *injectives*.

Sub. & quotient

Proposition $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$: ex-$\triangle$

(1) If $\mathcal{D} \subseteq \mathscr{C}$ is extension-closed,

$\Rightarrow$ $\mathcal{D}$ has an ex-$\triangle$ structure.

(2) If $\mathcal{I} \subseteq \mathscr{C}$ is full additive and $\mathcal{I} \subseteq \mathrm{Proj}(\mathscr{C}) \cap \mathrm{Inj}(\mathscr{C})$,

$\Rightarrow$ $\mathscr{C}/\mathcal{I}$ has an ex-$\triangle$ structure.

Proposition $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$: Frobenius ex-$\triangle$ (i.e., has enough inj.& proj., $\mathrm{Proj}(\mathscr{C}) = \mathrm{Inj}(\mathscr{C})$)

$\Rightarrow$ For $\mathcal{I} = \mathrm{Proj}(\mathscr{C}) = \mathrm{Inj}(\mathscr{C})$,

$$\mathscr{C}/\mathcal{I} \text{ : triangulated}$$

Remark

- [Happel '88] for exact category
- [Iyama & Yoshino '08] mutation pair on triangulated category

Proposition If $\mathscr{C}$: additive category, $[1] \colon \mathscr{C} \xrightarrow{\simeq} \mathscr{C}$ automorphism,

then for $\mathbb{E} = \mathscr{C}(-, -[1])$, TFAE:

(1) $\mathscr{C}$ is triangulated, with shift $[1]$.

(2) $(\mathscr{C}, \mathbb{E}, {}^{\exists}\mathfrak{s})$: ex-$\triangle$.

Proposition $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$: ex-$\triangle$

(1) $\mathscr{C}$ is exact

$\Leftrightarrow$ any inflation is monomorphic,
and any deflation is epimorphic.

(2) $\mathscr{C}$ is triangulated

$\Leftrightarrow$ $\mathscr{C}$ is Frobenius, and $\mathrm{Proj}(\mathscr{C}) = \mathrm{Inj}(\mathscr{C}) = 0$.

Correspodence:

$$\text{Hovey TCP} \quad \overset{1:1}{\longleftrightarrow} \quad \begin{matrix}\text{admissible} \\ \text{model structure}\end{matrix}$$

Cotorsion pair

In the rest, $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$: ex-$\triangle$

Definition $(\mathcal{U}, \mathcal{V})$: *cotorsion pair*

$\overset{\Longleftrightarrow}{\text{def}}$

- $\mathcal{U}, \mathcal{V} \subseteq \mathscr{C}$: full additive subcategories, closed by $\cong$ and direct summands.
- $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$.
- $\mathscr{C} = \mathrm{Cone}(\mathcal{V}, \mathcal{U}) = \mathrm{CoCone}(\mathcal{V}, \mathcal{U})$, i.e.,

  ${}^{\forall}C \in \mathscr{C}$, ${}^{\exists}$conflations

  $$V^C \to U^C \to C \quad (U^C \in \mathcal{U}, V^C \in \mathcal{V}),$$

  $$C \to V_C \to U_C \quad (U_C \in \mathcal{U}, V_C \in \mathcal{V}).$$

Definition

$$\mathfrak{CP}(\mathscr{C}) := \{(\mathcal{U}, \mathcal{V}) \mid \text{cotorsion pairs on } \mathscr{C}\}$$

Hovey TCP

Definition

(1) $\mathcal{P} = ((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$: *twin cotorsion pair* (*TCP* for short)

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet \ (\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}) \in \mathfrak{CP}(\mathscr{C}) \\ \bullet \ \mathbb{E}(\mathcal{S},\mathcal{V}) = 0 \quad (\Leftrightarrow \mathcal{S} \subseteq \mathcal{U} \ \Leftrightarrow \mathcal{V} \subseteq \mathcal{T}) \end{cases}$$

In addition,

(2) $\mathcal{P}$: *Hovey* TCP

$$\underset{\text{def}}{\Longleftrightarrow} \quad \mathrm{Cone}(\mathcal{V},\mathcal{S}) = \mathrm{CoCone}(\mathcal{V},\mathcal{S})$$

Example

1. $(\mathcal{U},\mathcal{V})$: CP $\Rightarrow$ $((\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V}))$: Hovey TCP

2. $\mathscr{C}$:triangulated, $\mathcal{I} \subseteq \mathscr{C}$ functorially finite rigid

   $\Rightarrow$ $((\mathcal{I},\mathcal{I}[-1]^{\perp}),({}^{\perp}\mathcal{I}[1],\mathcal{I}))$: TCP (not Hovey in general)

Hovey TCP

Definition

(1) $\mathcal{P} = ((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$: *twin cotorsion pair* (*TCP* for short)

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet \quad (\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}) \in \mathfrak{CP}(\mathscr{C}) \\ \bullet \quad \mathbb{E}(\mathcal{S},\mathcal{V}) = 0 \quad (\Leftrightarrow \mathcal{S} \subseteq \mathcal{U} \ \Leftrightarrow \mathcal{V} \subseteq \mathcal{T}) \end{cases}$$

In addition,

(2) $\mathcal{P}$: *Hovey* TCP

$$\underset{\text{def}}{\Longleftrightarrow} \quad \mathrm{Cone}(\mathcal{V},\mathcal{S}) = \mathrm{CoCone}(\mathcal{V},\mathcal{S})$$

Remark

If $\mathscr{C}$: triangulated, and if $\mathcal{P}$: Hovey TCP,

then $\mathcal{N} := \mathrm{Cone}(\mathcal{V},\mathcal{S})$ becomes thick in $\mathscr{C}$.

In particular, $\mathscr{C}_{\mathcal{N}}$: Verdier quotient, is triangulated.

Example of Hovey TCP, from recollement

$\mathscr{C}$: triangulated

Definition [Beĭlinson-Bernstein-Deligne]

A diagram of triangulated categories and triangle functors

$$\begin{tikzcd}[column sep=huge]
\mathcal{N} \arrow[r, hook, "i_*"'] & \mathscr{C} \arrow[r, "j^*"'] \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] & \mathscr{C}_{\mathcal{N}} \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"]
\end{tikzcd}$$

is a *recollement* if it satisfies

1. $i^* \dashv i_* \dashv i^!$ and $j_! \dashv j^* \dashv j_*$ are adjoint triplets.
2. $i_*, j_!, j_*$ are fully faithful.
3. $j^* \circ i_* = 0$ holds.
4. For any $C \in \mathscr{C}$, the units and counits of the above adjoints give distinguished triangles

$$i_*i^!C \to C \to j_*j^!C \to (i_*i^!C)[1] \quad \text{and} \quad j_!j^*C \to C \to i_*i^*C \to (j_!j^*C)[1].$$

Example of Hovey TCP, from recollement

Fact [Chen]

$$\begin{tikzcd}[column sep=huge] \mathcal{N} \arrow[r, hook, "i_*"] & \mathscr{C} \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^*"] & \mathscr{C}_{\mathcal{N}} \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"] \end{tikzcd} \quad : \text{ recollement}$$

- We can *glue* $\boxed{(\mathcal{S},\mathcal{V}) \in \mathfrak{CP}(\mathcal{N})}$ and $\boxed{(\mathscr{L},\mathscr{R})} \in \mathfrak{CP}(\mathscr{C}_{\mathcal{N}})$ to obtain $\boxed{(\mathcal{A},\mathcal{B})} \in \mathfrak{CP}(\mathscr{C})$

- If $\boxed{(\mathcal{A},\mathcal{B})} \in \mathfrak{CP}(\mathscr{C})$ satisfies

  ★ $$ii^*(\mathcal{A}) \subseteq \mathcal{A}, \quad ii^!(\mathcal{B}) \subseteq \mathcal{B} \quad \text{and} \quad j_*j^*(\mathcal{B}) \subseteq \mathcal{B},$$

  we can find $\boxed{(\mathcal{S},\mathcal{V})}$ and $\boxed{(\mathscr{L},\mathscr{R})}$ which glue to $(\mathcal{A},\mathcal{B})$.

Observation If we $\boxed{\text{fix } (\mathcal{S},\mathcal{V})}$, this gives a $\boxed{\text{bijection}}$

$$\mathfrak{CP}(\mathscr{C}_{\mathcal{N}}) \overset{\text{bij.}}{\longleftrightarrow} \left\{ (\mathcal{A},\mathcal{B}) \in \mathfrak{CP}(\mathscr{C}) \;\middle|\; \begin{array}{c}\text{some}\\ \text{condition}\end{array} \right\} =: \mathcal{M}_{(\mathcal{S},\mathcal{V})}(\mathscr{C})$$

Example of Hovey TCP, from recollement

Observation

$$\begin{tikzcd}[column sep=huge] \mathcal{N} \arrow[r, hook, "i_*" description] & \mathscr{C} \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^*" description] & \mathscr{C}_{\mathcal{N}} \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"] \end{tikzcd}$$: recollement

$(\mathcal{S}, \mathcal{V}) \in \mathfrak{CP}(\mathcal{N})$: given

$$\boxed{\mathfrak{CP}(\mathscr{C}_{\mathcal{N}}) \overset{\text{bij.}}{\longleftrightarrow} \left\{ (\mathcal{A}, \mathcal{B}) \in \mathfrak{CP}(\mathscr{C}) \;\middle|\; \substack{\mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U} \\ \mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T} \\ \operatorname{Ext}^1(j^*\mathcal{A}, j^*\mathcal{B}) = 0} \right\} =: \mathcal{M}_{(\mathcal{S},\mathcal{V})}(\mathscr{C})}$$

In particular,

$$\left.\begin{array}{l} (0, \mathscr{C}_{\mathcal{N}}) \mapsto (\mathcal{S}, \mathcal{T}) \\ (\mathscr{C}_{\mathcal{N}}, 0) \mapsto (\mathcal{U}, \mathcal{V}) \end{array}\right\} \in \mathfrak{CP}(\mathscr{C})$$

$\rightsquigarrow$ $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$: Hovey TCP on $\mathscr{C}$

Condition $\triangle$

$(\mathscr{C}, \mathbb{E}, \mathfrak{s})$: ex-$\triangle$

In the rest, assume:

Condition $\triangle$

For any $h = g \circ f$ in $\mathscr{C}$,

- h: inflation $\Rightarrow$ f: inflation
- h: deflation $\Rightarrow$ g: deflation

Remark

If $\mathscr{C}$ is exact, $\triangle$ $\Leftrightarrow$ weak idempotent completeness

If $\mathscr{C}$ is triangulated, $\triangle$ is always satisfied

Theorem 1 ([N-,Palu])

$\mathcal{P}$: Hovey TCP

- $Fib := \{f : \text{defl.} \mid \text{CoCone}(f) \in \mathcal{T}\}$,
- $wFib := \{f : \underline{\qquad} \mid \underline{\qquad\qquad\qquad} \in \mathcal{V}\}$,
- $Cof := \{f : \text{infl.} \mid \text{Cone}(f) \in \mathcal{U}\}$,
- $wCof := \{f : \underline{\qquad} \mid \underline{\qquad\qquad} \in \mathcal{S}\}$,
- $\mathbb{W} := wFib \circ wCof$

$\Longrightarrow$ $(Fib, Cof, \mathbb{W})$: model structure on $\mathscr{C}$

Model structure $\rightsquigarrow$ TCP

$(\mathit{Fib}, \mathit{Cof}, \mathbb{W})$: model structure on $\mathscr{C}$

- $\mathcal{T} := \{C \in \mathscr{C} \mid (C \to 0) \in \mathit{Fib}\}$,
- $\mathcal{V} := \{ \underline{\qquad\quad} \mid \underline{\qquad\qquad\quad} \in w\mathit{Fib}\}$,
- $\mathcal{U} := \{C \in \mathscr{C} \mid (0 \to C) \in \mathit{Cof}\}$,
- $\mathcal{S} := \{ \underline{\qquad\quad} \mid \underline{\qquad\qquad\quad} \in w\mathit{Cof}\}$,

Definition $(\mathit{Fib}, \mathit{Cof}, \mathbb{W})$ is *admissible*

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}$morphism f,

- $f \in \mathit{Fib} \quad \Leftrightarrow \quad f$:defl. $\quad \& \quad \mathrm{CoCone}(f) \in \mathcal{T}$
- $f \in w\mathit{Fib} \quad \Leftrightarrow \quad \underline{\qquad\qquad\qquad\qquad\qquad\qquad} \in \mathcal{V}$
- $f \in \mathit{Cof} \quad \Leftrightarrow \quad f$:infl. $\quad \& \quad \mathrm{Cone}(f) \in \mathcal{U}$
- $f \in w\mathit{Cof} \quad \Leftrightarrow \quad \underline{\qquad\qquad\qquad\qquad\qquad} \in \mathcal{S}$

Model structure $\rightsquigarrow$ TCP

Theorem 2 ([N-,Palu])

$(\mathit{Fib}, \mathit{Cof}, \mathbb{W})$: admissible model str.

$\Longrightarrow \quad ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$: Hovey TCP

Thm 1 & **Thm 2**

$$\boxed{\text{Hovey TCP} \xleftrightarrow{1:1} \text{admissible model str.}}$$

Remark Known for:

1. $\mathscr{C}$: abel [Hovey '02,07] *abelian model structure*
2. $\mathscr{C}$: exact [Gillespie '11] *exact model structure*
3. $\mathscr{C}$: triangulated [Yang '15] *triangulated model structure*

Reduction & Mutation of CP

Homotopy category

Corollary $\mathcal{P} = ((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$: Hovey TCP

$$\Rightarrow \quad \left\{\begin{matrix}\text{fibrant-cofibrant}\\ \text{objects}\end{matrix}\right\} \Big/ \underset{\text{htpy}}{\sim} \quad \simeq \quad \mathscr{C}[\mathbb{W}^{-1}]$$

localization

$$||$$

$$\mathcal{Z} =: (\mathcal{T}\cap\mathcal{U}) \big/ (\mathcal{S}\cap\mathcal{V}) =: \mathcal{I}$$

ideal quotient

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, hook] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{C} \arrow[d, "\ell"] \\
\mathcal{Z}/\mathcal{I} \arrow[r, "\simeq"'] & \mathscr{C}[\mathbb{W}^{-1}]
\end{tikzcd}$$

Remark

If $\mathscr{C}$: triangulated,

$$\mathscr{C}[\mathbb{W}^{-1}] \simeq \mathscr{C}_{\mathcal{N}}$$

Reduction of CP

Theorem 3 ([N-,Palu]) $\mathcal{P}$: Hovey TCP

$\Rightarrow$ (1) $\mathscr{C}[\mathbb{W}^{-1}]$: triangulated

(2) $$\mathfrak{M}_{\mathcal{P}}(\mathscr{C}) := \left\{ (\mathcal{A},\mathcal{B}) \in \mathfrak{CP}(\mathscr{C}) \;\middle|\; \begin{array}{l} \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U} \quad (\Leftrightarrow \;\; \mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T}) \\ \mathrm{Ext}^1_{\mathscr{C}[\mathbb{W}^{-1}]}(\ell(\mathcal{A}),\ell(\mathcal{B})) = 0 \end{array} \right\}$$

$$1{:}1 \updownarrow$$

$$\mathfrak{CP}(\mathscr{C}[\mathbb{W}^{-1}]) \;\cong \mathfrak{CP}(\mathcal{Z}/\mathcal{I})$$

Reduction of CP

Theorem 3 ([N-,Palu]) $\mathcal{P}$: Hovey TCP

$\Rightarrow$ (1) $\mathscr{C}[\mathbb{W}^{-1}]$: triangulated

(2)

$$\begin{array}{ccccc}
\mathfrak{M}_{\mathcal{P}}(\mathscr{C}) & \ni & (\mathcal{A},\mathcal{B}) & , & (\mathcal{U}\cap\ell^{-1}(\mathscr{L}),\mathcal{T}\cap\ell^{-1}(\mathscr{R})) \\
1{:}1 \updownarrow & & \downarrow & & \uparrow \\
\mathfrak{CP}(\mathscr{C}[\mathbb{W}^{-1}]) & \ni & (\ell(\mathcal{A}),\ell(\mathcal{B})) & , & (\mathscr{L},\mathscr{R})
\end{array}$$

Mutation of CP

Theorem 3 ([N-,Palu]) $\mathcal{P}$: Hovey TCP

$\Rightarrow$ (1) $\mathscr{C}[\mathbb{W}^{-1}]$: triangulated

(2) $\mathfrak{M}_{\mathcal{P}}(\mathscr{C}) \overset{1:1}{\longleftrightarrow} \mathfrak{CP}(\mathcal{Z}/\mathcal{I})$

Remark If $\mathscr{C}$: triangulated,

- This bij. holds for slightly more general TCPs. [N- '15] (arxiv)
- If $(\mathcal{Z},\mathcal{Z})$: $\mathcal{I}$-mutation pair ([Iyama-Yoshino '10])

  then $((\mathcal{I},\mathcal{Z}),(\mathcal{Z},\mathcal{I}))$: TCP on $\mathscr{C}$ of this type
- This recovers the bij. in [Zhou-Zhu '11]

$$\{(\mathcal{A},\mathcal{B}) \in \mathfrak{CP}(\mathscr{C}) \mid \mathcal{I} \subseteq \mathcal{A}, \mathcal{B} \subseteq \mathcal{Z}\} \overset{1:1}{\longleftrightarrow} \mathfrak{CP}(\mathcal{Z}/\mathcal{I})$$

Mutation of CP

Theorem 3 ([N-,Palu]) $\mathcal{P}$: Hovey TCP

$\Rightarrow$ (1) $\mathscr{C}[\mathbb{W}^{-1}]$: triangulated

(2) $\mathfrak{M}_{\mathcal{P}}(\mathscr{C}) \overset{1:1}{\longleftrightarrow} \mathfrak{CP}(\mathscr{C}[\mathbb{W}^{-1}])$

Corollary

Shift $\mathbb{Z}$-action on $\mathfrak{CP}(\mathscr{C}[\mathbb{W}^{-1}])$ induces a $\boxed{\mathbb{Z}\text{-action on } \mathfrak{M}_{\mathcal{P}}(\mathscr{C})}$

$= \textit{mutation}$

Triangulation of $\mathscr{C}[\mathbb{W}^{-1}]$

$\because)$ $\mathcal{P}$: Hovey TCP

- Shift $\forall A \in \mathscr{C}$,

$$\begin{tikzcd} \mathscr{C} \arrow[rd, "{[1]}"] \arrow[d, "\ell"'] \arrow[d, phantom, shift left=6ex, "\circlearrowright"] & \\ \mathscr{C}[\mathbb{W}^{-1}] \arrow[r, "{[1]}"'] & \mathscr{C}[\mathbb{W}^{-1}] \end{tikzcd}$$

$$A[1]$$
$$\mathbin{/\!/}$$
$$A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A \text{: conflation}$$
$$(\mathfrak{s}(\rho_A) = [A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A])$$

- "Connecting morphism"

$$\widetilde{\ell}\colon \mathbb{E}(C,A) \to \mathscr{C}[\mathbb{W}^{-1}](C, A[1]) \;\; ; \;\; \delta \mapsto \ell(d) \circ \ell(w)^{-1}$$

by taking any span $C \underset{w}{\longleftarrow} C' \underset{d}{\longrightarrow} U_A$ with $\begin{cases} w \in wFib \\ w^*\delta = d^*\rho_A \end{cases}$

- Triangles

For any ${}_A\delta_C$,

$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$,

Standard triangle

$$A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C \xrightarrow{\widetilde{\ell}(\delta)} A[1]$$

■