Subgraph Counting in Series-Parallel Graphs and Infinite Dimensional Systems of Functional Equations

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joint work with L. Ramos and J. Rue

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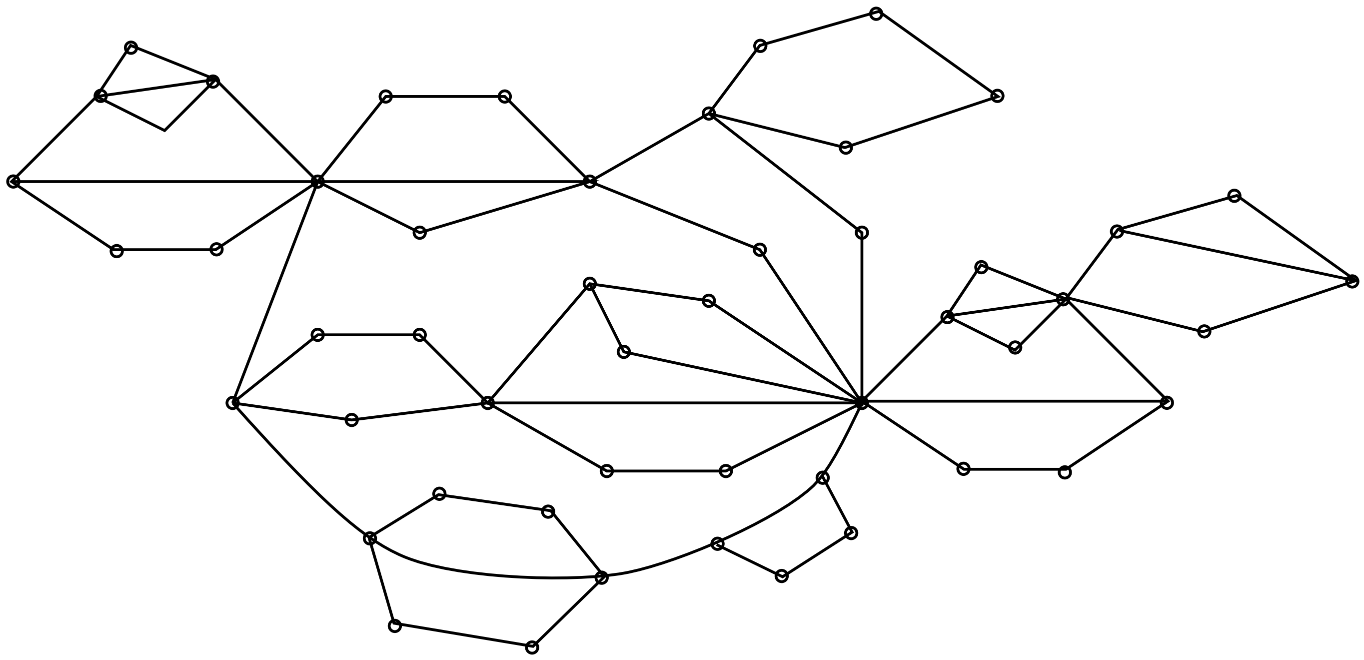
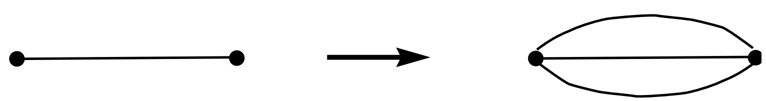
Series-Parallel Graphs



Series-parallel extension of a tree (if we restict to connected graphs)

Series-extension: 

Parallel-extension: 

Series-Parallel Graphs

Equivalent Definitions

- Ex(K_4)
- tree-width ≤ 2
- nested ear decomposition (if connected)

Series-Parallel Graphs

Generating functions

$b_{n,m}$ ... number of **2-connected vertex labelled series-parallel** graphs with n vertices and m edges

$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m$$

$c_{n,m}$ ... number of **connected vertex labelled series-parallel** graphs with n vertices and m edges

$$C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m$$

Series-Parallel Graphs

Generating functions

$$\boxed{\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right)},$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},$$

$$D(x,y) = y + S(x,y) + P(x,y),$$

$$S(x,y) = \frac{x(P(x,y)+y)^2}{1-x(P(x,y)+y)},$$

$$P(x,y) = (e^{S(x,y)} - 1 - S(x,y)) + y(e^{S(x,y)} - 1).$$

Series-Parallel Graphs

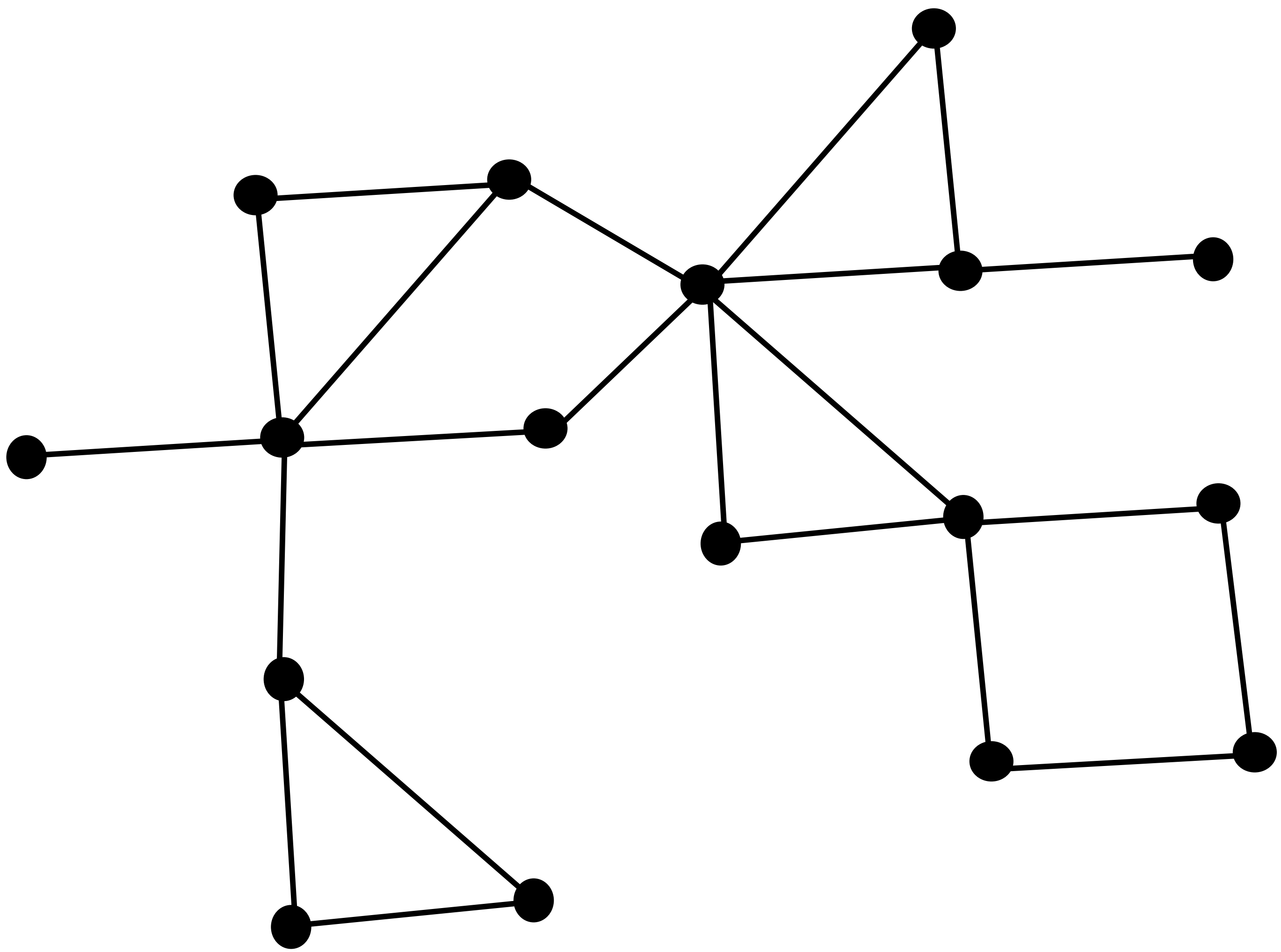
Asymptotic enumeration

[Bodirsky+Gimenez+Kang+Noy 2007]

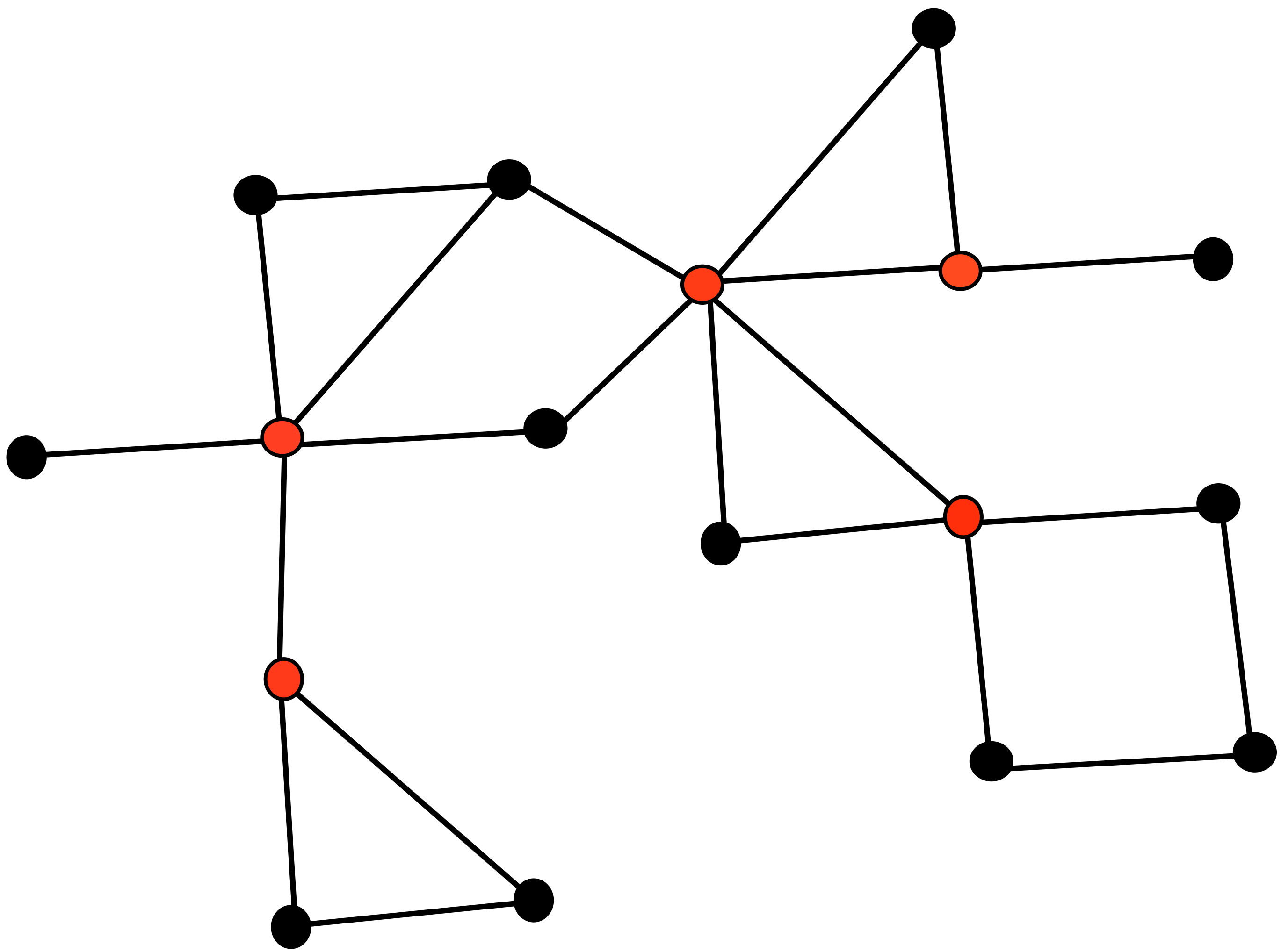
$$\boxed{c_n = \sum_m c_{n,m} \sim c\, n^{-5/2}\, \rho^{-n}\, n!}$$

with $c = 0.0067912...$ and $\rho = 0.11021...$

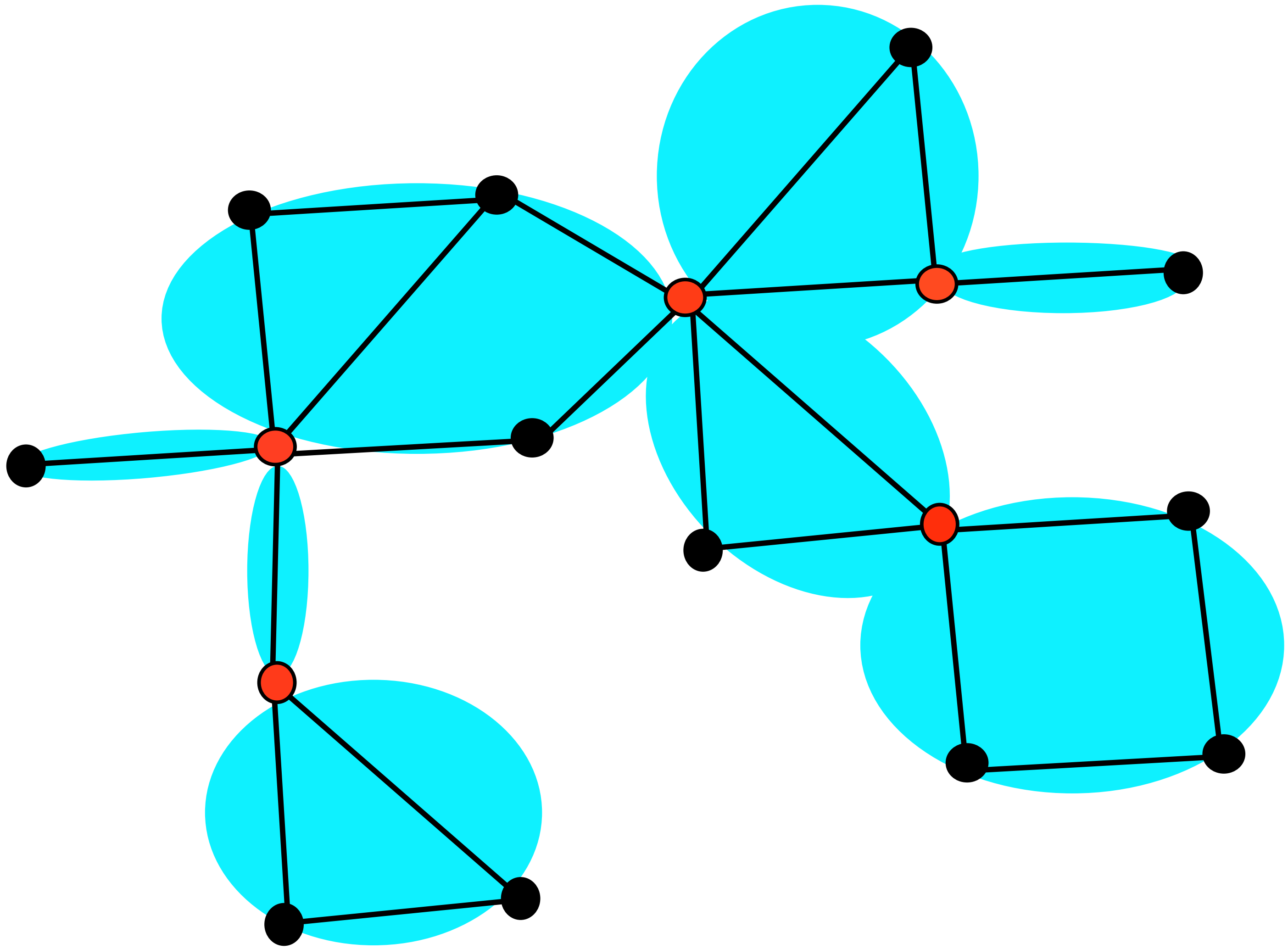
Block-Decomposition



Block-Decomposition



Block-Decomposition



Block-Decomposition

block: 2-connected component (= maximal 2-connected subgraph)

Block-stable graph class $\mathcal{G}$: $\mathcal{G}$ contains the one-edge graph and $G \in \mathcal{G}$ if and only if all blocks of G are contained in $\mathcal{G}$.

Equivalently, the 2-connected graphs of $\mathcal{G}$ and the one-edge graph generate all graphs of $\mathcal{G}$.

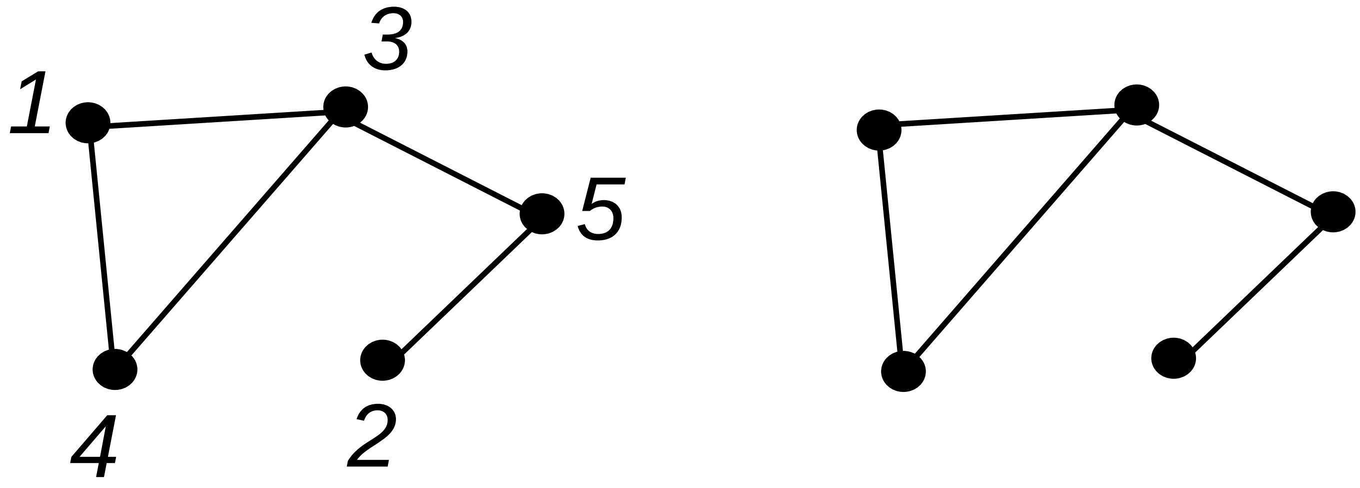
Examples: Planar graphs, series-parallel graphs, minor-closed graph classes etc.

$B(x)$ ... GF for 2-connected graphs in $\mathcal{G}$

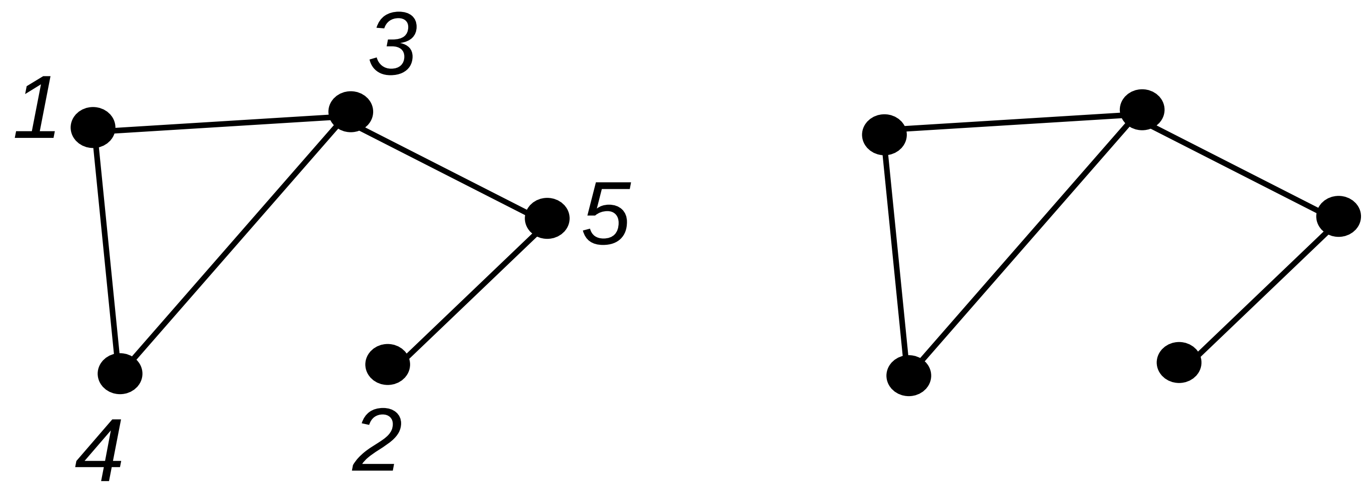
$C(x)$ ... GF for connected graphs in $\mathcal{G}$

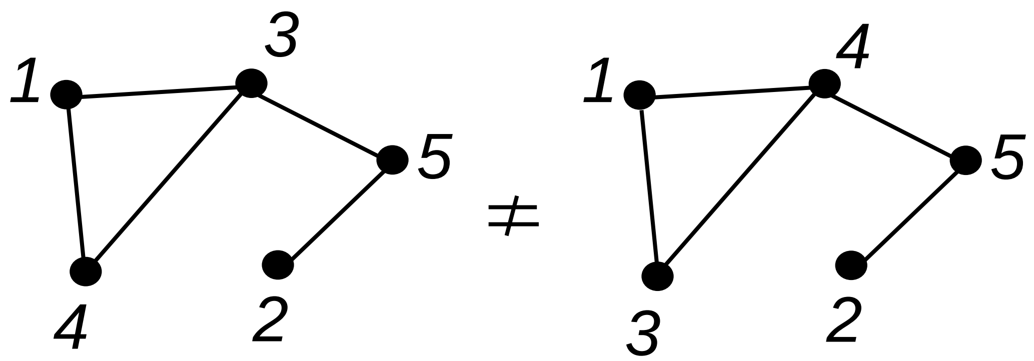
[We will consider here only connected graphs]

Labelled vs. Unlabelled Graphs

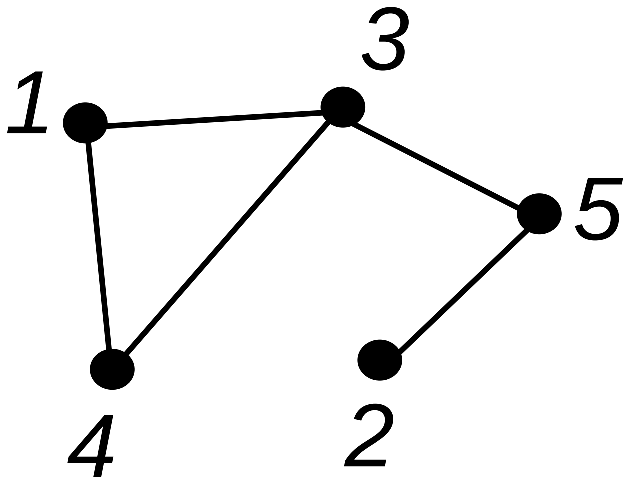


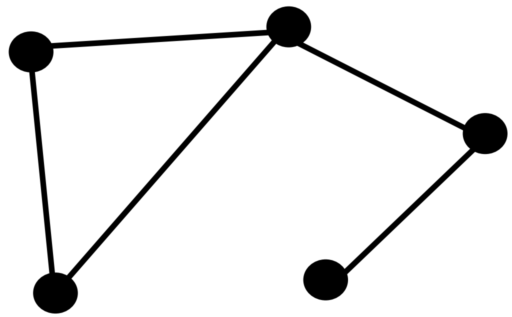
Labelled vs. Unlabelled Graphs





Labelled vs. Unlabelled Graphs





$$\frac{x^5}{5!}$$

$$x^5$$

Generating Functions

g_n ... number of graphs of size n (in a given graph class)

Labelled Graphs

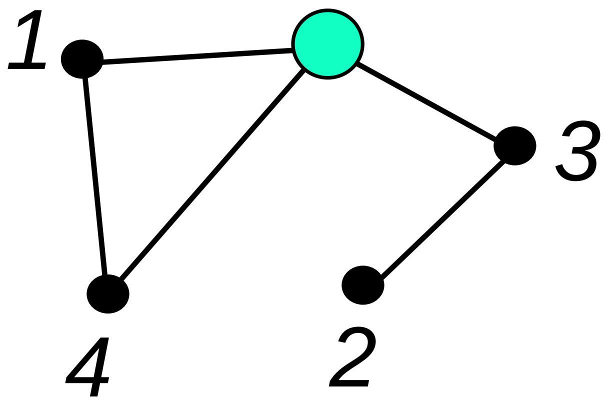
$$\boxed{G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!}}$$

Unlabelled Graphs

$$\boxed{G(x) = \sum_{n \geq 0} g_n x^n}$$

Generating Functions for Block-Decomposition

Vertex-rooted graphs: one vertext (the **root**) is distinguished (and usually discounted, that is, it gets no label)

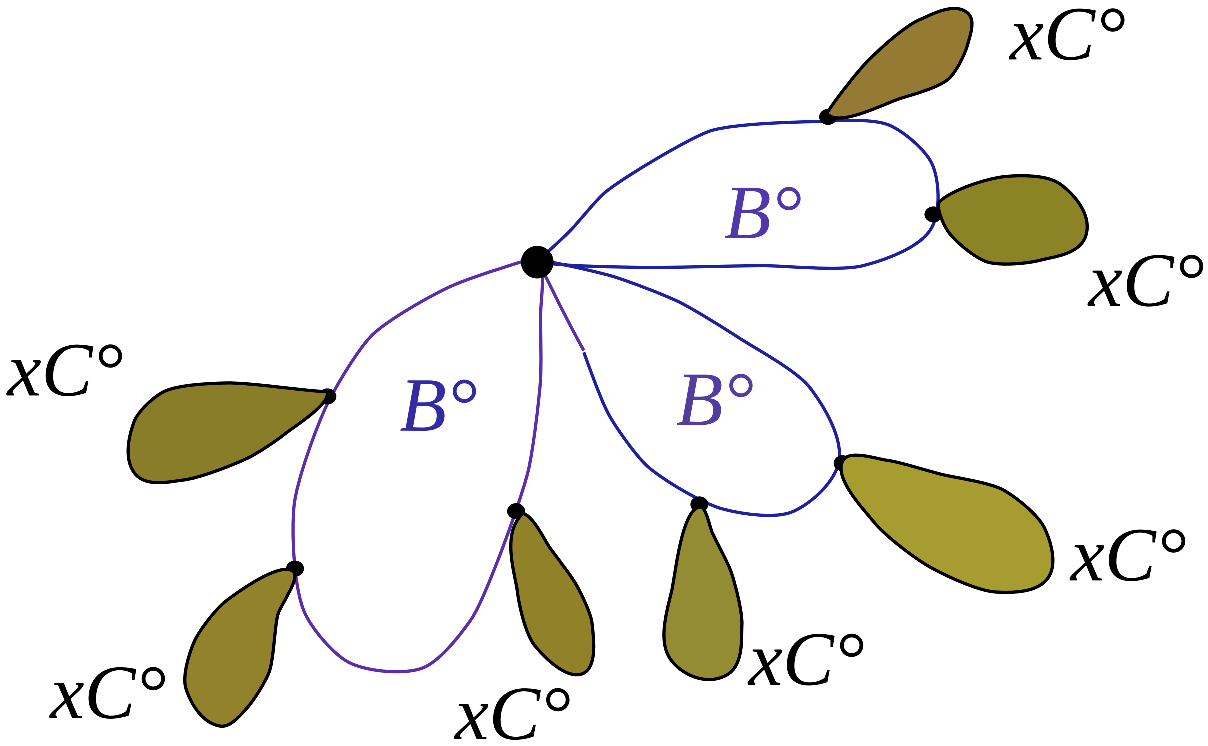


Generating function: (in den **labelled** case)

$$\boxed{G^{\bullet}(x) = G'(x)}$$

Generating Functions for Block-Decomposition

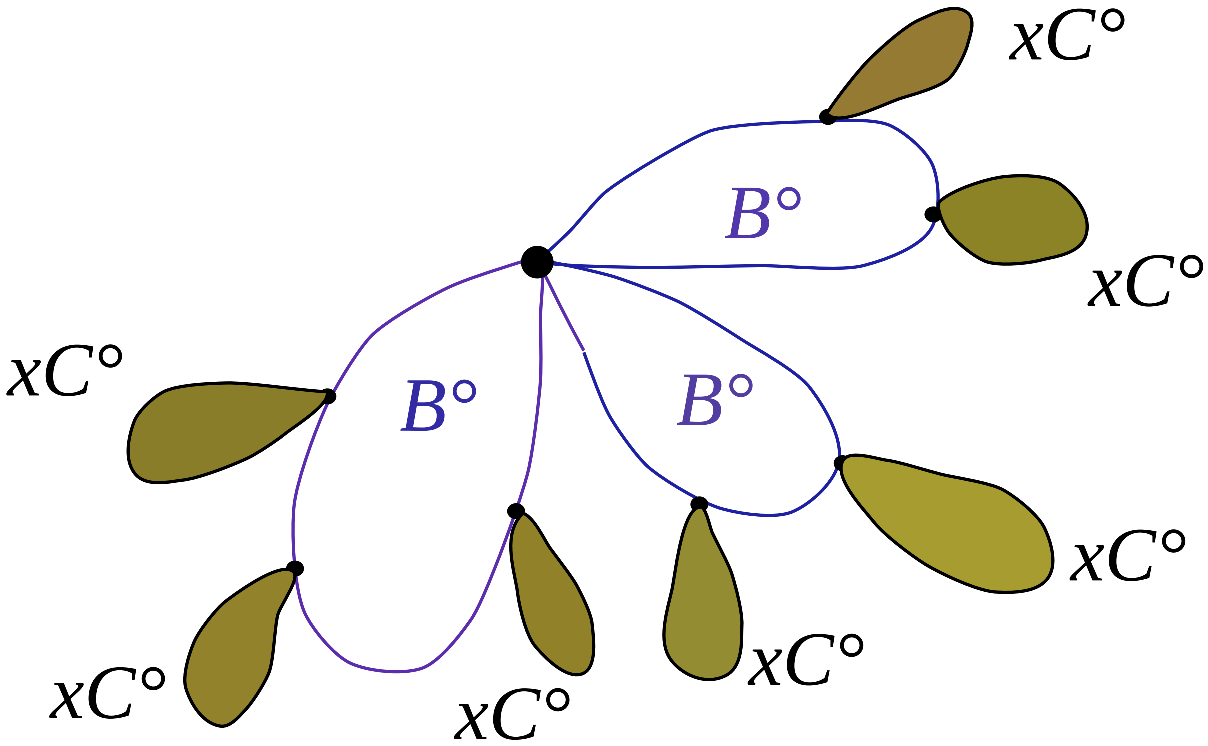
(in the **labelled** case)



$$\boxed{C^\bullet(x) = e^{B^\bullet(xC^\bullet(x))}}$$

Generating Functions for Block-Decomposition

(in the **labelled** case)



$$\boxed{\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right)}$$

Labelled Trees

Rooted Trees:

$$B^{\bullet}(x) = x$$

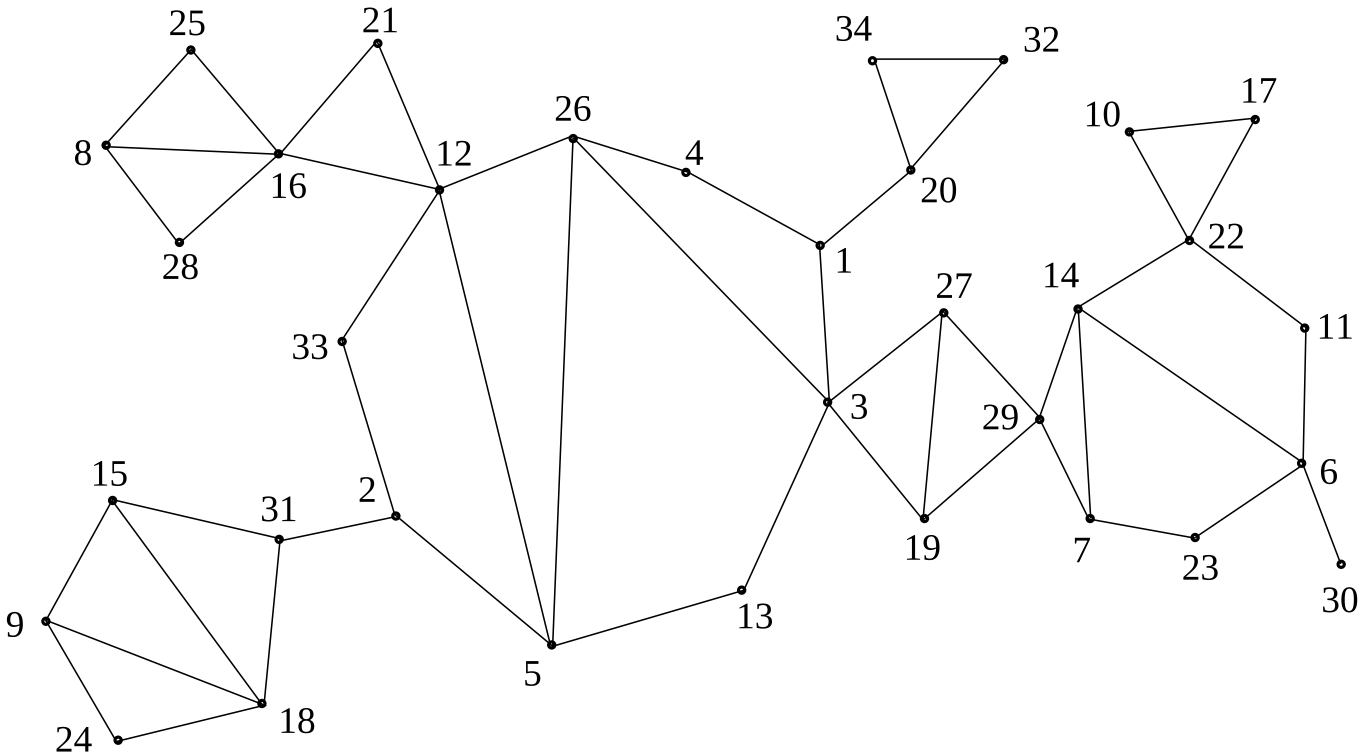
$T(x) = xC^{\bullet}(x)$ ... generating function of **rooted, labelled trees**

$$\boxed{C^{\bullet}(x) = e^{B^{\bullet}(xC^{\bullet}(x))}} \Longrightarrow \boxed{T(x) = xe^{T(x)}}$$

Remark: $\tilde{T}(x)$ ... GF for unrooted labelled trees:

$$\tilde{T}(x)' = \frac{1}{x}T(x) \quad \Longrightarrow \quad \tilde{T}(x) = T(x) - \frac{1}{2}T(x)^2$$

Outerplanar Graphs



All vertices are on the infinite face.

Outerplanar Graphs

Generating functions

$$\boxed{C^\bullet(x) = e^{B^\bullet(xC^\bullet(x))}},$$

$$B^\bullet(x) = \frac{1 + 5x - \sqrt{1 - 6x + x^2}}{8}.$$

2-connected outerplanar graphs = dissections of the n-gon

Series-Parallel Graphs

Generating functions

$$\boxed{\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right)},$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},$$

$$D(x,y) = y + S(x,y) + P(x,y),$$

$$S(x,y) = \frac{x(P(x,y)+y)^2}{1-x(P(x,y)+y)},$$

$$P(x,y) = (e^{S(x,y)} - 1 - S(x,y)) + y(e^{S(x,y)} - 1).$$

Labelled Planar Graphs

$$\boxed{\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right)},$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},$$

$$\frac{M(x,D)}{2x^2 D} = \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD},$$

$$M(x,y) = x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right),$$

$$U(x,y) = xy(1+V(x,y))^2,$$

$$V(x,y) = y(1+U(x,y))^2.$$

Sub-critical Graphs

Functional equations

Suppose that $\boxed{A(x) = \Phi(x, A(x))}$, where $\Phi(x, a)$ has a power series expansion at $(0, 0)$ with non-negative coefficients and $\Phi_{aa}(x, a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (**inside the region of convergence of** Φ) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0, a_0), \quad 1 = \Phi_a(x_0, a_0)}.$$

Then there exists analytic function $g(x), h(x)$ such that locally

$$\boxed{A(x) = g(x) - h(x)\sqrt{1 - \frac{x}{x_0}}}.$$

Remark. If there is no x_0, a_0 inside the region of convergence of Φ then the singular behaviour of Φ determines the singular behaviour of $A(x)$!!!

Sub-critical Graphs

$A(x) = xC^{\bullet}(x)$, $\Phi(x,a) = xe^{B^{\bullet}(a)}$, $\boxed{xC^{\bullet}(x) = xe^{B^{\bullet}(xC^{\bullet}(x))}}$

$$\Longrightarrow \quad \boxed{A(x) = \Phi(x, A(x))}$$

Case 1: the sub-critical case. The system (note that $B^{\bullet}(x) = B'(x)$)

$$a_0 = x_0 e^{B'(a_0)}, \qquad 1 = x_0 e^{B'(a_0)} B''(a_0)$$

has positive solutions x_0, a_0 such that a_0 **is smaller than the radius of convergence** η **of** $B^{\bullet}$. Eliminating x_0: $\boxed{a_0 B''(a_0) = 1}$. Thus

$$\boxed{\eta B''(\eta) > 1}$$

Case 2: the critical case. The other case:

$$\boxed{\eta B''(\eta) \leq 1}.$$

Here the singular behaviour of $B^{\bullet}$ determines the singular behaviour of $C^{\bullet}(x)$.

Sub-critical Graphs

- **Trees** are sub-critical
- **Outerplanar graphs** are sub-critical
- **Series-parallel graphs** are sub-critical
- **Planar graphs** are **critical**

Sub-critical Graphs

Conjecture [M. Noy]

Let $\mathcal{G}$ be a **minor closed graph class**, that is, $\boxed{\mathcal{G} = \text{Ex}(H_1, \ldots, H_k)}$.

$\boxed{\mathcal{G} \text{ is sub-critical} \iff \text{one of the excluced minors } H_1, \ldots, H_k \text{ is planar.}}$

- Trees: $\text{Ex}(K_3)$
- Outerplanar Graphs: $\text{Ex}(K_4, K_{2,3})$
- Series parallel Graphs: $\text{Ex}(K_4)$

- Planar Graphs: $\text{Ex}(K_5, K_{3,3})$

Sub-critical Graphs

Lemma. Suppose that $B(x)$ has radius of convergence $\eta \in (0, \infty]$.

$$\lim_{x \to \eta} B''(x) = \infty \quad \Longrightarrow \quad \textbf{sub-critical}.$$

Corollary If $B^\bullet(x) = B'(x)$ is entire or has a squareroot singularity:

$$B^\bullet(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\eta}},$$

then we are in the **sub-critical** case.

This applies for **outerplanar** and **series-parallel** graphs.

Sub-critical Graphs

What does "sub-critical" mean?

In a sub-critical graph class the **average size of the 2-connected components is bounded**.

$\Longrightarrow$ This leads to a **tree like structure**.

$\Longrightarrow$ The **law of large numbers** should apply so that we can expect **universal behaviors** that are independent of the the precise structure of 2-connected components.

Unlabelled Graph Classes

Cycle index sums

$$\boxed{Z_{\mathcal{G}}(s_1, s_2, \ldots) := \sum_n \frac{1}{n!} \sum_{\substack{\sigma, g \in \mathfrak{S}_n \times \mathcal{G}_n \\ \sigma \cdot g = g}} s_1^{c_1(\sigma)} s_2^{c_2(\sigma)} \cdots s_n^{c_n(\sigma)}}$$

where $c_j(\sigma)$ denotes the number of cycles of size j in $\sigma \in \mathfrak{S}_n$

$$G(x) = Z_{\mathcal{G}}(x, x^2, x^3, \cdots)$$

$$Z_{\mathcal{G}^\bullet}(s_1, s_2, \ldots) = \frac{\partial}{\partial s_1} Z_{\mathcal{G}}(s_1, s_2, \ldots)$$

$$G^\bullet(x) = Z_{\mathcal{G}^\bullet}(x, x^2, x^3, \cdots) = \frac{\partial}{\partial s_1} Z_{\mathcal{G}}(x, x^2, x^3, \cdots)$$

Unlabelled Graph Classes

Block decomposition

$$\boxed{C^\bullet(x) = \exp\left(\sum_{i\geq 1}\frac{1}{i}Z_{B^\bullet}(x^iC^\bullet(x^i), x^{2i}C^\bullet(x^{2i}), \ldots)\right)}$$

- Dichotomy between **sub-critical** and **critical** can be defined in a natural way.

- Unlabelled **trees** are **sub-critical**.

- Unlabelled **outerplanar graphs** are sub-critical

- Unlabelled **series-parallel graphs** are **sub-critical**.

Sub-critical Graphs

Universal properties

- **Asymptotic enumeration**:

Labelled case:

$$\boxed{c_n \sim c\, n^{-5/2} \rho^{-n} n!}$$

Unlabelled case:

$$\boxed{c_n \sim c\, n^{-5/2} \rho^{-n}}$$

($c > 0$, ρ ... radius of convergence of $C(z)$)

[D.+Fusy+Kang+Kraus+Rue 2011]

Sub-critical Graphs

- **Asymptotic enumeration**:

$$C^{\bullet}(x) = e^{B^{\bullet}(xC^{\bullet}(x))}$$

$$\longrightarrow \quad xC^{\bullet}(x) = xC'(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\rho}}$$

$$\longrightarrow \quad [x^n]xC'(x) = \frac{n\,c_n}{n!} \sim c\,n^{-3/2}\rho^{-n}$$

$$\longrightarrow \quad \boxed{c_n \sim c\,n^{-5/2}\rho^{-n}n!}.$$

Additive Parameters in Subcritical Graph Classes

Theorem 1 [D.+Fusy+Kang+Kraus+Rue]

X_n ... number of **edges** / number of **blocks** / number of **cut-vertices** / number of **vertices of degree k**

$$\Longrightarrow \quad \boxed{\frac{X_n - \mu n}{\sqrt{n}} \to N(0, \sigma^2)}$$

with $\mu > 0$ and $\sigma^2 \geq 0$.

Remark. There is an easy to check "combinatorial condition" that ensures $\sigma^2 > 0$.

Additive Parameters in Subcritical Graph Classes

Proof Methods:
Refined versions of the functional equation $C^\bullet(x) = e^{B^\bullet(xC^\bullet(x))}$,
+ singularity analysis (**always squareroot singularity**)

E.g: number of edges:

$$C^\bullet(x,y) = e^{B^\bullet(xC^\bullet(x,y),y)}$$

or number of 2-connected components:

$$C^\bullet(x,y) = e^{yB^\bullet(xC^\bullet(x,y))}$$

$$\longrightarrow \quad C^\bullet(x,y) = g(x,y) - h(x,y)\sqrt{1 - \frac{x}{\rho(y)}}$$

$$\longrightarrow \quad [x^n]C^\bullet(x,y) \sim c(y)\rho(y)^{-n}n^{-3/2}$$

+ application of Quasi-Power-Theorem (by Hwang).

Graph Limits

$\mathcal{T}_e$ ... **continuum random tree** (CRT)

Theorem 2 [Panagiotou+Stufler+Weller]

$\mathcal{C}$ ... sub-critical graph class of connected graphs

$$\Longrightarrow \quad \boxed{\frac{c}{\sqrt{n}}\mathcal{C}_n \to \mathcal{T}_e}$$

with respect to the Gromov-Hausdorff metric, where $c > 0$ is a constant.

Corollary. The diameter D_n as well as a typical distance in a sub-critical graph is or order $\sqrt{n}$.

Graph Limits

Theorem 3 [Stufler, Georgakopoulos+Wagner]

$\mathcal{C}$ ... sub-critical graph class of connected graphs

Then there exists a random rooted graph $\widehat{C}^\bullet$ such that for all $R > 0$ the R-neighborhood of a random vertex of a random graph in $\mathcal{C}$ has in the limit the same distribution as the R-neighborhood of the root of $\widehat{C}^\bullet$.

Remark. $\widehat{C}^\bullet$ is the Benjamini-Schramm limit. All local structures *stabilize*.

Graph Limits

Corollary [Stufler]

$\mathcal{C}$ ... sub-critical graph class of connected graphs

H ... fixed graph

$X_n^{(H)}$ ... number of occurences of H as a subgraph in graphs of size n

$$\Longrightarrow \quad \boxed{X_n^{(H)}/n \to c \qquad \text{in prob.}}$$

for some constant c.

Subgraph Counting

Theorem [D.+Ramos+Rue]

$\mathcal{G}$ ... sub-critial graph class, $\boxed{H \in \mathcal{G}}$ fixed.
$X_n^{(H)}$ ... number of occurences of H as a subgraph in graphs of size n

$$\Longrightarrow \quad \boxed{\frac{X_n^{(H)} - \mu n}{\sqrt{n}} \to N(0, \sigma^2)}$$

with $\mu > 0$ and $\sigma^2 \geq 0$.

Remark. The proof is easy if H is 2-connected = **additive parameter!!!**

Subgraph Counting

$\boxed{H = P_2}$ ... path of length 2

$B_j^\bullet(w_1, w_2, w_3, \ldots; u)$ .... generating function of blocks in $\mathcal{G}$, where the root has degree j, where w_i counts the number of non-root vertices of degree i, and where u counts the number of occurrences of $H = P_2$.

$C_j^\bullet(x, u)$ ... generating function of connected rooted graphs in $\mathcal{G}$, where the root vertex has degree j, where x counts the number of (all) vertices and u the number of occurrences of $H = P_2$.

Subgraph Counting

System of infinite number of equations

$$C_j^\bullet(x,u) = \sum_{s\ge 0} \frac{1}{s!} \sum_{j_1+\cdots+j_s=j} u^{\sum_{i_1<i_2} j_{i_1} j_{i_2}}$$
$$\times \prod_{i=1}^{s} B_{j_i}^\bullet \left(x \sum_{\ell_1\ge 0} u^{\ell_1} C_{\ell_1}^\bullet(x,u), x \sum_{\ell_2 \ge 0} u^{2\ell_2} C_{\ell_2}^\bullet(x,u), \ldots ; u \right),$$
$$(j \ge 0)$$

$$C_j^\bullet(x,1) = \sum_{s\ge 0} \frac{1}{s!} \sum_{j_1+\cdots+j_s=j} \prod_{i=1}^{s} B_{j_i}^\bullet \left(xC^\bullet(x), xC^\bullet(x), \ldots ; 1 \right)$$
$$C^\bullet(x) = \sum_{\ell \ge 0} C_\ell^\bullet(x,1)$$

Subgraph Counting

System of infinite number of equations

Lemma [D.+Gittenberger+Morgenbesser]

Suppose that $\boxed{\mathbf{A}(z) = (A_j(z))_{j\ge 0} = \mathbf{\Phi}(z, \mathbf{A}(z))}$ is a **positive**, **non-linear**, **infinite** and **strongly connected** system such that the **Jacobian** $\mathbf{\Phi}_{\mathbf{a}}(z, \mathbf{a})$ is **compact** for $z > 0$ and $\mathbf{a} > \mathbf{0}$.

Let $z_0 > 0$, $\mathbf{a}_0 = (a_{j,0})_{j\ge 0}$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{\mathbf{a}_0 = \mathbf{\Phi}(z_0, \mathbf{a}_0), \quad r(\mathbf{\Phi}_{\mathbf{a}}(z_0, \mathbf{a}_0)) = 1},$$

where $r(\cdot)$ denotes the spectral radius.

Then there exists analytic function $g_j(z), h_j(z) \ne 0$ such that locally

$$\boxed{A_j(z) = g_j(z) - h_j(z)\sqrt{1 - \frac{z}{z_0}}}.$$

with $g_j(z_0) = a_{j,0}$ and $h_j(z_0) > 0$.

Subgraph Counting

Extension [D.+Gittenberger+Morgenbesser]

Suppose that $\boxed{\mathbf{A}(z,u) = (A_j(z,u))_{j\ge 0} = \mathbf{\Phi}(z,u,\mathbf{A}(z,u))}$ is a **positive**, **non-linear**, **infinite** and **strongly connected** system such that the **Jacobian $\Phi_{\mathrm{a}}(z,1,\mathrm{a})$ is compact** for $z>0$ and $\mathbf{a} > \mathbf{0}$.

Let $z_0 > 0$, $\mathbf{a}_0 = (a_{j,0})_{j\ge 0}$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{\mathbf{a}_0 = \mathbf{\Phi}(z_0,1,\mathbf{a}_0), \quad r(\mathbf{\Phi}_{\mathbf{a}}(z_0,1,\mathbf{a}_0)) = 1},$$

where $r(\cdot)$ denotes the spectral radius.

Then there exists analytic function $g_j(z,u), h_j(z,u) \ne 0$ and $\rho(u)$ such that locally

$$\boxed{A_j(z,u) = g_j(z,u) - h_j(z,u)\sqrt{1-\frac{z}{\rho(u)}}}.$$

with $g_j(z_0,1) = a_{j,0}$, $h_j(z_0,1) > 0$, and $\rho(1) = z_0$.

Subgraph Counting

Central Limit Theorem

$$\Longrightarrow \qquad A(z,u) = g(z,u) - h(z,u)\sqrt{1 - \frac{z}{\rho(u)}}$$

$$\longrightarrow \qquad [z^n]A(z,u) \sim C(u)\rho(u)^{-n}n^{-3/2}$$

+ application of Quasi-Power-Theorem (by Hwang) implies CLT.

Subgraph Counting

Special case of infinite system

$$A_j = \Phi_j(z, u, A_0, A_1, \ldots), \qquad j \geq 0,$$

with

$$\boxed{\Phi_j(z, \mathbf{1}, A_0, A_1, \ldots) = \tilde{\Phi}_j(z, A_0 + A_1 + \cdots)},$$

so that $\boxed{A = A_0 + A_1 + \cdots}$ satisfies

$$\boxed{A = \tilde{\Phi}(z, A)},$$

where

$$\tilde{\Phi}(z, A) = \sum_{j \geq 0} \tilde{\Phi}_j(z, A) = \sum_{j \geq 0} \Phi(z, 1, A_0, A_1, \ldots)$$

$$\Longrightarrow \quad \frac{\partial \Phi_j}{\partial a_i}(z, 1, \mathbf{a}) \quad \text{does not depend on } i$$

$$\Longrightarrow \quad \boxed{\mathbf{\Phi_a}(z, 1, \mathbf{a}) \ \ \text{is compact}}$$

Thank You for Your Attention!