LIMIT LAWS OF THE COEFFICIENTS OF POLYNOMIALS WITH ONLY UNIT ROOTS

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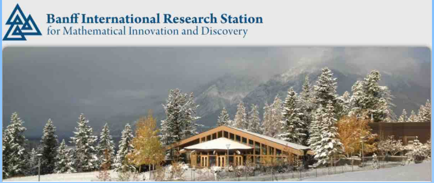
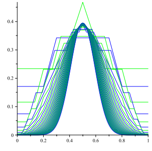
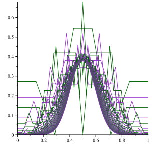
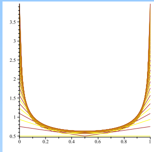
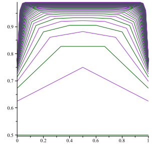










POLYNOMIALS WITH NONNEGATIVE COEFFS

$$P_n(z) = \sum_{0 \leqslant j \leqslant n} a_j z^j, \; a_j \geqslant 0$$

$$\boxed{\mathbb{P}(X_n = j) = \frac{a_j}{P_n(1)} \qquad (j = 0, \ldots, n)}$$

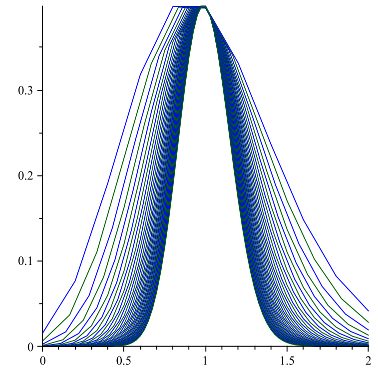
This talk: Zero distribution $\Longrightarrow$ Coeff. distribution

$$(\Im(\textbf{root}) = 0, \Re(\textbf{root}) \leqslant 0, \boxed{|\textbf{root}| = 1}, \ldots)$$

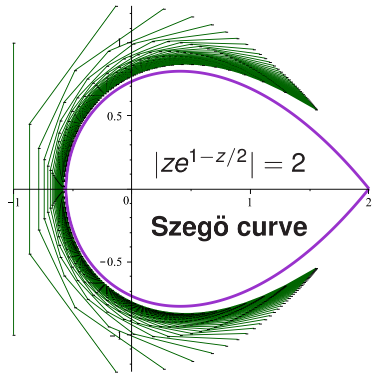
Coeff. distribution $\Longrightarrow$ Zero distribution: Harder in general

COEFF. DISTRIBUTION ⇔ ZERO DISTRIBUTION

Coefficients ($n = 5, \ldots, 40$)



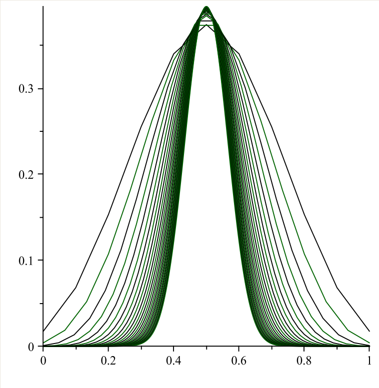
Zeros ($n = 1, \ldots, 100$)



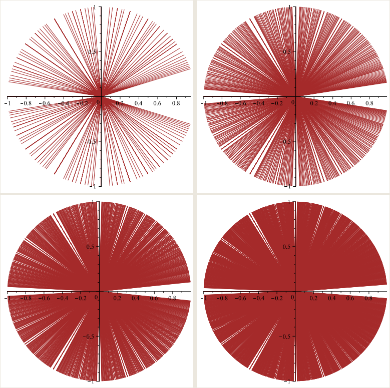
Truncated exponential: $\displaystyle\sum_{0 \leqslant j \leqslant 2n} \frac{(nz)^j}{j!}$

INVERSIONS OF PERMUTATIONS (KENDALL'S τ)

Coefficients ($n = 5, \ldots, 30$)



Zeros ($n = 20, 40, 60, 80$)



$$\text{PGF} = \prod_{1 \leqslant j \leqslant n} \frac{1 + z + \cdots + z^{j-1}}{j}$$

POLYNOMIALS WITH ONLY REAL ROOTS

$P_n(z) = \prod_{1\leqslant j\leqslant n}(z+\alpha_{n,j}) \implies X_n = \sum \text{Bernoulli}$

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \sim \mathscr{N}(0,1) \quad \textbf{iff} \quad \mathbb{V}(X_n) \to \infty$$

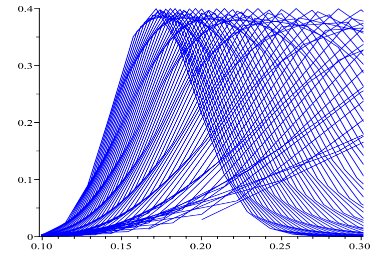
see Pitman's (1997) survey

Proving all roots real $\implies$ challenging

Frieze-Pittel (1994):
for $n \geqslant 1$

$$f_n(z) = z \sum_{0\leqslant j<n} \frac{(n-j)n!}{n^{n-j+1}j!} f_j(z)$$

with $f_0(z) = 1$.



COMBINATORIAL EXAMPLES

All roots are real (negative)

- **Binomial** $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

  $$(1+z)^n$$

- **Stirling first kind** $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$

  $$\frac{z(z+1)\cdots(z+n-1)}{n!}$$

- **Stirling second kind** $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$

  $$P_n(z) = zP'_{n-1}(z) + P_{n-1}(z)$$

- **Eulerian** $\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$

  $$P_n(z) = z(1-z)P'_{n-1}(z) + nzP_{n-1}(z)$$

- **Eulerian second** $\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$

  $$P_n(z) = z(1-z)P'_{n-1}(z) + (1+2z(n-1))P_{n-1}(z)$$

Examples are abundant

MORE EXAMPLES FROM PROBABILITY, COMBINATORICS, ALGEBRA, ALGORITHMICS

Brenti (1992, 1994), Pitman (1997)

- **Hypergeometric** $\frac{\binom{N}{k}\binom{M}{n-k}}{\binom{N+M}{n}}$
- **Occupancy** $\binom{N}{k}\frac{k}{N^n}S(n,k)$
- **Leaves of random trees**
- **Matchings (graphs)**
- **Random mappings**
- **Partitions of multisets**
- **Finite Coxeter system (algebra)**
- **Chromatic polynomials**
- $\cdots$

POLYNOMIALS WITH UNIT ROOTS ($|\text{roots}| = 1$)

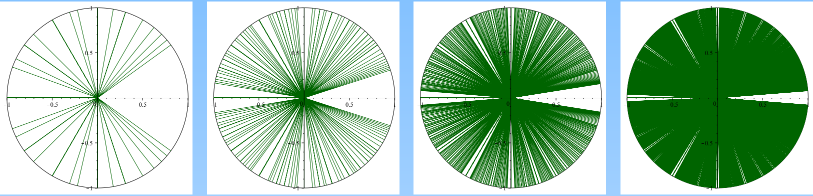
Unit roots are important in many areas

- **Unit-root test: whether a time series is *non-stationary* (using an autoregressive model)**
- **Control theory: *stability* of a system**
- **Lee-Yang's Theorem (1952; 1957 Nobel Laureates): all zeroes of the *partition function* of the ferromagnetic Ising model in a magnetic field lie on the unit circle** $(x_{i,j} = x_{j,i} \in \mathbb{R},\ |x_{i,j}| \leqslant 1)$

$$\sum_{0 \leqslant m \leqslant n} \frac{z^m}{m!(n-m)!} \sum_{(i_1,\dots,i_m,j_1,\dots,j_{n-m}) \in \mathbb{S}_n} \prod_{\substack{1 \leqslant \ell \leqslant m \\ 1 \leqslant r \leqslant n-m}} x_{i_\ell, j_r} = 0$$

- **Many examples below**

POLYNOMIALS WITH ONLY UNIT ROOTS & COEFFs $\geqslant 0$



Root-unitary: polynomials with only unit roots

MAIN THEOREM: 4TH MOMENTS & LIMIT LAWS

If $\mathbb{E}(z^{X_n})$ is root-unitary, then $\left(X_n^* := \dfrac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}}\right)$

$$1 \leqslant \mathbb{E}(X_n^*)^4 < 3$$

$$X_n^* \sim \mathscr{N}(0,1) \quad \textbf{iff} \quad \mathbb{E}(X_n^*)^4 \to 3$$

$$X_n^* \sim \mathscr{B}(-1,1) \quad \textbf{iff} \quad \mathbb{E}(X_n^*)^4 \to 1$$

$\mathscr{B}(-1,1) =$ **Bernoulli assuming** ± 1 **with probability** $\frac{1}{2}$ **each**

Most previous papers need all moments for CLT

Define the kurtosis $\mathbf{Kurt}(X) := \dfrac{\mathbb{E}(X - \mathbb{E}(X))^4}{\mathbb{V}(X)^2}$

$$1 \leqslant \mathbf{Kurt}(X_n) < 3$$

$$X_n^* \sim \mathscr{N}(0,1) \quad \textbf{iff} \quad \mathbf{Kurt}(X_n) \to 3$$

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Usual kurtosis $= \textit{\textbf{Kurt}}(X) - 3 = \dfrac{\kappa_4}{\kappa_2^2}$

ANOTHER FOURTH MOMENT THEOREM

Nualart and Piccati (*Ann. Prob.*, 2004)

***A sequence of random variables belonging to a level of Wiener chaos with unit variance is asymptotically normally distributed iff the sequence of the fourth moments converge to* 3.**

Central limit theorems for sequences of multiple stochastic integrals
D **Nualart**, G **Peccati** - The Annals of Probability, 2005 - projecteuclid.org
Abstract We characterize the convergence in distribution to a standard normal law for a sequence of multiple stochastic integrals of a fixed order with variance converging to 1. Some applications are given, in particular to study the limiting behavior of quadratic ...
Cited by 252 Related articles All 14 versions Cite Save

Many follow-up papers

Tao & Vu's Four Moment Theorem (random matrices)
$\Longrightarrow$ very different

SIMPLE EXAMPLES

Normal: Kendall's τ: $\prod_{1 \leqslant j \leqslant n} \frac{1 + z + \cdots z^{j-1}}{j}$

$$\textbf{Kurt}(X_n) = 3 - \frac{9(6n^2 + 15n + 16)}{25n(n-1)(n+1)} \to 3$$

$$\textbf{MGF} = e^{s^2/2}$$

Bernoulli: $\frac{1 + z^{2n}}{2}$

$$\textbf{MGF} = \frac{e^s + e^{-s}}{2} = \cosh(s) = \prod_{k \geqslant 1} \left(1 + \frac{4s^2}{(2k-1)^2 \pi^2}\right)$$

Uniform: $\frac{1 + z + \cdots + z^{2n}}{2n+1}$ (kurtosis $= \frac{9}{5} - \frac{3}{5n(n+1)}$)

$$\textbf{MGF} = \frac{1}{2\sqrt{3}} \int_{\sqrt{3}}^{\sqrt{3}} e^{xs}\, \mathrm{d}x = \frac{\sinh(\sqrt{3}\, s)}{\sqrt{3}\, s} = \prod_{k \geqslant 1} \left(1 + \frac{3s^2}{k^2 \pi^2}\right)$$

SIMPLE EXAMPLES

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INFINITE-PRODUCT REPRESENTATION FOR THE MGFs OF LIMIT LAWS

Assume $\mathbb{E}(z^{X_n})$ are root-unitary.

If $X_n^* \xrightarrow{d} X$, then

$$\mathbb{E}(e^{Xs}) = e^{qs^2/2} \prod_{k\geqslant 1} \left(1 + \frac{q_k}{2} s^2\right)$$

where $q, q_k \geqslant 0$ and $q + \sum_{k\geqslant 1} q_k = 1$.

PROPERTIES OF ROOT-UNITARY POLYNOMIALS

WLOG, consider polynomials of the form
$P_{2n}(z) := \sum_{0\leqslant k\leqslant 2n} p_k z^k$, $p_k \geqslant 0$

$$\mathbb{E}(z^{X_{2n}}) := \frac{P_{2n}(z)}{P_{2n}(1)}$$

If $P_{2n}(z)$ is root-unitary, then

- ***$P_{2n}(z)$ is self-inversive ($p_k = p_{2n-k}$)***
- $\mathbb{E}(X_{2n}) = n$
- $\mathbb{E}(X_{2n} - n)^{2m+1} = 0$
- $\frac{n}{2} \leqslant \mathbb{V}(X_{2n}) \leqslant n^2$

A key step

$$\mathbb{E}(z^{X_{2n}}) = \prod_{1\leqslant j\leqslant n} \frac{1 - 2z\cos\phi_j + z^2}{2(1-\cos\phi_j)}, \quad P_{2n}(e^{i\phi_j}) = 0$$

PROPERTIES OF ROOT-UNITARY POLYNOMIALS

$$\mathbb{E}(z^{X_{2n}}) = \prod_{1\leqslant j\leqslant n} \frac{1-2z\cos\phi_j+z^2}{2(1-\cos\phi_j)}$$

For $|s| \leqslant \frac{1}{4}\min\{\sigma_n, \omega_n^{-1/4}\}$

$$\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) = \exp\left(\frac{s^2}{2} + O\left(\frac{|s|^3}{\sigma_n} + \omega_n|s|^4\right)\right),$$

where $\omega_n := \frac{1}{\sigma_n^4}\sum_{1\leqslant j\leqslant n}\frac{1}{(1-\cos\phi_j)^2}$ $\left(\sigma_n^2 = \sum_{1\leqslant j\leqslant n}\frac{1}{1-\cos\phi_j}\right)$.

$$\boxed{\omega_n \to 0} \Rightarrow \boxed{\sum_{1\leqslant j\leqslant n}\phi_j^{-4} = o\left(\left(\sum_{1\leqslant j\leqslant n}\phi_j^{-2}\right)^2\right)}$$

ϕ_j **not too close to zero (say** $\phi_j \gg \sqrt{j}/n$**)** $\Longrightarrow$ **Normal; otherwise, non-Normal.**

INFINITE-PRODUCT REPRESENTATION

Two proofs

- **Convergence of all moments $\Longrightarrow$ *convergence of all cumulants***

$$\frac{\kappa_{2m}}{(2m)!} = \frac{(-1)^{m-1}}{m2^m} \lim_{n\to\infty} \frac{1}{\sigma^{2m}} \sum_{1\leqslant j\leqslant n} \frac{1}{(1-\cos\phi_j)^{2m}}$$
$$= \frac{(-1)^{m-1}}{m2^m} \int_0^1 x^{m-1}\,\mathrm{d}F(x),$$

where $F(x) = \begin{cases} q + \sum_{q_k<x} q_k, & \textbf{if } x \geqslant 0, \\ 0, & \textbf{if } x < 0. \end{cases}$

- ***Hadamard factorization theorem***

APPLICATIONS

A simple scheme for *normal limit law*

Assume $P_k(z) := \frac{(1-z^{b_1})(1-z^{b_2})\cdots(1-z^{b_k})}{(1-z^{a_1})(1-z^{a_2})\cdots(1-z^{a_k})}$ **is a polynomial with *nonnegative coeffs* (Chen et al. 2008).**

Define $\mathbb{E}(z^{X_n}) := \frac{P_k(z)}{P_k(1)}$, **where** $n := \sum_{1\leqslant j\leqslant k}(b_j - a_j)$.

$$X_n^* \sim \mathcal{N}(0,1) \quad \textbf{iff} \quad \lim_{k\to\infty} \frac{\sum_{1\leqslant j\leqslant k}(b_j^4 - a_j^4)}{\left(\sum_{1\leqslant j\leqslant k}(b_j^2 - a_j^2)\right)^2} = 0$$

Previous papers checked all cumulants $\frac{\kappa_{2m}(k)}{\kappa_2(k)^m} \to 0$

EXAMPLES

A large number of concrete examples

- **Inversions in permutations:** $\prod_{1\leqslant j\leqslant n} \frac{1-z^j}{1-z}$
- **Stirling permutations:** $\prod_{1\leqslant j\leqslant n} \frac{1-z^{r+(j-1)r^2}}{1-z^r}$
- **Gaussian polynomials:** $\prod_{1\leqslant j\leqslant n} \frac{1-z^{j+m}}{1-z^j}$
- **Generalized q-Catalan numbers:** $\prod_{2\leqslant j\leqslant n} \frac{1-z^{(m-1)n+j}}{1-z^j}$
- **Sums of uniform discrete RVs:** $\prod_{1\leqslant j\leqslant k} \frac{1-z^{d_j}}{1-z}$

$$\textbf{CLT} \quad \textbf{iff} \quad \frac{d_1^4 + d_2^4 + \cdots + d_k^4}{(d_1^2 + d_2^2 + \cdots + d_k^2)^2} \to 0,$$

which is equivalent to Olds (1952)

$$\textbf{CLT} \quad \textbf{iff} \quad \frac{\max_{1\leqslant j\leqslant k} d_j}{\sqrt{d_1^2 + d_2^2 + \cdots + d_k^2}} \to 0$$

EXAMPLES

More concrete examples: Poincaré polynomials, integer partitions, etc.

- **Mahonian statistics of multisets:** $\frac{\prod_{1\leqslant j\leqslant a_1+\cdots+a_m}(1-z^j)}{\prod_{1\leqslant j\leqslant m}\prod_{1\leqslant i\leqslant a_j}(1-z^i)}$

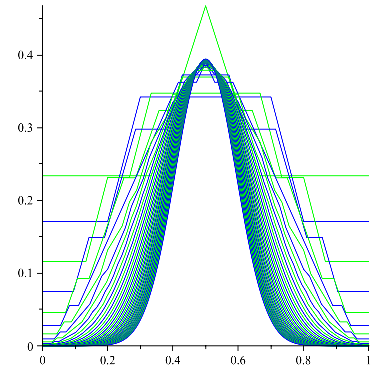
  CLT iff $a_1 \to \infty$ **and** $a_2 + a_3 + \cdots + a_m \to \infty$

  when $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m$ **(Canfield et al. 2011)**
- **Bimodal permutations**
- **Rank statistics:**

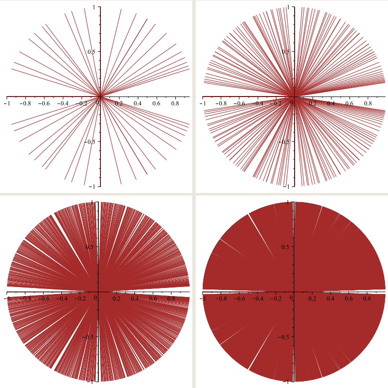
Kendall's τ	Inversions in permutations
Mann-Whitney test	Gaussian polynomials
Jonckheere-Terpstra test	Mahonian statistics

WILCOXON (1945): $\prod_{1\leqslant j\leqslant n} \frac{1+z^j}{2}$

Coefficients $n = 3, \ldots, 40$



Zeros ($n = 10, 20, 50, 100$)



POLICELLO & HETTMANSPERGER (1976): $\left(\frac{1+z^n}{2}\right)^{\lceil \frac{n}{2} \rceil} \prod_{1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor} \frac{1+z^j}{2}$

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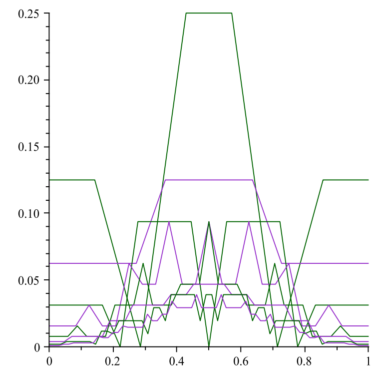

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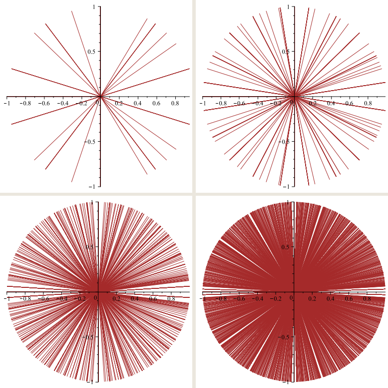

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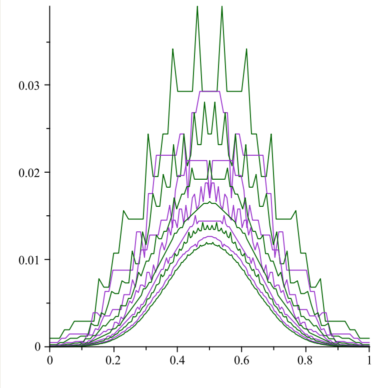

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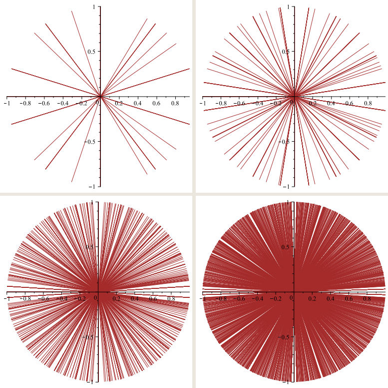

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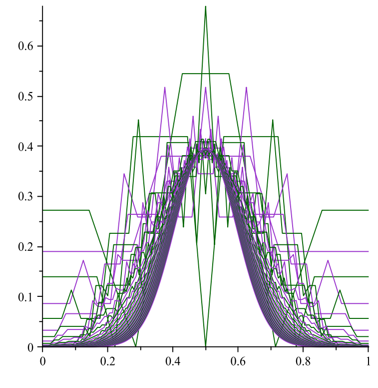

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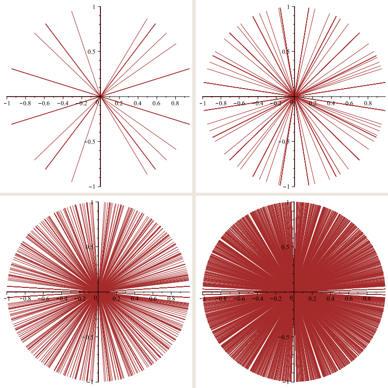
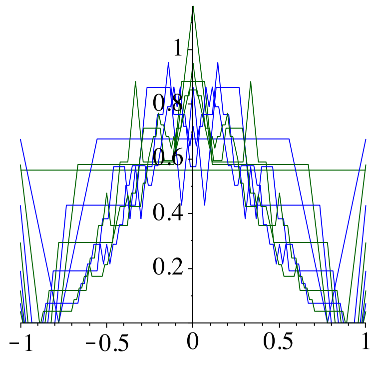


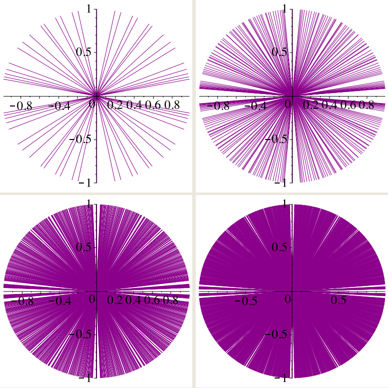
BAGAI, DESHPANDÉ & KOCHAR (1989):

$$z^{-\frac{1}{2}n(2n-1)} \prod_{2\leqslant k\leqslant n+1} \frac{1+z^{2(2n-k)}}{2}$$

Coefficients ($n = 2, \ldots, 10$)



Zeros ($n = 5, 10, 20, 30$)



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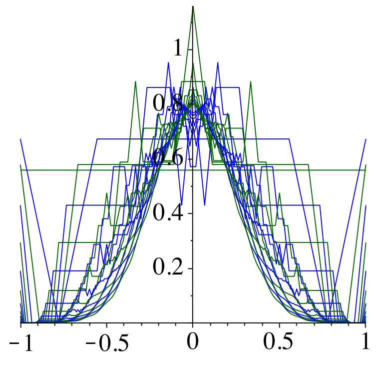
Coefficients $(n = 2, \ldots, 20)$



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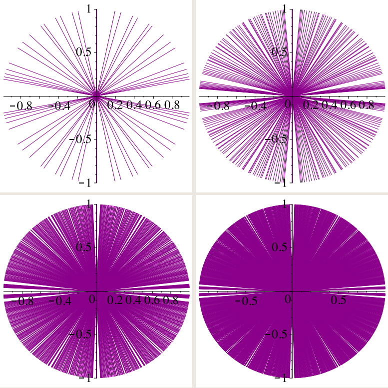

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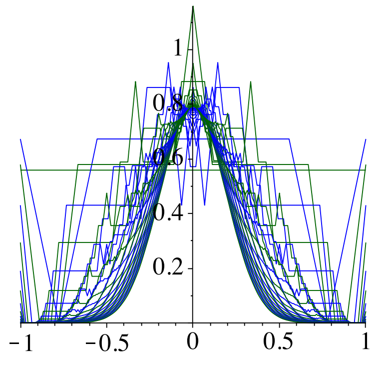
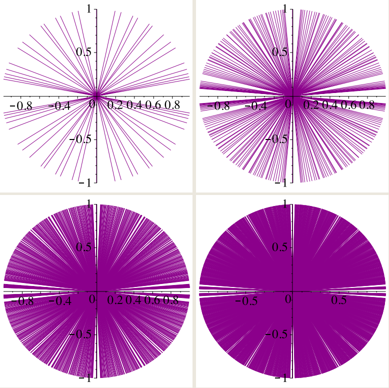


Zeros $(n = 5, 10, 20, 30)$



TURÁN-FEJÉR POLYNOMIALS: $\sum_{0\leqslant j\leqslant n-k} \frac{\binom{j+k}{k}\binom{n-j}{k}}{\binom{n+k+1}{2k+1}} z^j$

(i) not finite product; (ii) $\mathscr{N}$ and non-$\mathscr{N}$ (iii) quicksort

$X_{n,k}$

- **If $k = O(1)$, then**

$$\frac{X_{n,k}}{n} \xrightarrow{d} \mathbf{Beta}(k,k)$$

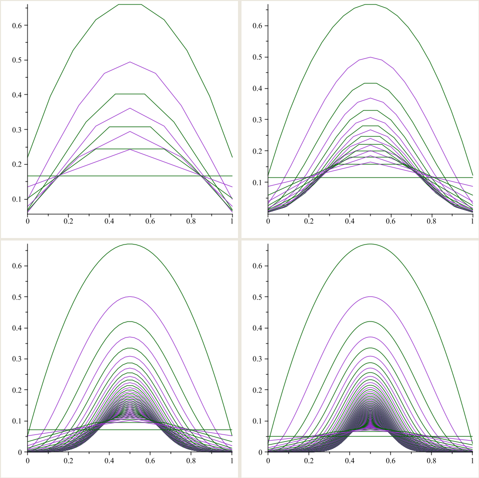
- **If $k, n-k \to \infty$, then**

$$\frac{X_{n,k} - \frac{n-k}{2}}{\sqrt{\frac{(n-k)(n+k+2)}{4(2k+3)}}} \xrightarrow{d} \mathscr{N}(0,1)$$

- **If $\ell := n-k = O(1)$, then**

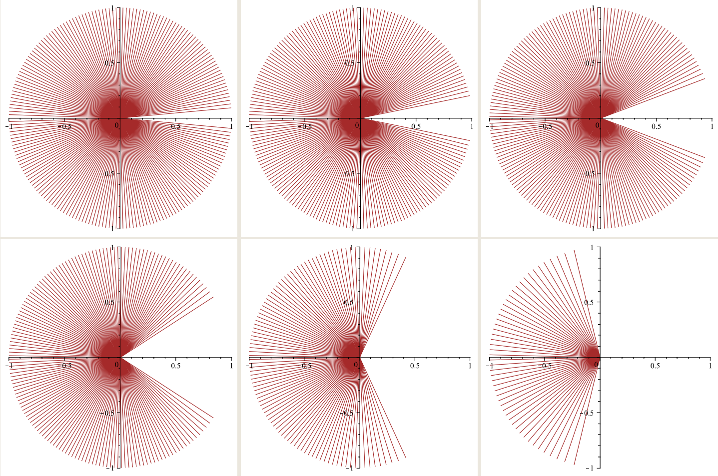
$$X_{n,k} \xrightarrow{d} \mathbf{Binom}(\ell; \tfrac{1}{2})$$

Coefficients ($n = 10, 20, 50, 100$)



TURÁN-FEJÉR POLYNOMIALS: $\sum_{0 \leqslant j \leqslant n-k} \frac{\binom{j+k}{k}\binom{n-j}{k}}{\binom{n+k+1}{2k+1}} z^j$

Zeros ($n = 200$; $k = 5, 15, 30, 50, 100, 150$)



INFINITE-PRODUCT REPRESENTATIONS

Beta(k,k) when $k = O(1)$

$$\mathbb{E}\left(e^{(\mathbf{Beta}(k,k)-1/2)s}\right) = \left(\frac{is}{4}\right)^{-k-\frac{1}{2}} \Gamma\left(k+\tfrac{3}{2}\right) J_{k+\frac{1}{2}}(is/2)$$
$$= \prod_{j\geqslant 1}\left(1+\frac{s^2}{4\zeta_{k+\frac{1}{2},j}^2}\right)$$

J_α = **Bessel function and** $\zeta_{\alpha,j}$ = **positive zeros of** $J_\alpha(z)$**.**

Binomial distribution when $\ell := n-k = O(1)$

$$\mathbb{E}\left(e^{\frac{\mathbf{Binom}(\ell;\frac{1}{2})-\ell/2}{\sqrt{\ell/4}}s}\right) = \cosh\left(\frac{s}{\sqrt{\ell}}\right)^{\ell}$$
$$= \prod_{j\geqslant 1}\left(1+\frac{4s^2}{(2j-1)^2\pi^2\ell}\right)^{\ell}.$$

Normal $\Rightarrow$ *non-Normal*

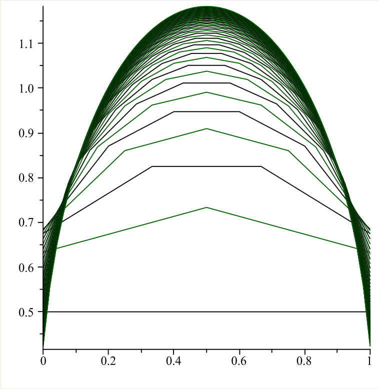
NON-NORMAL LIMIT LAWS

Reimer's (1969) polynomials

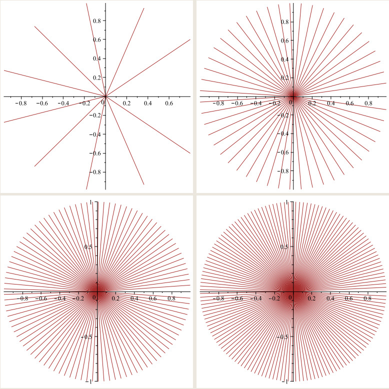
$$\mathbb{E}(z^{X_n}) := 12 \sum_{0 \leqslant k \leqslant n} \binom{n}{k} z^{n-k}(1-z)^k A_{k+2}$$

$A_k =$ **Cauchy numbers:** $A_k = \sum_{0 \leqslant j < k} \frac{-A_j}{k+1-j}$ $(A_0 = -1)$

Coefficients ($n = 1, \ldots, 50$)



Zeros ($n = 10, 50, 100, 150$)



REIMER'S POLYNOMIALS

The sequence A_k

$$\sum_{k\geqslant 0} A_k z^k = \frac{z}{\log(1-z)}$$

$$\{A_k\}_{k\geqslant 1} = \left\{\frac{1}{2}, \frac{1}{12}, \frac{1}{24}, \frac{19}{720}, \frac{3}{160}, \frac{863}{60480}, \frac{275}{24192}, \frac{33953}{3628800}, \dots\right\}$$

$$\mathbb{E}(X_n) = \frac{n}{2} \quad \textbf{and} \quad \mathbb{V}(X_n) = \frac{n}{60}(4n+11)$$

Convergence in distribution

$$\frac{X_n}{n} \xrightarrow{d} X \quad \textbf{where} \quad \mathbb{E}(X^m) = 12 \sum_{0\leqslant \ell \leqslant m} \binom{m}{\ell} (-1)^{\ell} A_{\ell+2}$$

Q: characterize the limit law?

CHUNG-FELLER'S (1949) ARCSIN LAW

Positive terms W_n of random walks $S_n = \sum_{1\leqslant j\leqslant n} \mathscr{B}_j(0,1)$

$$\mathbb{P}(W_n = k) = \binom{2k}{k}\binom{2n-2k}{n-k}4^{-n} \qquad (k = 0, \dots, n).$$

The limit distribution is an arcsine law

$$\frac{W_n}{n} \xrightarrow{d} W, \quad \textbf{where} \quad \mathbb{P}(W < x) = \frac{2}{\pi}\arcsin\sqrt{x}.$$

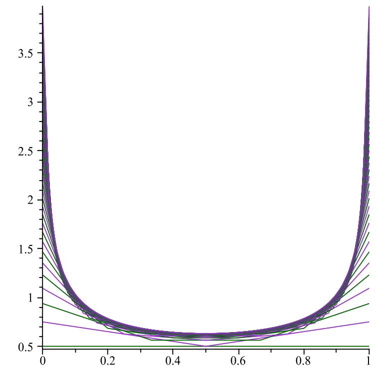
$\mathbb{E}(z^{W_n})$ is root-unitary

$$\mathbb{E}(e^{(W-1/2)s/\sqrt{2}}) = e^{-\sqrt{2}s}\left(1 + \sum_{k\geqslant 1}\binom{2k}{k}\frac{(s/\sqrt{2})^k}{k!}\right)$$

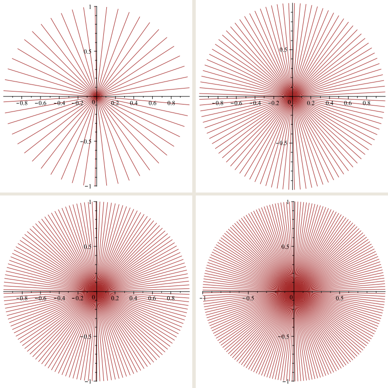
$$= J_0(\sqrt{2}\,is) = \prod_{j\geqslant 1}\left(1 + \frac{2s^2}{\zeta_{0,j}^2}\right)$$

CHUNG-FELLER'S ARCSIN LAW

Coefficients ($n = 1, \ldots, 50$)



Zeros ($n = 50, 100, 150, 200$)

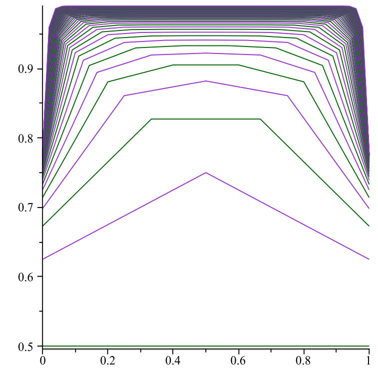


Many ways to produce root-unitary polys with uniform limit law

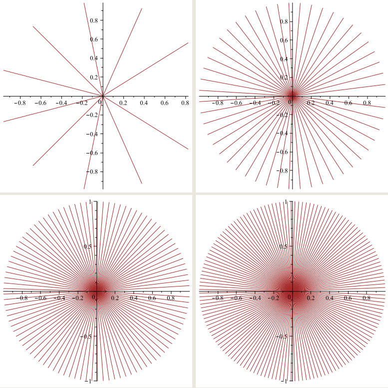
$$P_n(z) = (-1)^n \sum_{0\leqslant j\leqslant n} \binom{2n}{2j} E_{2j} E_{2n-2j} z^j \quad \textbf{is root-unitary}.$$

(Lalin & Rogers, 2011), $\sum_{j\geqslant 0} \frac{E_j}{j!} z^j = \frac{1}{\cosh(z)}$ **(Euler's numbers).**

Coefficients ($n = 1, \ldots, 50$)



Zeros ($n = 10, 50, 100, 150$)



Subfile-size after the partitioning stage of quicksort

$$\mathbb{E}(z^{X_n}) = \sum_{0\leqslant j<r} p_j \sum_k \frac{\binom{k}{j}\binom{n-1-k}{r-1-j}}{\binom{n}{r}} z^k \qquad \left(\begin{array}{c} \sum_j p_j = 1 \\ p_j = p_{r-1-j} \end{array} \right)$$

$$\frac{X_n}{n} \xrightarrow{d} X$$

$$\mathbb{E}(e^{Xs}) = r \sum_{0\leqslant j<r} p_j \binom{r-1}{j} \int_0^1 x^{r-1-j}(1-x)^j e^{xs}\mathbf{d}x$$

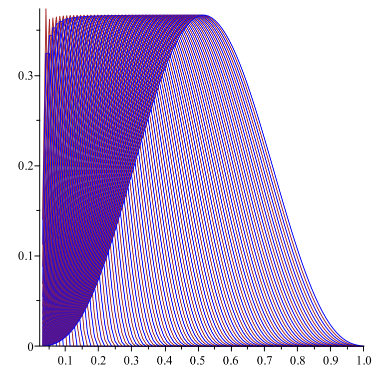
Proof of root-unitarity lacking

POLYS IN CONNECTION WITH QUICKSORT

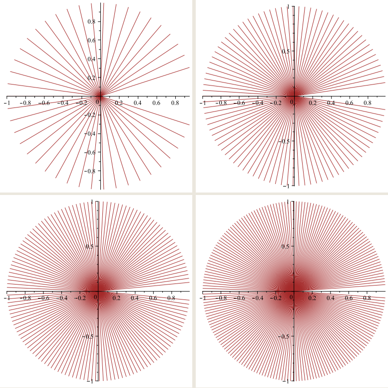
Tukey's ninther: $\{p_j\} = \{0, 0, 0, \frac{3}{14}, \frac{4}{7}, \frac{3}{14}, 0, 0, 0\}$

$$\mathbb{E}(z^{Y_n}) = \sum_{0 \leqslant j < r} p_j \sum_{j \leqslant k < n} \frac{\binom{k}{j}\binom{n-1-k}{r-1-j}}{\binom{n}{r}} z^k$$

Coefficients ($n = 9, \ldots, 100$)



Zeros ($n = 50, 100, 150, 200$)



A QUICK SUMMARY

Polys with only unit roots vs polys with only real roots

properties \ polynomials	all $\lvert\text{roots}\rvert = 1$	all $\Im(\text{roots}) = 0$
all roots bounded	Y	N
symmetric $(p_k = p_{2n-k})$ self-inversive	Y	N
asymptotic normality	kurtosis $\to 3$	var $\to \infty$

H. & Zacharovas (2013, *RSA*)

Open

Convergence rate? Local limit theorems? Large deviations? Characterize the infinite-product representation ($\kappa_{2m} = \frac{(-1)^{m-1}}{m2^m} \sum_{j\geqslant 1} q_j^m$)? . . .


THANK YOU