Total mean curvature, scalar curvature, and a variational analog of Brown-York mass

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Theorem (Shi - Tam, 02)

Let (Ω, g) be a compact, connected, Riemannian 3-manifold with nonnegative scalar curvature, and with nonempty boundary Σ. Suppose Σ has finitely many components Σ_j, $j = 1, \ldots, k$, so that each Σ_j is a topological 2-sphere which has positive Gauss curvature and positive mean curvature H. Then

$$\int_{\Sigma_j} H \, d\sigma \leq \int_{\Sigma_j} H_0 \, d\sigma, \tag{1}$$

where $d\sigma$ denotes the induced area element on Σ_j, and H_0 is the mean curvature of the isometric embedding of Σ_j in $\mathbb{R}^3$. Moreover, equality holds for some Σ_j if and only if $k = 1$ and (Ω, g) is isometric to a convex domain in $\mathbb{R}^3$.

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Suppose Σ is topologically a 2-sphere. There exists a constant $\Lambda > 0$, depending only on (Σ, γ), where γ is the induced metric on Σ from (Ω, g), such that $\int_\Sigma H d\sigma \le \Lambda$.

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- This fact was also independently derived and made used by Lu recently on isometric embeddings into Riemannian manifolds.

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- This fact was also independently derived and made used by Lu recently on isometric embeddings into Riemannian manifolds.
- The proof makes key use of results of Wang-Yau and Shi-Tam on boundary behavior of compact manifolds with a negative lower bound on scalar curvature.

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It follows that

$$\Lambda_{(\Sigma,\gamma)} := \sup\left\{ \frac{1}{8\pi}\int_{\partial\Omega} H d\sigma \mid (\Omega, g) \in \mathcal{F}_{(\Sigma,\gamma)} \right\} < \infty, \qquad (2)$$

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- On $\Sigma = S^2$, if γ is a metric with $\lambda_1(-\Delta + K) > 0$, then by the method of Mantoulidis-Schoen (2014), $\mathcal{F}_{(\Sigma,\gamma)} \neq \emptyset$.
- A related but different set of fill-ins was used by Jauregui (2011).

Recall that, given a compact 3-manifold (Ω, g) with boundary Σ being a 2-sphere, if the induced metric γ on Σ has $K > 0$, the Brown-York mass of Σ in (Ω, g) is defined as

$$\mathfrak{m}_{BY}(\Sigma; \Omega) := \frac{1}{8\pi} \int_{\Sigma} (H_0 - H) d\sigma. \tag{3}$$

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Without assuming γ on Σ has positive K, one may consider

Definition (Mantoulidis - M)

Given a compact 3-manifold (Ω, g) with $\Sigma = \partial\Omega$ being a 2-sphere, define

$$\tilde{\mathfrak{m}}_{BY}(\Sigma; \Omega) := \Lambda_{(\Sigma, \gamma)} - \frac{1}{8\pi} \int_{\Sigma} H d\sigma. \tag{4}$$

For those (Ω, g) which has $R \geq 0$ and $H > 0$, one has

a) $\tilde{\mathfrak{m}}_{BY}(\Sigma; \Omega) \geq 0$,

b) $\tilde{\mathfrak{m}}_{BY}(\Sigma; \Omega) = 0$ only if (Ω, g) is flat, and

c) $\tilde{\mathfrak{m}}_{BY}(\Sigma; \Omega) = \mathfrak{m}_{BY}(\Sigma; \Omega)$ when $K > 0$.

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The third property, which is equivalent to the assertion

$$\Lambda_{(\Sigma,\gamma)} = \frac{1}{8\pi} \int_{\Sigma} H_0 d\sigma \quad \text{when } K > 0,$$

is provided by the Shi-Tam theorem.

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$$\mathring{\mathcal{F}}_{(\Sigma,\gamma)} = \{(\Omega, g) \mid (\Omega, g) \text{ satisfies conditions imposed on elements in } \mathcal{F}_{(\Sigma,\gamma)}, \text{ except that } \partial\Omega \setminus \Sigma \text{ may consist of minimal surfaces}\}.$$

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If one lets $\mathring{\Lambda}_{(\Sigma,\gamma)} := \sup \left\{ \frac{1}{8\pi} \int_{\partial\Omega} H d\sigma \mid (\Omega, g) \in \mathring{\mathcal{F}}_{(\Sigma,\gamma)} \right\}$, then it can be shown

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So it does not matter whether one uses $\Lambda_{(\Sigma,\gamma)}$ or $\mathring{\Lambda}_{(\Sigma,\gamma)}$ in the definition of $\tilde{\mathfrak{m}}_{BY}(\Sigma; \Omega)$.

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be a collection of $k \geq 1$ closed, connected, orientable 2-surface Σ_j where γ_j is any metric on Σ_j, $j = 1, \ldots, k$. Denote by

$$\mathcal{F}_{(\Sigma_1,\gamma_1),\ldots,(\Sigma_k,\gamma_k)}$$

the set of all compact, connected 3-manifolds (Ω, g) satisfying:

- $\partial\Omega$, with the induced metric, is isometric to the disjoint union of (Σ_j, γ_j), $j = 1, \ldots, k$,
- $H > 0$, where H is the mean curvature of $\partial\Omega$, and
- $R(g) \geq 0$, where $R(g)$ is the scalar curvature of g.

Given $\mathcal{F} = \mathcal{F}_{(\Sigma_1,\gamma_1),\ldots,(\Sigma_k,\gamma_k)}$, let

$$\Lambda_{(\Sigma_1,\gamma_1),\ldots,(\Sigma_k,\gamma_k)} := \sup \left\{ \frac{1}{8\pi} \int_{\partial\Omega} H d\sigma \mid (\Omega, g) \in \mathcal{F} \right\}. \qquad (6)$$

If $\mathcal{F} = \emptyset$, $\Lambda_{(\Sigma_1,\gamma_1),\ldots,(\Sigma_k,\gamma_k)} := -\infty$.

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In this notation, the Shi-Tam theorem can be rephrased as:

"Suppose Σ is a 2-sphere and $\gamma, \gamma_1, \ldots, \gamma_k$ are metrics on Σ with $K > 0$, then

(I) $\Lambda_{(\Sigma,\gamma)} = \dfrac{1}{8\pi} \displaystyle\int_\Sigma H_0 d\sigma$;

(II) $\displaystyle\int_{\Sigma_j} H\, d\sigma \le 8\pi\Lambda_{(\Sigma,\gamma_j)},\ \forall\ (\Omega, g) \in \mathcal{F}_{(\Sigma,\gamma_1),\ldots,(\Sigma,\gamma_k)}$."

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Theorem (Mantoulidis - M)

Let $\Sigma_1, \ldots, \Sigma_k$ be $k \geq 1$ closed, connected, orientable surfaces endowed with metrics $\gamma_1, \ldots, \gamma_k$. Given $(\Omega, g) \in \mathcal{F}_{(\Sigma_1, \gamma_1), \ldots, (\Sigma_k, \gamma_k)}$, one has

$$\int_{\Sigma_j} H \, d\sigma \leq 8\pi \Lambda_{(\Sigma_j, \gamma_j)}, \quad \forall\, j = 1, \ldots, k. \tag{7}$$

Moreover, equality holds for some j only if $k = 1$ and (Ω, g) is isometric to a mean-convex handlebody with flat interior whose genus is that of Σ_1. In particular, if $\operatorname{genus}(\Sigma_1) = 0$ then (Ω, g) is a flat 3-ball.

Moreover, one can show that the functional $\Lambda_{(\Sigma_1,\gamma_1),\ldots,(\Sigma_k,\gamma_k)}$ satisfies an additivity property.

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Theorem (Mantoulidis - M)

Let $\Sigma_1, \ldots, \Sigma_k$ be $k \geq 2$ closed, connected, orientable surfaces endowed with metrics $\gamma_1, \ldots, \gamma_k$. One has

$$\Lambda_{(\Sigma_1,\gamma_1),\ldots,(\Sigma_k,\gamma_k)} = \sum_{j=1}^{k} \Lambda_{(\Sigma_j,\gamma_j)}, \tag{8}$$

provided each set $\mathcal{F}_{(\Sigma_j,\gamma_j)}$, $j = 1, \ldots, k$, is nonempty.

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provided each set $\mathcal{F}_{(\Sigma_j,\gamma_j)}$, $j = 1, \ldots, k$, is nonempty.

In the course of the proof, it is shown that $\mathcal{F}_{(\Sigma_1,\gamma_1),\ldots,(\Sigma_k,\gamma_k)} = \emptyset$ if and only if $\mathcal{F}_{(\Sigma_j,\gamma_j)} = \emptyset$ for some j.

Proof of "$\Lambda_{(\Sigma,\gamma)} < \infty$" when Σ is a 2-sphere

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Theorem (Wang - Yau, 06)

Let (Ω, g) *be a compact 3-manifold with scalar curvature* $R \geq -6\kappa^2$ *for some* $\kappa > 0$. *Suppose its boundary* Σ *is a topological 2-sphere which has Gauss curvature* $K > -\kappa^2$ *and positive mean curvature* H. *Then there exists a future-directed time-like vector-valued function* $W^0 : \Sigma \to \mathbb{R}^{3,1}$, *which depends on* H *and the embedding of* Σ *into* $\mathbb{H}^3_{-\kappa^2} \subset \mathbb{R}^{3,1}$ *such that*

$$\int_\Sigma (H_0 - H) W^0 d\sigma \tag{9}$$

is a future-directed non-spacelike vector. Here H_0 *is the mean curvature of the isometric embedding of* Σ *in* $\mathbb{H}^3_{-\kappa^2}$.

Shi and Tam later found that W^0 in Wang-Yau's theorem can be taken as $W^0 = (x_1, x_2, x_3, \alpha t)$ for some $\alpha > 1$ depending only on (Σ, γ).

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$$\int_\Sigma H \cosh \kappa r \ d\sigma \leq \int_\Sigma H_0 \cosh \kappa r \ d\sigma, \tag{10}$$

where H_0 is the mean curvature of the isometric embedding of Σ in $\mathbb{H}^3_{-\kappa^2}$ and $r(\cdot)$ denotes the distance to any fixed point in the interior of the image of Σ in $\mathbb{H}^3_{-\kappa^2}$. Moreover, equality in (10) holds if and only if (Ω, g) is isometric to a convex domain in $\mathbb{H}^3_{-\kappa^2}$.

Some open questions
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- If γ is a metric on a higher genus surface Σ, is it true

$$\Lambda_{(\Sigma,\gamma)} < \infty \ ?$$

- If $\Lambda_{(\Sigma,\gamma)} < \infty$ and if (Σ,γ) isometrically embeds in $\mathbb{R}^3$, does

$$\Lambda_{(\Sigma,\gamma)} = \frac{1}{8\pi} \int_\Sigma H_0 d\sigma \ ?$$