On positive mass conjecture in the asymptotically hyperbolic setting

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Mass in Riemannian Geometry

Suppose that (M, g) is asymptotic to a model space (M_0, g_0) at infinity. Under which minimal geometric assumptions can we tell them apart?

A model space is e.g. the Euclidean space or the hyperbolic space.

Intuitively, mass is an invariant 'at infinity' which measures the difference.

ADM mass

Let $(\mathbb{R}^n, \delta)$ be the Euclidean space.

We say that (M^n, g) is **asymptotically Euclidean** if there is a compact $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B_R}$ such that $e := \Phi_* g - \delta = O\left(|x|^{-\frac{n-2}{2}-\varepsilon}\right)$ for $\varepsilon > 0$.

If $\mathrm{Scal} \in L^1$ then **ADM mass**

$$m_{ADM} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \left(\operatorname{div}^\delta e - d(\operatorname{tr}^\delta e)\right)(\nu_r)\, d\mu^\delta$$

is a well-defined invariant.

Positive mass theorem

Theorem (Schoen&Yau 1979, Witten 1981)

Suppose a complete AE manifold (M^n, g) has $\mathrm{Scal}^g \geq 0$. If either $3 \leq n \leq 7$ or if (M, g) is spin then $m_{ADM} \geq 0$. Moreover, $m_{ADM} = 0$ iff (M, g) is isometric to $(\mathbb{R}^n, \delta)$.

Conformally compact asymptotically hyperbolic manifolds

Conformally compact approach is motivated by AdS/CFT. The prototype is ball model of hyperbolic space:

$$(B_1(0), \phi^{-2}\delta), \text{ where } \phi = \tfrac{1-|x|^2}{2}.$$

X. Wang 2001: (M^n, g) is **asymptotically hyperbolic** if it is conformally compact with round sphere (S^{n-1}, σ) at infinity and near conformal infinity we have an expansion

$$g = \sinh^{-2}\rho \left(d\rho^2 + \sigma + \frac{\rho^n}{n}\mathbf{m} + O(\rho^{n+1}) \right).$$

Here $\mathbf{m}$ is a symmetric 2-tensor on S^{n-1}, the so-called **mass aspect tensor**, its trace $\operatorname{tr}^\sigma \mathbf{m}$ is the so-called **mass aspect function**.

Conformally compact asymptotically hyperbolic manifolds

Define the **mass vector** $\vec{P} = (P_0, P_1, \dots, P_n)$ by

$$P_0 = \int_{S^{n-1}} \mathrm{tr}^\sigma \mathbf{m} \, d\mu^\sigma,$$

$$P_i = \int_{S^{n-1}} x^i \, \mathrm{tr}^\sigma \mathbf{m} \, d\mu^\sigma, \qquad i = 1, 2, \dots, n.$$

Theorem (X. Wang 2001)

The Minkowskian length of $\vec{P}$ is an invariant. If (M^n, g) is complete and spin with $\mathrm{Scal}^g \geq -n(n-1)$ then $\vec{P}$ is timelike future directed, i.e.

$$P_0 > \sqrt{\sum_{i=1}^{n} P_i^2}$$

unless $(M, g) \cong (\mathbb{H}^n, b)$.

Chart-dependent asymptotically hyperbolic manifolds

More generally, let $(\mathbb{H}^n, g_{hyp}) = (\mathbb{R} \times S^{n-1}, \frac{dr^2}{1+r^2} + r^2\sigma)$.

(M, g) is **asymptotically hyperbolic** if there is a compact $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to \mathbb{H}^n \setminus \overline{B_R}$ such that $e := \Phi_* g - g_{hyp} = O(r^{-\frac{n}{2}-\varepsilon})$ for $\varepsilon > 0$.

Define $\vec{P} = (H_\Phi(\sqrt{1+r^2}), H_\Phi(x^1), \ldots, H_\Phi(x^n))$, where

$$H_\Phi(V) = \lim_{r\to\infty} \int_{S_r} \left(V(\operatorname{div} e - d\operatorname{tr} e) + (\operatorname{tr} e)dV - e(\nabla V, \cdot)\right)(\nu_r)\, d\mu.$$

Theorem (Chrusciel-Herzlich 2003)

If $r\,\mathrm{Scal} \in L^1$ then Minkowskian length of $\vec{P}$ is a well-defined invariant. If (M^n, g) is complete and spin with $\mathrm{Scal} \geq -n(n-1)$ then $\vec{P}$ is timelike future directed unless $(M, g) \cong (\mathbb{H}^n, g_{hyp})$.

Can spin assumption be removed?

Theorem (Andersson-Cai-Galloway 2008)

Let (M^n, g), $3 \leq n \leq 7$, be a complete asymptotically hyperbolic manifold in the sense of X. Wang with $\mathrm{Scal}^g \geq -n(n-1)$. Then the mass aspect function cannot be negative, and if it is zero, then $(M, g) \cong (\mathbb{H}^n, b)$.

That is, positive mass theorem holds if mass aspect function **has sign**.

But apparently we cannot force mass aspect function to have sign through a choice of chart at infinity...

Non-spinor proof for general asymptotics?

Given an asymptotically hyperbolic (M, g) with $\mathrm{Scal}^g \geq -n(n-1)$ and mass m, can we deform it to an asymptotically Euclidean manifold $(\bar{M}, \bar{g})$ with $\mathrm{Scal}^{\bar{g}} \geq 0$ and mass $\bar{m} \leq m$?

Then $m \geq 0$ would follow by positive mass theorem for asymptotically Euclidean manifolds.

Spacetime positive mass theorem

Idea is borrowed from the proof of

Theorem (Schoen-Yau 1981, Eichmair 2012)

Let (M^n, g, k), $3 \leq n \leq 7$, be a complete asymptotically Euclidean initial data set satisfying the dominant energy condition. Then the ADM mass of (M, g) is positive unless (M, g, k) is a slice of Minkowski spacetime.

Dominant energy condition: $\mu \geq |J|_g$, where

$$2\mu = \mathrm{Scal}^g + |k|_g^2 - (\mathrm{tr}^g\, k)^2,$$
$$J = \mathrm{div}^g(k - (\mathrm{tr}^g\, k)g).$$

Important observation Schoen and Yau, and Jang

Given initial data (M, g, k) satisfying $\mu \geq |J|_g$ consider a graph of $f : M \to \mathbb{R}$ in $(M \times \mathbb{R}, g + dt^2)$. Set $k(\cdot, \partial_t)$ =0.

It turns out that if f is chosen to satisfy the **Jang equation**

$$\left(g^{ij} - \frac{\nabla^i f \nabla^j f}{1 + |\nabla f|^2}\right)\left(\frac{\nabla_i \nabla_j f}{\sqrt{1 + |\nabla f|^2}} - k_{ij}\right) = 0,$$

then $(\bar{M}, \bar{g}) = \mathrm{graph}\, f$ has

$$\mathrm{Scal}^{\bar{g}} \geq 2|q|^2_{\bar{g}} - \mathrm{div}^{\bar{g}} q,$$

for a 1-form q. This is a consequence of **Schoen and Yau's identity**.

Asymptotically hyperbolic initial data

Prototype: hyperboloid $t = \sqrt{1 + r^2}$ in Minkowski, umbilic $k = g = g_{hyp}$.

For the mass vector to be well-defined, the following fall-off is required:

- $\Phi_* g - g_{hyp} = O\left(r^{-\frac{3}{2}-\varepsilon}\right)$
- $\Phi_*(k - g) = O\left(r^{-\frac{3}{2}-\varepsilon}\right)$
- $r\Phi_*(|\mu| + |J|_g) \in L^1$

Mass vector of AH initial data set

Mass vector of (M, g, k) is

$$\vec{P} = (H_\Phi(\sqrt{1+r^2}), H_\Phi(x^1), \dots, H_\Phi(x^n))$$

where

$$H_\Phi(V) = \lim_{r\to\infty} \int_{S_r} \left(V(\operatorname{div} e - d \operatorname{tr} e) + (\operatorname{tr} e) dV - (e + 2\eta)(\nabla V, \cdot)\right)(\nu_r)\, dA,$$

where $e = \Phi_* g - g_{hyp}$, $\eta = \Phi_*(k - g)$.

This is Trautman-Bondi mass of Chruściel et al 2004.

Positive mass theorem for AH initial data

Theorem (S)

Let (M^3, g, k) be an asymptotically hyperbolic initial data set with general asymptotics satisfying $\mu \geq |J|_g$. Then the mass vector of (M, g, k) is non-spacelike future directed, i.e $P_0 \geq \sqrt{\sum_{i=1}^3 P_i^2}$.

In particular, if (M^3, g) is an asymptotically hyperbolic manifold with $\mathrm{Scal} \geq -6$ the mass vector of (M, g) is non-spacelike future directed.

Proof of main theorem

- It suffices to show that the first component of the mass vector is nonnegative.
- By **density theorem** [Dahl and S. 2015] it suffices to assume that initial data has simple (Wang's style) asymptotics and that $\mu \geq (1+\gamma)|J|_g$ for $\gamma > 0$.

In particular,

$$g = \frac{dr^2}{1+r^2} + r^2(\sigma + \mathbf{m} r^{-3} + o(r^{-3})),$$

$$P_0 = \tfrac{1}{16\pi} \int_{S^2} \mathrm{tr}_\sigma(\mathbf{m} + 2\mathbf{k}) d\mu^\sigma,$$

where $\mathbf{k}$ comes from the expansion of the 'angular' components of k:

$$k_{\mu\nu} = \mathbf{k}_{\mu\nu} r^{-1} + o(r^{-1}).$$

Proof of main theorem

- The solution of the Jang equation on hyperboloid in Minkowski is $f = \sqrt{1 + r^2}$. Inspection shows that in general the solution expands at infinity as
$$f = \sqrt{1 + r^2} + 2P_0 \ln r + \psi + o(1),$$
where $\psi : S^2 \to \mathbb{R}$ is determined by the mass aspect function.
- In this case $g + df \otimes df$ has asymptotically Euclidean end with $m_{ADM} = 2P_0$, Schoen and Yau's identity shows that we 'almost have' $\mathrm{Scal} \geq 0$.
- So we just need to make it more precise... in particular solve the Jang equation with the above asymptotics and make sure that the positive mass theorem for AE manifolds does apply on its graph.

Solving the Jang equation: barriers

Barriers are super-/subsolutions $f_\pm$ on $\{r \geq R\}$ with $\partial_r f_\pm|_{\{r=R\}} = \mp\infty$.

Complicated asymptotics makes it difficult to find them by inspection.

Here we rely on the fact that we are 'almost' in spherical symmetry because of Wang's asymptotics.

In spherical symmetry there is a substitution which allows to rewrite the Jang equation as a Riccati type ODE.

We use this substitution to obtain equations for barriers.

Geometric solution

The **geometric solution** is constructed by taking the limit of graphs of solutions of regularized BVP

$$\begin{aligned} J(f_n) = \tau_n f_n \quad & \text{in} \quad \{r \leq R_n\} \\ f_n = \phi \quad & \text{on} \quad \{r = R_n\} \end{aligned}$$

where $\tau_n \to 0$, $R_n \to \infty$ as $n \to \infty$. Here $J(f)$ is the LHS of the Jang equation, $f_- < \phi < f_+$.

It is not an entire graph, but a union of graphical and cylindrical components. May have **asymptotically cylindrical ends**!

Asymptotically Euclidean structure of the end

From this construction, we only know that near infinity $f_{-} < f < f_{+}$.

Estimates for derivatives of $f : M \to \mathbb{R}$ required for showing that $g + df \otimes df$ is asymptotically Euclidean as $r \to \infty$ are not obvious.

In particular, the rescaling technique does not work on asymptotically hyperbolic manifolds.

However, it does work on asymptotically Euclidean manifolds and the graph of the lower barrier is asymptotically Euclidean.

Writing the Jang equation $H_{\bar{M}} = \operatorname{tr}_{\bar{M}} k$ in terms of the distance function to the lower barrier yields the required estimates.

If there are no asymptotically cylindrical ends...

Then using Schoen-Yau's identity one shows that there exists $\varphi > 0$ such that

- $\hat{g} = \varphi^4 \bar{g}$ is asymptotically Euclidean
- $\mathrm{Scal}^{\hat{g}} = 0$
- $m^{\hat{g}} \leq \frac{1}{2} m^{\bar{g}} = P_0$.

Hence $P_0 \geq 0$ by positive mass theorem for asymptotically Euclidean manifolds.

If there are asymptotically cylindrical ends...

At this step it is important to have $\mu > |J|_g$ near the surface $\Sigma \subset M$ where the asymptotically cylindrical blow up occurs.

Let φ_1 be the solution of

$$-\Delta^{\bar{g}}\varphi + \frac{1}{8}\mathrm{Scal}^{\bar{g}}\varphi = 0$$

such that $\varphi_1 \to 1$ in the AE end and $\varphi_1 \to 0$ in AC end. For any $\varepsilon > 0$ there is another solution $\varphi_2 > 0$ such that $\varphi_2 \to 1$ in the AE end, $\varphi_2 \to \infty$ in AC end and $|m^{\varphi_2^4 \bar{g}} - m^{\varphi_1^4 \bar{g}}| < \varepsilon$. Since

$$m^{\varphi_1^4 \bar{g}} \leq \tfrac{1}{2} m^{\bar{g}} = P_0$$

and $m^{\varphi_2^4 \bar{g}} \geq 0$ it follows that $P_0 \geq 0$.

Rigidity

This method only allows to prove

Theorem (S)

Let (M^3, g, k) be asymptotically hyperbolic initial data with Wang's asymptotics satisfying $\mu \geq |J|_g$. If $P_0 = 0$ then (M, g, k) is a slice of Minkowski spacetime.

The argument will not directly apply to general asymptotics, because barriers are not available.

By analogy with asymptotically Euclidean setting (Chrusciel-Maerten 2005) one would expect that initial data embeds in Minkowski spacetime if merely its mass vector is null. However, so far all available proofs (Maerten 2006) require $P_0 = 0$ as well.

Other applications

Solution of the Jang equation w.r.t. asymptotically hyperbolic initial data can be used for other purposes, e.g.

- Reduction arguments for other geometric inequalities (Cha-Khuri-S 2015)
- Initial data topological censorship (cf. Andersson-Dahl-Galloway-Pollack 2015)

Thank you!