Towards quantum chiral algebras

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Outline

Overview

In 1988 I. Frenkel and N. Jing published the seminal paper titled "Vertex Representations of quantum affine algebras".
Recall that there is the Drinfeld realization of the quantum affine algebras, which parallels the current realization of the affine Lie algebras in the form $\hat{\mathfrak{g}} \simeq \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$.
It is given in terms of generators $\gamma^{1/2}, \gamma^{-1/2}$ and generating series

$$x_i^{\pm}(z) = \sum_{k\in\mathbb{Z}} x_{ik}^{\pm} z^{-k}; \quad \phi_i(z) = \sum_{m\in-\mathbb{Z}_+} \phi_{im} z^{-m}; \quad \psi_i(z) = \sum_{n\in\mathbb{Z}_+} \psi_{in} z^{-n};$$

modulo relations, including:

$$(z - q^{\pm A_{ij}} w) x_i^{\pm}(z) x_j^{\pm}(w) = (q^{\pm A_{ij}} z - w) x_j^{\pm}(w) x_i^{\pm}(z)$$

Frenkel and Jing then constructed the vertex representations, starting with the quantum Heisenberg operators $a_{ik}, k \in \mathbb{Z}$ satisfying the relations ($\gamma = e^{tc/2} = q^c$)

$$[a_i(m), a_i(n)] = \delta_{i,-m}\frac{1}{mt^2}(q^{mA_{ij}} - q^{-mA_{ij}})(q^m - q^{-m})$$

and defining the vertex operators

$$Y_i^{\pm}(z) = exp\Big(\pm t\sum_{n\geq 1}\frac{q^{\mp\frac{n}{2}}}{q^n - q^{-n}}a_i(-n)z^n\Big)$$
$$\times\, exp\Big(\mp t\sum_{n\geq 1}\frac{q^{\mp\frac{n}{2}}}{q^n - q^{-n}}a_i(n)z^{-n}\Big)a_i^{\pm 1}z^{\pm a_i(0)+1}.$$

These vertex operators (together with $\Phi_i(z)$ and $\Psi_i(z)$) define a representation of the quantum affine algebras at $c = 1$.

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Recall, in a chiral algebra (super vertex algebra), we have the following OPE formula for any two fields $a(z), b(w)$ in a super vertex algebra:

$$a(z)b(w) = \sum_{j=0}^{N-1} i_{z,w}\frac{c^j(w)}{(z-w)^{j+1}} + :a(z)b(w):,$$

where $:a(z)b(w):$ denotes the nonsingular part of the expansion of $a(z)b(w)$ as a Laurent series in $(z-w)$, and is referred to as *normal ordered product* of $a(z)$ and $b(z)$.
Moreover, $Res_{(z-w)}a(z)b(w)(z-w)^j = c^j(w) = (a_{(j)}b)(w)$.

This in particular implies that the coefficients in the OPEs are vertex operators in the **same** super vertex algebra, which property is referred to as *completeness with respect to OPEs*. The term "chiral algebra" will refer to the fact that we require completeness with respect to the OPEs.

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For example, in the quantum affine case (FJ):

$$Y_i^{\mp}(z)Y_i^{\pm}(w) = (1-wq/z)^{-1}(1-w/qz)^{-1}(w/z) : Y_i^{\pm}(z)Y_i^{\mp}(w) :$$

and so OPEs such as

$$Y_i^{\mp}(z)Y_i^{\pm}(w) \sim \frac{q^2}{q^2-1}\frac{w : Y_i^{\pm}(qw)Y_i^{\mp}(w) :}{z-qw} - \frac{w : Y_i^{\pm}(q^{-1}w)Y_i^{\mp}(w) :}{(q^2-1)(z-q^{-1}w)}$$

In any chiral algebra (vertex algebra) one needs to address the question of "the descendent fields", i.e., given two fields $a(z)$, $b(z)$ that are "in" the vertex algebra V (i.e. $a(z) = Y(a, z)$ for some $a \in V$, and $b(z) = Y(b, z)$ for some $b \in V$), which of fields "descending" from $a(z)$ and $b(z)$ do we also want to be "in" the vertex algebra V.

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These field descendants are addressed in the classical vertex algebra case by Dong's Lemma, where the goal was to prove that these type of descendants are mutually local.

This reflects one of the approaches in constructing vertex algebras, (H. Li): consider sets of fields which are compatible (local, quasi-local, *S*-local, etc..) in an a priory defined way, and the resulting types of quantum vertex algebras and modules, quasi-modules, ϕ-coordinated modules, ϕ-coordinated quasi-modules, etc...

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The (A, H, S) structure deals with the singular product, and thus is responsible for the singularities in the product of fields.

Note: the (A, H, S) structure on its own doesn't produce fields, one needs auxiliary maps: in general, a projection map, an exponential map and evaluation map (producing the expansions, i.e., the fields, and the coordination between the expansions and the Hopf algebra H action).

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$$Y(a, z)1 = e^{zD}a, \quad \text{where} \quad e^{zD} = \sum_{n\geq 0} z^n D^{(n)}.$$

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Practically though, this means we have to address two types of operations on field descendants:

- OPE coefficients and normal ordered products
- Hopf algebra action

Consequences from completeness with respect to OPEs

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This has profound consequences: in particular if both the fields $a(z)$ and $a(\gamma z)$ are to be incorporated in the **same chiral algebra structure**, this would result in the the state-field correspondence becoming non-invertible!

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$V \ni \tilde{a}(z) \mapsto a := \tilde{a}(z)|0\rangle|_{z=0} \in W$.
If both the fields $a(z)$ and $a(\gamma z)$ have to be incorporated in the same chiral algebra , then the field-state correspondence map will send the different fields $a(z)$ and $a(\gamma z)$ to the same state element $a \in W$:

$$a(\gamma z)|0\rangle|_{z=0} = a(z)|0\rangle|_{z=0} = a \in W.$$

Thus the space of fields V will be a (ramified) cover of the space of states W.

Since we are already considering descendant fields of the type $T_\gamma a(z) := a(\gamma z)$, where T_γ is group-like, we need to consider Hopf algebras with group-like subalgebras Γ acting on the fields. The simplest example of such is the Hopf algebra $H_{D,\Gamma}$ with a primitive generator D and grouplike elements T_γ corresponding to each element $\gamma \in \Gamma$, subject to the relations:

$$DT_\gamma = \gamma T_\gamma D,$$

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$\mathbf{F}^\Gamma(z, w)(z, w)^{+,w}$: nonsingular at $w = 0$.

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Chiral algebra with Γ-type singularities is a collection of the following data $(V, W, Y, \pi_s, \Gamma, R)$:

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- a group of singularities Γ;
- braiding-map R, dictating braided commutativity.

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2. shift conditions with respect to the Hopf algebra ($H_{D,\Gamma}$):

$$\triangle(h)_{1,z}R(z,w)=R(z,w)\circ(h\otimes 1),$$
$$\triangle(h)_{2,w}R(z,w)=R(z,w)\circ(1\otimes h),$$

where $\triangle(h)_{1,z}=(1\otimes\tau\otimes 1)\circ(\triangle(h)\otimes 1\otimes 1)$ and similarly for $\triangle(h)_{2,w}$, the third and fourth factors act on $\mathbf{F}^{\Gamma}(z,w)$.

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3. unitarity condition

$$\tau\circ R(w,z)\circ\tau=R(z,w)^{-1}, \text{where } \tau(a\otimes b)=b\otimes a.$$

Satisfying the following set of axioms:

- vacuum axiom: $Y(|0\rangle, z) = Id_W$;
- modified creation axiom (field-state correspondence): $Y(a, z)|0\rangle|_{z=0} = \pi_s(a)$, for any $a \in V$;

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- transfer of action: $Y(ha, z) = h_z \cdot Y(a, z)$ for any $h \in H_{D,\Gamma}$;
- analytic continuations of arbitrary products exist: $Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_k, z_k)1$ converges in the domain $|z_1| \gg \dots \gg |z_k|$ and can be continued to a meromorphic vector valued function

$$X_{z_1,z_2,\dots,z_k}(a_1 \otimes a_2 \otimes \dots \otimes a_k) : V^{\otimes k} \to W \otimes \mathbf{F}^{\Gamma}(z_1, z_2, \dots, z_k)^{+,z_k}, \tag{1}$$

so that $Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_k, z_k)|0\rangle = i_{z_1,z_2,\dots,z_k} X_{z_1,z_2,\dots,z_k}(a_1 \otimes a_2 \otimes \dots \otimes a_k)$

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  $X_{w,z,0}(b \otimes a \otimes c) = X_{z,w,0}(R_{z,w}(a \otimes b) \otimes c)$;
- completeness with respect to Operator Product Expansions (OPE's): For each $\gamma \in \Gamma$, $k \in \mathbb{Z}$, any $a, b, c \in V$, a, b–homogeneous with respect to the grading by D, we have

$$Res_{z=\gamma w} X_{z,w,0}(a \otimes b \otimes c)(z-\gamma w)^k = \sum_{s}^{\text{finite}} w^{l^s_{k,i}} Y(v^s_{k,i}, w)\pi_s(c) \tag{2}$$

for some homogeneous elements
$v^s_{k,i} \in V$, $l^s_{k,i} \in \mathbb{Z}$, $|l^s_{k,i}| < min(k, |\Gamma|)$.

Classes of examples

- A chiral algebra of type $(V, V, \pi_f = Id_V, Y, \Gamma = \{1\}, R(a\otimes b) = (-1)^{p(a)p(b)} a\otimes b)$ is a super vertex algebra.

Classes of examples

- A chiral algebra of type $(V, V, \pi_f = Id_V, Y, \Gamma = \{1\}, R(a\otimes b) = (-1)^{p(a)p(b)} a\otimes b)$ is a super vertex algebra.
- The chiral algebra of type $(V, W, \pi_f, Y, \Gamma = \{1, \epsilon, \dots, \epsilon^{N-1}\}, R(a\otimes b) = (-1)^{p(a)p(b)} a\otimes b)$. We call this particular subclass of chiral algebras ”twisted vertex algebras” (IA). They represent the ”baby” examples of what happens when the OPE singularities form a nontrivial group. Nevertheless they are interesting because of the boson-fermion correspondences of types B, C and D, which are all isomorphisms of twisted vertex algebras.

- The deformed chiral algebras (including the deformed Virasoro chiral algebra) of Reshetikhin and E. Frenkel are chiral algebras with Γ-type singularities, where Γ is an integer lattice, and the braiding map is non-trivial.

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- The vertex representations of the quantum affine algebras can be "described" by chiral algebras with Γ-type singularities, where Γ is an integer lattice, and the braiding map is non-trivial.
  Although such a description based on $H_{D,\Gamma}$ is possible, we believe the Hopf algebra $H_{D,\Gamma}$ should be replaced with H_{D_q,Γ_q}.

Bosonization and boson-fermion correspondences

Bosonization is the representation of given chiral fields (Fermi or Bose) via bosonic fields.

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aka the bosonization of the charged free fermions.
Fermionization refers to the representation of given fields in terms of fermionic fields.
Boson-fermion correspondences are equivalences between two chiral field theories (only in two dimensions): one bosonic and one fermionic. I.e., a boson-fermion correspondence is (should be) an isomorphism of (twisted, quantum) chiral algebras.

Bosonization and boson-fermion correspondences: why do we care?

Applications to many areas, besides chiral algebra theory:

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Applications to many areas, besides chiral algebra theory:

- representation theory
- integrable systems
- symmetric polynomials
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- many others

The double infinite-rank algebra a_∞ and the boson-fermion correspondence of type A

The Lie algebra a_∞.

The Lie algebra $\bar{a}_\infty$ is the Lie algebra of infinite matrices

$$\bar{a}_\infty = \{(a_{ij})|\ i,j \in \mathbb{Z},\ a_{ij} = 0 \text{ for} |i-j| \gg 0\}. \tag{3}$$

As usual denote the elementary matrices by E_{ij}.

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As usual denote the elementary matrices by E_{ij}.
The algebra a_∞ is a central extension of $\bar{a}_\infty$ by a central element c, $a_\infty = \bar{a}_\infty \oplus \mathbb{C}c$, with cocycle given by

$$C(A, B) = Trace([J, A]B), \qquad (4)$$

where the matrix $J = \sum_{i\leq 0} E_{ii}$. In particular

$$C(E_{ij}, E_{ji}) = -C(E_{ji}, E_{ij}) = 1, \quad \text{if } \ i \leq 0, \ \ j \geq 1$$
$$C(E_{ij}, E_{kl}) = 0 \quad \text{in all other cases.}$$

The commutation relations in a_∞ are

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj} + C(E_{ij}, E_{kl})c.$$

We can arrange the non-central generators in a generating series

$$E^A(z, w) = \sum_{i,j\in\mathbb{Z}} E_{i,j} z^{i-1} w^{-j}. \quad (5)$$

The generating series $E^A(z,w)$ obeys the relations

$$E^A(z,w) = -E^A(w,z)$$

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$$[E^A(z_1, w_1), E^A(z_2, w_2)]$$
$$= E^A(z_1, w_2)\delta(z_2 - w_1) - E^A(z_2, w_1)\delta(z_1 - w_2)$$
$$+ \iota_{z_1,w_2}\frac{1}{z_1 - w_2}\iota_{w_1,z_2}\frac{1}{w_1 - z_2}c - \iota_{z_2,w_1}\frac{1}{z_2 - w_1}\iota_{w_2,z_1}\frac{1}{w_2 - z_1}c$$

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Notation:

$$i_{z,w}\frac{1}{z-w} = \sum_{n\geq 0}\frac{w^n}{z^{n+1}};$$

$$\delta(z-w) = i_{z,w}\frac{1}{z-w} - i_{w,z}\frac{1}{z-w} = \sum_{n\in\mathbb{Z}}\frac{w^n}{z^{n+1}}$$

"Factorization problem"

Question: Can we "factorize" the generating series $E^A(z,w)$, i.e., write

$$E^A(z,w) = :\psi(z)\psi(w):$$

i.e., write the two-variable series $E^A(z,w)$ as a "normal ordered product" of a single-variable generating series $\psi(z)$.

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Answer: not quite, but we can write

$$E^A(z, w) =: \psi^+(z)\psi^-(w) :$$

From

$$E^A(z,w) = -E^A(w,z)$$

we can see that the single variable generating series $\psi(z)$ are "fermionic", i.e., they obey anti-commutation relations:

$$:\psi(z)\psi(w): = -:\psi(w)\psi(z):$$

Since we have two such, $\psi^+(z), \psi^-(z)$, each one is fermionic.

We can then read the anti-commutation relations for the generating series $\psi^+(z), \psi^-(z)$ from the commutation relations for the two-variable series $E^A(z,w)$:

$$\{\psi^+(z), \psi^+(w)\} = 0, \quad \{\psi^+(z), \psi^+(w)\} = 0,$$

$$\{\psi^+(z), \psi^-(w)\} = \psi^+(z)\psi^-(w) + \psi^-(w)\psi^+(z)$$

$$= i_{z,w}\frac{1}{z-w} + i_{w,z}\frac{1}{w-z} = \delta(w-z)$$

We write this as an Operator Product Expansion (OPE):

$$\psi^+(z)\psi^-(w) \sim \frac{1}{z-w}$$

The three stages of bosonization

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- Construct a (bosonic) Heisenberg (twisted or untwisted) field descendant (this is often a fermionization);

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- Construct a (bosonic) Heisenberg (twisted or untwisted) field descendant (this is often a fermionization);
- Decompose the Fock space (the space of states of your chiral algebra) into irreducible Heisenberg modules;
- Write the original (generating) fields in terms of (exponential) bosonic fields (lattice vertex operators)

Charged free fermion–boson correspondence; a.k.a. type A (I. Frenkel, Date/Jimbo/Kashiwara/Miwa, ..)

The fermion side of the boson-fermion correspondence of type A is generated by the two nontrivial odd fields—two charged fermions: the fields $\psi^+(z)$ and $\psi^-(z)$ with only nontrivial operator product expansion (OPE):

$$\psi^+(z)\psi^-(w) \sim \frac{1}{z-w} \sim \psi^-(z)\psi^+(w), \tag{6}$$

where the 1 above denotes the identity map Id.

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$$\psi^+(z)\psi^-(w) \sim \frac{1}{z-w} \sim \psi^-(z)\psi^+(w), \tag{6}$$

where the 1 above denotes the identity map Id.
The fields $\psi^+(z)$ and $\psi^-(z)$ are indexed as

$$\psi^+(z) = \sum_{n\in\mathbf{Z}} \phi_n^+ z^{-n-1}, \quad \psi^-(z) = \sum_{n\in\mathbf{Z}} \psi_n^- z^{-n-1}, \tag{7}$$

The modes of the fields $\psi^+(z)$ and $\psi^-(z)$ generate a Clifford algebra Cl_A with relations

$$\{\psi^+_m, \psi^-_n\} = \delta_{m+n,-1}1, \quad \{\psi^+_m, \psi^+_n\} = \{\psi^+_m, \psi^+_n\} = 0. \qquad (8)$$

This Clifford algebra has a canonical Fock space representation F_A—the fermionic Fock space — which is the highest weight representation of Cl_A generated by the vacuum vector $|0\rangle$, so that $\psi^+_n|0\rangle = \psi^-_n|0\rangle = 0$ for $n \geq 0$.

The boson-fermion correspondence of type A is determined by the images of the generating fields $\phi(z)$ and $\psi(z)$ under the correspondence. An essential ingredient is the free boson field $h(z)$ given by

$$h(z) =: \psi^+(z)\psi^-(z) : \tag{9}$$

This is, in fact, the **fermionization** of the free bosonic current.

It follows that the field $h(z)=\sum_{n\in\mathbb{Z}} h_n z^{-n-1}$ has OPEs with itself given by:

$$h(z)h(w)\sim \frac{1}{(z-w)^2}, \quad \text{in modes: } [h_m, h_n]=m\delta_{m+n,0}1. \tag{10}$$

and so is an untwisted Heisenberg field (i.e., its modes $h_n,\ n\in\mathbb{Z}$, generate a Heisenberg algebra $\mathcal{H}_{\mathbb{Z}}$). This completes the first stage of the bosonization.

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The second stage is accomplished by the decomposition

$$F_A\cong\oplus_{m\in\mathbb{Z}}B_m,\quad B_m\cong\mathbb{C}[x_1,x_2,\dots,x_n,\dots],\quad \forall\ m\in\mathbb{Z}$$

Now we can write the images of the generating fields $\phi(z)$ and $\psi(z)$ under the correspondence (the third stage):

$$\phi(z) \mapsto e_A^{\alpha}(z), \quad \psi(z) \mapsto e_A^{-\alpha}(z), \tag{11}$$

where the generating fields $e_A^{\alpha}(z)$, $e_A^{-\alpha}(z)$ for the bosonic part of the correspondence are given by

$$e_A^{\alpha}(z) = \exp(\sum_{n\geq 1} \frac{h_{-n}}{n} z^n) \exp(-\sum_{n\geq 1} \frac{h_n}{n} z^{-n}) e^{\alpha} z^{\partial_\alpha}, \tag{12}$$

$$e_A^{-\alpha}(z) = \exp(-\sum_{n\geq 1} \frac{h_{-n}}{n} z^n) \exp(\sum_{n\geq 1} \frac{h_n}{n} z^{-n}) e^{-\alpha} z^{-\partial_\alpha}, \tag{13}$$

The three algebras $\bar{b}_\infty$, $\bar{c}_\infty$ and $\bar{d}_\infty$ are all defined as subalgebras of $\bar{a}_\infty$, each preserving different bilinear form (Kac).

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Question: what are these boson-fermion correspondences, i.e., isomorphisms between what objects?

The answer is isomorphisms of twisted vertex algebras (twisted chiral algebras).

The infinite dimensional Lie algebra b_∞

The infinite dimensional Lie algebra $\bar{b}_\infty$ is the subalgebra of $\bar{a}_\infty$ consisting of the infinite matrices preserving the bilinear form $B(v_i, v_j) = (-1)^i \delta_{i,-j}$, i.e.,

$$\bar{b}_\infty = \{(a_{ij}) \in \bar{a}_\infty |\ a_{ij} = (-1)^{i+j-1} a_{-j,-i}\}. \tag{14}$$

Denote by b_∞ the central extension of $\bar{b}_\infty$ by a central element c, $b_\infty = \bar{b}_\infty \oplus \mathbb{C}c$, where we use C (from (4)) as a cocycle for b_∞, see (Kac).

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Denote by b_∞ the central extension of $\bar{b}_\infty$ by a central element c, $b_\infty = \bar{b}_\infty \oplus \mathbb{C}c$, where we use C (from (4)) as a cocycle for b_∞, see (Kac). The commutation relations are inherited

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj} + C(E_{ij}, E_{kl})c.$$

The generators for the algebra b_∞ can be written as:

$$\{(-1)^j E_{i,-j} - (-1)^i E_{j,-i},\ \ c \,|\, i,j \in \mathbb{Z}\}.$$

We can arrange the non-central generators in a generating series

$$E^B(z,w) = \sum_{i,j\in\mathbb{Z}} ((-1)^j E_{i,-j} - (-1)^i E_{j,-i}) z^{i-1} w^{-j}. \tag{15}$$

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$$[E^B(z_1,w_1),E^B(z_2,w_2)] \tag{16}$$

$$\begin{aligned}&=-z_2E^B(z_1,w_2)\delta(w_1+z_2)+w_2E^B(z_1,z_2)\delta(w_1+w_2)\\&+z_2E^B(w_1,w_2)z_1\delta(z_1+z_2)-w_2E^B(w_1,z_2)\delta(z_1+w_2)\\&+ci_{w_1,z_2}\frac{w_1-z_2}{w_1+z_2}i_{z_1,w_2}\frac{z_1-w_2}{z_1+w_2}-ci_{w_2,z_1}\frac{w_2-z_1}{z_1+w_2}i_{z_2,w_1}\frac{z_2-w_1}{z_2+w_1}\\&-ci_{w_1,w_2}\frac{w_1-w_2}{w_1+w_2}i_{z_1,z_2}\frac{z_1-z_2}{z_1+z_2}+ci_{w_2,w_1}\frac{w_2-w_1}{w_1+w_2}i_{z_2,z_1}\frac{z_2-z_1}{z_1+z_2}\end{aligned} \tag{17}$$

The infinite dimensional Lie algebra c_∞

The infinite dimensional Lie algebra $\bar{c}_\infty$ is the subalgebra of $\bar{a}_\infty$ consisting of the infinite matrices preserving the bilinear form $C(v_i, v_j) = (-1)^i\delta_{i,1-j}$, i.e.,

$$\bar{c}_\infty = \{(a_{ij}) \in \bar{a}_\infty |\ a_{ij} = (-1)^{i+j-1}a_{1-j,1-i}\}. \tag{18}$$

The algebra c_∞ is a central extension of $\bar{c}_\infty$ by a central element c, $c_\infty = \bar{c}_\infty \oplus \mathbb{C}c$, with C the same cocycle as for a_∞, (4) (see Kac, Wang). The commutation relations in c_∞ are inherited.
The generators for the algebra c_∞ can be written as:

$$\{(-1)^j E_{i,j} - (-1)^i E_{1-j,1-i},\ i,j \in \mathbb{Z}; \text{and}\ \ c\}.$$

We arrange the non-central generators in a generating series

$$E^C(z,w)=\sum_{i,j\in\mathbb{Z}}((-1)^jE_{ij}-(-1)^iE_{1-j,1-i})z^{i-1}w^{-j}. \qquad (19)$$

The generating series $E^C(z,w)$ obeys the relations:
$E^C(z,w)=+E^C(w,z)$

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The generating series $E^C(z,w)$ obeys the relations:
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$$\begin{aligned}
&[E^C(z_1,w_1), E^C(z_2,w_2)] \\
&= E^C(z_1,w_2)\delta(z_2+w_1) - E^C(z_2,w_1)\delta(z_1+w_2) \\
&\qquad - E^C(w_2,w_1)\delta(z_1+z_2) + E^C(z_1,z_2)\delta(w_2+w_1) \\
&+ 2\iota_{z_1,w_2}\frac{1}{z_1+w_2}\iota_{w_1,z_2}\frac{1}{w_1+z_2}c - 2\iota_{w_2,z_1}\frac{1}{w_2+z_1}\iota_{z_2,w_1}\frac{1}{z_2+w_1}c \\
&+ 2\iota_{z_1,z_2}\frac{1}{z_1+z_2}\iota_{w_1,w_2}\frac{1}{w_1+w_2}c - 2\iota_{z_2,z_1}\frac{1}{z_2+z_1}\iota_{w_2,w_1}\frac{1}{w_2+w_1}c.
\end{aligned}$$

The infinite dimensional Lie algebra d_∞

The infinite dimensional Lie algebra $\bar{d}_\infty$ is the subalgebra of $\bar{a}_\infty$ consisting of the infinite matrices preserving the bilinear form $D(v_i, v_j) = \delta_{i,1-j}$, i.e.,

$$\bar{d}_\infty = \{(a_{ij}) \in \bar{a}_\infty |\ a_{ij} = -a_{1-j,1-i}\}. \quad (20)$$

Denote by d_∞ the central extension of $\bar{d}_\infty$ by a central element c, $d_\infty = \bar{d}_\infty \oplus \mathbb{C}c$, with C (from (4)) as a cocycle for d_∞, see (Kac, Wang). The commutation relations for the elementary matrices in d_∞ are inherited.
The generators for the algebra d_∞ can be written as:

$$\{E_{i,j} - E_{1-j,1-i},\ i,j \in \mathbb{Z}; \text{and}\ \ c\}.$$

We can arrange the non-central generators in a generating series

$$E^D(z,w) = \sum_{i,j\in\mathbb{Z}} (E_{ij} - E_{1-j,1-i}) z^{i-1} w^{-j}. \tag{21}$$

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The generating series $E^D(z,w)$ obeys the relations:

$E^D(z,w)=-E^D(w,z)$

$$\begin{aligned}
&[E^D(z_1,w_1),E^D(z_2,w_2)]\\
&=E^D(z_1,w_2)\delta(z_2-w_1)-E^D(z_2,w_1)\delta(z_1-w_2)\\
&\qquad+E^D(w_2,w_1)\delta(z_1-z_2)-E^D(z_1,z_2)\delta(w_1-w_2)\\
&\quad+\iota_{z_1,w_2}\frac{1}{z_1-w_2}\iota_{w_1,z_2}\frac{1}{w_1-z_2}c-\iota_{z_2,w_1}\frac{1}{z_2-w_1}\iota_{w_2,z_1}\frac{1}{w_2-z_1}c\\
&\quad-\iota_{z_1,z_2}\frac{1}{z_1-z_2}\iota_{w_1,w_2}\frac{1}{w_1-w_2}c+\iota_{z_2,z_1}\frac{1}{z_1-z_2}\iota_{w_2,w_1}\frac{1}{w_2-w_1}c.
\end{aligned}$$

The twisted neutral fermion–boson correspondence; a.k.a. type B (Date/Jimbo/Kashiwara/Miwa; You, IA)

The fermion side is generated by a single (neutral) field $\phi^B(z) = \sum_{n\in \mathbf{Z}} \phi_n z^n$, with OPE with itself given by:

$$\phi^B(z)\phi^B(w) \sim \frac{z-w}{z+w}, \quad \text{in modes: } [\phi^B_m, \phi^B_n]_\dagger = 2(-1)^m \delta_{m,-n} 1. \tag{22}$$

Thus the modes generate a Clifford algebra Cl_B, and the underlying space of states, which is a highest weight module for Cl_B is denoted by F_B.

The boson-fermion correspondence of type B is again determined once we write the image of the generating field $\phi^B(z)$ under the correspondence. In order to do that, an essential ingredient is once again the field $h(z)$ given by:

$$h(z) = \frac{1}{4}(:\phi^B(z)\phi^B(-z): -1) \qquad (23)$$

It follows that this field, which has only odd-indexed modes, $h(z) = \sum_{n\in\mathbb{Z}} h_{2n+1} z^{-2n-1}$, has OPEs with itself given by:

$$h(z)h(w) \sim \frac{zw(z^2 + w^2)}{2(z^2 - w^2)^2}, \qquad (24)$$

and is thus a twisted Heisenberg field.

For stage two, we have the decomposition (You):
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Now we can write the image of the generating field $\phi^B(z) \mapsto e^\alpha_B(z)$, which will determine the correspondence of type B:

$$e^\alpha_B(z) = \exp\Big(\sum_{k\geq 0} \frac{h_{-2k-1}}{k+1/2} z^{2k+1}\Big) \exp\Big(-\sum_{k\geq 0} \frac{h_{2k+1}}{k+1/2} z^{-2k-1}\Big) e^\alpha, \tag{25}$$

The fields $e^\alpha_B(z)$ and $e^\alpha_B(-z) = e^{-\alpha}(z)$ (observe the symmetry) generate a resulting twisted vertex algebra:

$$\phi^B(z) \mapsto e^\alpha_B(z) \qquad \phi^B(-z) \mapsto e^\alpha(-z) = e^{-\alpha}_B(z) \tag{26}$$

The neutral fermion–boson correspondence; a.k.a. type D (IA)

The fermion side is generated by a single (neutral) field $\phi^D(z) = \sum_{n\in \mathbf{Z}+\mathbf{1/2}} \phi^D_n z^{-n-1/2}$, with OPEs with itself given by:

$$\phi^D(z)\phi^D(w) \sim \frac{1}{z-w}, \quad \text{in modes: } [\phi^D_m, \phi^D_n]_\dagger = \delta_{m,-n}1. \quad (27)$$

Thus the modes generate a Clifford algebra Cl_D, with underlying space of states, denoted by $F^{\otimes \frac{1}{2}}$, the highest weight representation of Cl_D with the vacuum vector $|0\rangle$.

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It is important to note that this field generates on its own a super-vertex algebra $F^{\otimes\frac{1}{2}}$, called free neutral fermion vertex algebra.

The boson-fermion correspondence of type D-A is again determined once we write the image of the generating field $\phi^D(z)$ under the correspondence. In order to do that, an essential ingredient is once more the field $h(z)$ given by:

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It follows that this field, which due to the symmetry above has only odd-indexed modes, $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-2n-1}$, (note the different indexing), has OPEs with itself given by:

$$h(z)h(w) \sim \frac{zw}{(z^2 - w^2)^2}, \qquad (29)$$

Its modes, $h_n, \ n \in \mathbb{Z}$, generate an ordinary, untwisted Heisenberg algebra $\mathcal{H}_\mathbb{Z}$ with relations $[h_m, h_n] = m\delta_{m+n,0}1$, m, n integers.

For stage two of the bosonization, we prove that (IA)

$$F^{\otimes\frac{1}{2}} \oplus_{m\in\mathbb{Z}} B_m, \quad B_m \cong \mathbb{C}[x_1, x_2, \dots, x_n, \dots], \quad \forall\, m \in \mathbb{Z}$$

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Observe then that as Heisenberg $\mathcal{H}_\mathbb{Z}$ modules

$$F^{\otimes \frac{1}{2}} \cong F_A, \text{ i.e. } F^{\otimes \frac{1}{2}} \cong F^{\otimes 1} \qquad (F_A \text{ is often denoted } F^{\otimes 1})!$$

The image of the generating fields $\phi^D(z)$ a which will determine the correspondence of type D is given as follows:

$$\phi^D(z) = e_A^{-\alpha}(z^2) + z e_A^{\alpha}(z^2) \tag{30}$$

where recall $e_A^{\alpha}(z)$ and $e_A^{\alpha}(z)$ were the bosonic (lattice) vertex operators of type A; e.g.,

$$e_\phi^{-\alpha}(z^2) = \exp(-\sum_{n\geq 1} \frac{h_{-n}}{n} z^{2n}) \exp(\sum_{n\geq 1} \frac{h_n}{n} z^{-2n}) e^{-\alpha} z^{-2\partial_\alpha},$$

Note that we can go back in the boson-fermion correspondence by

$$\phi^D(z) = e_A^{-\alpha}(z^2) + z e_A^{\alpha}(z^2) \quad \phi^D(-z) = e_A^{-\alpha}(z^2) - z e_A^{\alpha}(z^2)$$

Boson-fermion correspondence of type D and order $N \in \mathbb{N}$ (IA, Rehren/Tedesco

The boson fermion correspondence of type D can be generalized to arbitrary order $N \in \mathbb{N}$:
Let ϵ be a N-th order primitive root of unity; Consider the field $h(z)$

$$h(z) = \frac{1}{N} \sum_{i=0}^{N-1} \epsilon^{i-1} : \phi^D(\epsilon^{i-1} z) \phi^D(\epsilon^i z) := \sum_{n \in \mathbb{Z}} h_n z^{-Nn-1} \tag{31}$$

Its OPE is

$$h(z)h(w) \sim \frac{z^{N-1} w^{N-1}}{(z^N - w^N)^2}, \tag{32}$$

and thus $h(z)$ is an untwisted Heisenberg field.

For the generating fields we have

$$e_\phi^{\epsilon^k\alpha}(w) = \frac{1}{N}\left(\sum_{i=0}^{N-1}\epsilon^{(k-1)i}\phi^D(\epsilon^i w)\right) = \frac{1}{N}\left(\sum_{i=0}^{N-1}\epsilon^{(k-1)i}T^i\phi^D(w)\right);$$

where

$$e_\phi^{\epsilon^k\alpha}(z) = \exp\left(\epsilon^{-k}\sum_{n\geq 1}\frac{h_{-n}}{n}z^{Nn}\right)\exp\left(\epsilon^k\sum_{n\geq 1}\frac{h_n}{n}z^{-Nn}\right)e_\phi^{\epsilon^k\alpha}z^{1-k+N\partial_\alpha}$$

Bosonization of type C and $\beta - \gamma$ system

The bosonization of type C was also completed, and for $N = 2$ we prove that as twisted chiral algebras the chiral algebra generated by the field $\chi(z)$

$$\chi(z) = \sum_{n\in\mathbb{Z}+1/2} \chi_n z^{-n-1/2} \tag{33}$$

with OPE

$$\chi(z)\chi(w) \sim \frac{1}{z+w}. \tag{34}$$

is isomorphic to the twisted vertex algebra with space of fields generated by the $\beta - \gamma$ system, but at changed gauge: $\mathfrak{FD}\{\beta(z^2), \gamma(z^2); 2\}$.

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That has interesting consequences: each of those chiral algebras inherits the structures form the other.

Thank you.