A maximal dissipation condition for dynamic fracture: an existence result in a constrained case

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- The mathematical models for dynamic fracture are based on
  - elastodynamics out of the crack, with suitable boundary conditions on the crack,
  - a rule that couples elastodynamics with crack growth.
- In this talk we consider only linearly elastic fracture mechanics with no forces on the crack, so we use the standard linear system of elastodynamics with homogeneous Neumann boundary conditions on the crack.
- The coupling between elastodynamics and crack growth is obtained through an energy criterion, which goes back to Griffith (1920) in the quasistatic case, and was extended to the dynamic case by Mott (1948).
- The process of crack production dissipates energy. Even if we neglect thermal effects, we have to take into account the energy spent to break the interatomic bonds. In the isotropic case, this leads to an energy dissipation proportional to the area of the crack. The proportionality constant is a material property, called toughness.

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Dynamic energy-dissipation balance

- For every time t we define

$$\text{total energy} := \text{kinetic energy} + \text{elastic energy}$$
$$+ \text{energy dissipated by the crack between } 0 \text{ and } t$$

The last term is proportional to the increase of area of the crack from 0 to t.

- The energy criterion which connects elastodynamics with crack growth is given by the dynamic energy-dissipation balance:

$$\text{total energy at time } t := \text{total energy at time } 0$$
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This leads to the well known equality between dynamic energy release rate and toughness, and the formulas to compute the energy release rates provide a relationship between dynamic stress intensity factors and toughness.

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Need for an additional principle

- The dynamic energy-dissipation balance is not sufficient to determine the evolution of a crack, since elastodynamics with a stationary crack will always satisfy energy balance.
- In the phase-field approach to dynamic fracture the crack is replaced by a phase-field approximation: a function v which takes the value 0 near the crack and the value 1 far from it.
- In these models, an energy minimization condition on v provides a principle that can require the crack to grow (so that stationary cracks are not always solutions).
- This idea has no natural extension to sharp crack models. We propose different criterion, a maximal dissipation criterion, as an additional principle for crack growth.
- Although this criterion could be formulated in a general setting, we prefer to give a precise formulation only within a specific two dimensional model with a prescribed crack path.

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Importance of existence results

- Indeed, we want to test this idea on a model of dynamic crack growth with a prescribed crack path, where a solution of the dynamic crack problem is defined as a crack-displacement pair such that the displacement satisfies the system of elastodynamics out of the crack set and the pair satisfies the dynamic energy-dissipation balance and the maximal dissipation condition.
- We want to prove that, under suitable assumptions on the initial and boundary conditions, this problem has a solution.
- This is not a mathematical luxury – a formulation that prescribes too many properties runs a strong risk of not having solutions.
- The proof of the existence of a solution in the framework of a model, under suitable assumptions on the data, guarantees that this model has no internal contradictions. Only in this case one can use it to compute approximate solutions and then compare the predictions of the model with the outcome of experiments.

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Main features of the model

- The model we consider here is linearly elastic with antiplane displacement. Therefore, the reference configuration Ω is contained in the plane, the displacement u is scalar, and the system of elastodynamics reduces to the scalar wave equation.
- The crack follows a sufficiently regular prescribed path Γ.
- We consider only the problem of crack growth, assuming that an initial crack Γ_0 is already present.
- We neglect all thermal effects, as well as other sources of dissipation, except for the energy spent to produce new crack.
- Our point is that the main mathematical difficulties to obtain an existence result are already present in this simplified model, and we expect that more realistic models could be studied later by adapting similar ideas and techniques.

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Precise hypotheses on Ω and Γ

- The reference configuration $\Omega \subset \mathbb{R}^2$ is a bounded open set with Lipschitz boundary $\partial\Omega$
- The prescribed crack path Γ is a simple curve of class $C^{2,1}$ contained in Ω except for its end-points, which belong to $\partial\Omega$. We also assume that Γ divides Ω into two subsets Ω^+ and Ω^-, both having a Lipschitz boundary (transversality condition).
- $\partial\Omega$ is the union of two disjoint Borel sets $\partial_D\Omega$ and $\partial_N\Omega$; on $\partial_D\Omega$ we prescrive a time dependent Dirichlet boundary condition, on $\partial_N\Omega$ we prescribe the homogeneous Neumann boundary condition.
- Let $\gamma\colon [a,b] \to \overline{\Omega}$ be an arc-length parametrization of the crack path Γ, with $a < 0 < b$ and $\gamma(a), \gamma(b) \in \partial\Omega$. The initial crack tip corresponds to $s = 0$.
- For every $s \in [a,b]$ we set $\Gamma_s = \gamma([a,s])$ and $\Omega_s := \Omega \setminus \Gamma_s$.
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Precise hypotheses on the data

- For every $s \in [a,b]$ we set $H^1_D(\Omega_s) := \{u \in H^1(\Omega_s) : u = 0 \text{ on } \partial_D\Omega\}$, endowed with he norm of $H^1(\Omega_s)$; its dual is denoted by $H^{-1}_D(\Omega_s)$.
- Given a function $u \in H^1(\Omega_s)$, let $\widehat{\nabla} u = \nabla u$ on Ω_s and $\widehat{\nabla} u = 0$ on Γ_s. Note that $\widehat{\nabla} u \in L^2(\Omega;\mathbb{R}^2)$.
- The crack problem is studied in a bounded time interval $[0,T]$.
- The body force f satisfies $f \in L^2((0,T);L^2(\Omega))$.
- The Dirichlet boundary condition is prescribed using a function $w \in L^2((0,T);H^2(\Omega_0)) \cap H^1((0,T);H^1(\Omega_0)) \cap H^2((0,T);L^2(\Omega_0))$, where $\Omega_0 := \Omega \setminus \Gamma_0$ is the cracked domain corresponding to $s = 0$.
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Wave equation in cracking domains

- We now suppose that the crack growth is prescribed through a nondecreasing function $s\colon [0,T] \to [0,b]$.
- To find the corresponding displacement u, we have to solve the wave equation in a time dependent domain

$$\ddot{u}(t,x) - \Delta u(t,x) = f(t,x) \quad \text{for } t \in (0,T) \text{ and } x \in \Omega_{s(t)}.$$

- This equation is complemented by Dirichlet boundary conditions

$$u(t,x) = w(t,x) \quad \text{for } t \in (0,T) \text{ and } x \in \partial_D \Omega,$$

- Neumann boundary conditions

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- The classical results on the wave equation in time-dependent domains cannot be applied directly, since they require suitable regularity assumptions on the boundary, which are clearly not satisfied here.

Wave equation in cracking domains

- We now suppose that the crack growth is prescribed through a nondecreasing function $s\colon [0,T] \to [0,b]$.
- To find the corresponding displacement u, we have to solve the wave equation in a time dependent domain

$$\ddot{u}(t,x) - \Delta u(t,x) = f(t,x) \quad \text{for } t \in (0,T) \text{ and } x \in \Omega_{s(t)}\,.$$

- This equation is complemented by Dirichlet boundary conditions

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Regularity conditions on s

- The existence of a solution to this problem in domains with a prescribed growing crack was proved in DM-Larsen 2011 under very general assumptions.
- The uniqueness, however, is an open problem in this general setting.
- Since uniqueness is crucial in our treatment of the problem, in our model with a prescribed crack path we assume more regularity on s in order to apply the uniqueness result proved in DM-Lucardesi 2015
- More precisely, we fix two parameters $0 < \delta < 1$ and $M > 0$, and consider the class $\mathcal{C}_{\delta,M}([0,T])$ composed of all functions satisfying the following conditions:
  - $s \in C^{2,1}([0,T];[0,b])$,
  - $0 \leq \dot{s}(t) \leq 1-\delta$ for every $t \in [0,T]$,
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Weak solution in cracked domains

Under the previous assumptions on w, u^0, u^1, and s, we say that u is a weak solution of the wave equation (with boundary and initial conditions) on the time-dependent cracking domains $t \mapsto \Omega_{s(t)}$ if

$u \in C^1([0,T]; L^2(\Omega))$,

$u(t) - w(t) \in H^1_D(\Omega_{s(t)})$ for every $t \in [0,T]$,

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$\frac{1}{h}(\dot{u}(t+h) - \dot{u}(t)) \underset{h\to 0}{\rightharpoonup} \ddot{u}(t)$ weakly in $H^{-1}_D(\Omega_{s(t)})$, for a.e. $t \in (0,T)$,

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Theorem (DM-Lucardesi, 2015)

Under the previuous assumptions there exists a unique weak solution on the time-dependent cracking domains $t \mapsto \Omega_{s(t)}$.

We also have the continuous dependence of the solutions on the data, in particular on the function $t \mapsto s(t)$.

Theorem (DM-Lucardesi, 2015)

Assume that $s_k \in \mathcal{C}_{\delta,M}([0,T])$ *converges uniformly to some* $s \in \mathcal{C}_{\delta,M}([0,T])$. *Let* u_k *and* u *be the weak solutions on the cracking domains* $t \mapsto \Omega_{s_k(t)}$ *and* $t \mapsto \Omega_{s(t)}$. *Then for every* $t \in [0,T]$

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Existence and uniqueness

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Piecewise regular functions

- Besides the class $\mathcal{C}_{\delta,M}([0,T])$, we can consider the class $\mathcal{C}^{\text{piec}}_{\delta,M}([0,T])$ defined in the following way: $s \in \mathcal{C}^{\text{piec}}_{\delta,M}([0,T])$ if and only if $s \in C^0([0,T])$ and there exist $0 = T_0 < T_1 < \cdots < T_k = T$ such that $s|_{[T_{j-1},T_j]} \in \mathcal{C}_{\delta,M}([T_{j-1},T_j])$ for every $j = 1, \ldots, k$.
- If $s \in \mathcal{C}^{\text{piec}}_{\delta,M}([0,T])$, then we can still define a weak solution of the wave equation (with boundary and initial conditions) in the time-dependent cracking domains $t \mapsto \Omega_{s(t)}$.
- The existence and uniqueness of such a solution is a direct consequence of the theorem for $\mathcal{C}_{\delta,M}([0,T])$, applied to each interval $[T_{j-1},T_j]$ of the subdivision.

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- The existence and uniqueness of such a solution is a direct consequence of the theorem for $\mathcal{C}_{\delta,M}([0,T])$, applied to each interval $[T_{j-1},T_j]$ of the subdivision.

Piecewise regular functions

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Energy and work

- The sum of the kinetic and elastic energies of a solution u at time t is given by $\mathcal{E}(\dot{u}(t), \widehat{\nabla} u(t))$, where $\mathcal{E}(v, \Psi) := \frac{1}{2}\|v\|^2 + \frac{1}{2}\|\Psi\|^2$ for every $v \in L^2(\Omega)$ and $\Psi \in L^2(\Omega; \mathbb{R}^2)$.
- The work of the external forces on the solution u over a time interval $[t_1, t_2] \subset [0, T]$ is given by

$$\mathcal{W}_{load}(u; t_1, t_2) := \int_{t_1}^{t_2} \langle f(t), \dot{u}(t) \rangle dt.$$

- The work on the solution u due to the varying boundary conditions w over a time interval $[t_1, t_2] \subset [0, T]$ is given by

$$\mathcal{W}_{bdry}(u; t_1, t_2) = \int_{t_1}^{t_2} \langle \partial_\nu u(t), \dot{w}(t) \rangle_{\partial_D \Omega} dt,$$

when $u(t)$ is regular enough. Integrating by parts, it can be written by means of a longer expression that makes sense for every weak solution.
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- Under the previous assumption on w, u^0, u^1, and s, given $s_0 \in [0, b]$ we consider the class $\mathcal{S}_{s_0}([0,T])$ (resp. $\mathcal{S}^{\rm piec}_{s_0}([0,T])$) of all functions $s \in \mathcal{C}_{\delta,M}([0,T])$ (resp. $s \in \mathcal{C}^{\rm piec}_{\delta,M}([0,T])$), with $s(0) = s_0$, such that the unique weak solution u of the wave equation (with initial conditions u^0 and u^1, and boundary condition w) on the time-dependent cracking domains $t \mapsto \Omega_{s(t)}$ satisfies the dynamic energy-dissipation balance
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- These classes describe all sufficiently regular crack evolutions satisfying the dynamic energy-dissipation balance and with initial crack corresponding to s_0.
- Note that the classes $\mathcal{S}_{s_0}([0,T])$ and $\mathcal{S}^{\rm piec}_{s_0}([0,T])$ are not empty: they contain at least the constant function $s(t) = s_0$ for all $t \in [0, T]$.

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Maximal dissipation

- The spirit of the maximal dissipation condition is simply that the crack must run as fast as possible, consistent with energy balance.
- To give a formal definition, for every $s \in \mathcal{S}_0^{piec}([0,T])$ and $\tau \in [0,T]$ we introduce the class $\mathcal{A}(s,\tau)$ of admissible comparison functions, defined as the class of functions $\sigma \in \mathcal{S}_0^{piec}([0,T])$, with $\sigma(t) \geq s(t)$ for all $t \in [0,\tau]$, such that $\dot\sigma$, $\ddot\sigma$ are continuous where $\dot s$, $\ddot s$ are continuous.
- We say that s is a maximal dissipation solution of the dynamic crack evolution problem if $s \in \mathcal{S}_0^{piec}([0,T])$ and for every $\tau \in [0,T]$ there is no $\sigma \in \mathcal{A}(s,\tau)$ such that $\sigma(\tau) > s(\tau)$.
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The existence result

Theorem (DM-Larsen-Toader 2015)

Assume that f, w, u^0, u^1 satisfy the previous hypotheses and let $\eta > 0$. Then there exists an η-maximal dissipation solution of the dynamic crack evolution problem corresponding to these data.

- The main difficulty in the definition of an η-maximal dissipation solution is the variability of the interval $[0, \tau]$ where the comparison function $\sigma \in \mathcal{S}_0^{piec}([0, T])$ satisfies the inequality $\sigma(t) \geq s(t)$.
- To overcome this problem we discretize time and in each time interval we prove the existence of a maximal function s among all functions satisfying our regularity requirements and the energy equality.
- Then we prove that the function obtained by glueing together these maximal functions is an η-maximal dissipation solution, provided that the length of each time interval is less that η (recall that the speed of sound is normalized to one).

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For every $s_0 \in [0, b]$ *exists* $s \in \mathcal{S}_{s_0}([0,T])$ *such that*

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for every interval $[t_1, t_2] \subset [0, T]$.

End of the proof of the lemma

- By the continuous dependence on the data for every interval $[t_1, t_2] \subset [0, T]$ we can pass to the limit in

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- This gives

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where u is the solution of the wave equation corresponding to s.

- Hence u and s satisfy the dynamic energy-dissipation balance, i.e., $s \in \mathcal{S}_{s_0}([0, T])$.

- Recalling that s_n is a maximizing sequence, we have

$$\sup_{\sigma \in \mathcal{S}_{s_0}([0,T])} \int_0^T \sigma(t)\,dt = \lim_{n \to \infty} \int_0^T s_n(t)\,dt = \int_0^T s(t)\,dt.$$

- Since $s \in \mathcal{S}_{s_0}([0, T])$, this shows that s is a maximizer, and concludes the proof of the lemma.

End of the proof of the lemma

- By the continuous dependence on the data for every interval $[t_1, t_2] \subset [0, T]$ we can pass to the limit in
$\mathcal{E}(\dot{u}_n(t_2), \widehat{\nabla} u_n(t_2)) - \mathcal{E}(\dot{u}_n(t_1), \widehat{\nabla} u_n(t_1)) + s_n(t_2) - s_n(t_1) = \mathcal{W}(u_n; t_1, t_2)$.
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Proof of the theorem (part 1)

- To prove the theorem we fix a finite subdivision $0 = T_0 < T_1 < \cdots < T_k = T$ such that $T_j - T_{j-1} \le \eta$ for $j = 1, \dots, k$. The solution will be constructed recursively in the intervals $[T_{j-1}, T_j]$.
- By applying the lemma on the interval $[0, T_1]$ we find $s_1 \in \mathcal{S}_0([0, T_1])$ such that
$$\int_0^{T_1} s_1(t)\,dt = \max_{s \in \mathcal{S}_0([0,T_1])} \int_0^{T_1} s(t)\,dt,$$
and we consider the unique solution u_1 corresponding to s_1.
- By applying the lemma on the interval $[T_1, T_2]$, with initial conditions $u_1(T_1)$ and $\dot u_1(T_1)$, we find $s_2 \in \mathcal{S}_{s_1(T_1)}([T_1, T_2])$ such that
$$\int_{T_1}^{T_2} s_2(t)\,dt = \max_{s \in \mathcal{S}_{s_1(T_1)}([T_1,T_2])} \int_{T_1}^{T_2} s(t)\,dt,$$
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- By applying the lemma on the interval $[T_j, T_{j+1}]$, with initial conditions $u_j(T_j)$ and $\dot{u}_j(T_j)$, we find $s_{j+1} \in \mathcal{S}_{s_j(T_j)}([T_j, T_{j+1}])$ such that
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and we consider the unique solution u_{j+1} corresponding to s_{j+1}.
- We now set $s(t) := s_j(t)$ and $u(t) := u_j(t)$ for $t \in [T_{j-1}, T_j]$, $j = 1, \ldots, k$.
- It follows from the construction that $s \in C^{piec}_{\delta,M}([0, T])$, that $s(0) = 0$, and that u is the unique solution of the wave equation corresponding to s.
- Since the energy-dissipation balance is satisfied on every subinterval of the intervals $[T_{j-1}, T_j]$, it is satisfied on every subinterval of $[0, T]$, hence $s \in \mathcal{S}^{piec}_0([0, T])$.

Proof of the theorem (part 2)

- By applying the lemma on the interval $[T_j, T_{j+1}]$, with initial conditions $u_j(T_j)$ and $\dot{u}_j(T_j)$, we find $s_{j+1} \in \mathcal{S}_{s_j(T_j)}([T_j, T_{j+1}])$ such that
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Proof of the theorem (part 3)

- It remains to prove that the maximal dissipation condition is satisfied.
- Suppose the contrary. Then there exist $\tau \in [0,T]$ and $\sigma \in \mathcal{S}_0^{\mathrm{piec}}([0,T])$, with $\sigma(t) \geq s(t)$ for all $t \in [0,\tau]$ and $\sigma(\tau) > s(\tau) + \eta$, such that $\dot\sigma$ and $\ddot\sigma$ are continuous where $\dot s$ and $\ddot s$ are continuous.
- Let $\tau_0 := \inf\{t \in [0,\tau] : \sigma(t) > s(t)\}$ and let j be the index such that $\tau_0 \in [T_j, T_{j+1})$. We claim that $T_{j+1} \leq \tau$.
- If not, the inequality $\dot\sigma(t) \leq 1-\delta$ implies $\sigma(\tau)-\sigma(\tau_0) \leq T_{j+1}-T_j < \eta$, hence $\sigma(\tau) < \sigma(\tau_0) + \eta = s(\tau_0) + \eta \leq s(\tau) + \eta$, which contradicts $\sigma(\tau) > s(\tau) + \eta$. This proves that $T_{j+1} \leq \tau$.
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- Therefore $\sigma|_{[T_j, T_{j+1}]}$ is a competitor in the maximum problem which defines s_{j+1}.

Proof of the theorem (part 3)

- It remains to prove that the maximal dissipation condition is satisfied.
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- Therefore $\sigma|_{[T_j,T_{j+1}]}$ is a competitor in the maximum problem which defines s_{j+1}.

Proof of the theorem (conclusion)

- Since $\sigma|_{[T_j,T_{j+1}]}$ is a competitor in that maximum problem, we have
$$\int_{T_j}^{T_{j+1}} s(t)\,dt = \int_{T_j}^{T_{j+1}} s_{j+1}(t)\,dt \geq \int_{T_j}^{T_{j+1}} \sigma(t)\,dt.$$
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Example of growing dynamic crack

- Let us consider the special case in which Ω is a circle, the Dirichhlet part of the boundary $\partial_D\Omega$ coincides with the whole boundary $\partial\Omega$, the crack path Γ is a straight line segment, and the body force is $f = 0$.
- Given $c > 0$ we can find an explicit example of Dirichlet boundary conditions w and of initial conditions u^0 and u^1 such that a crack growing with constant velocity c (which corresponds to $s(t) = ct$), satisfies the dynamic energy-dissipation balance.
- This shows that in this case the η-maximal dissipation solution is not stationary, although the stationary crack $s(t) = 0$ always satisfies the dynamic energy-dissipation balance.

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THANK YOU FOR YOUR ATTENTION!