Gradient damage models coupled with plasticity

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joint work with

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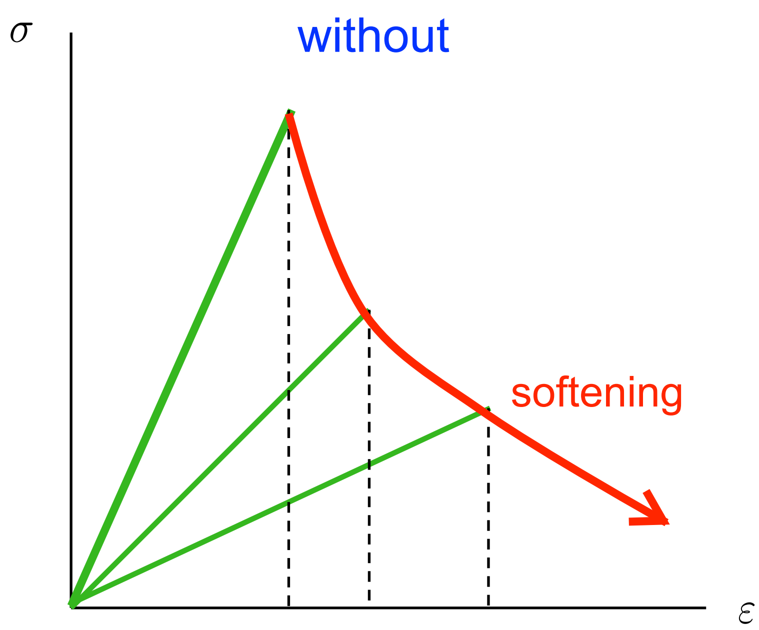
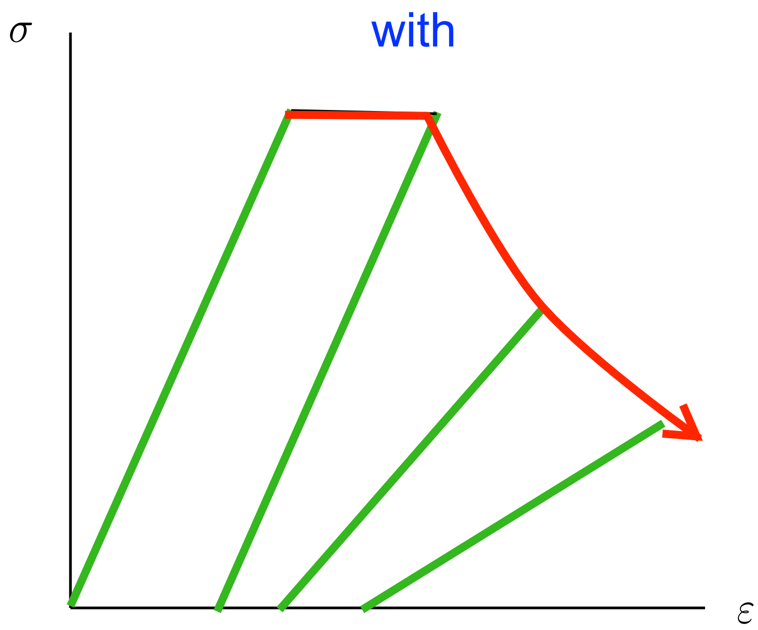
Damage models without or with plasticity

- quasi-static, rate independent evolution law
- scalar damage variable
- variational approach



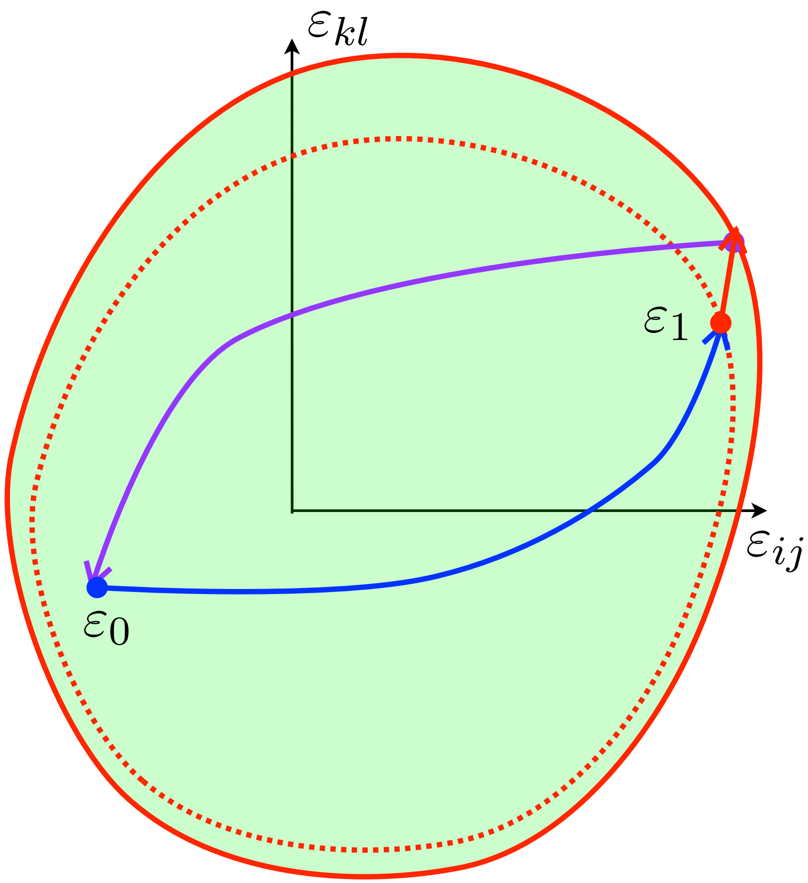


Justification of "standard" laws

✓ Drucker-Ilyushin Postulate

The strain work must be non negative in every strain cycle

$$\boxed{\oint_{\mathcal{C}} \sigma \cdot d\varepsilon \geq 0, \quad \forall \mathcal{C}}$$



✓ In perfect plasticity

The D-I postulate is equivalent to the Hill principle of maximal plastic work which is equivalent to the convexity of the yield surface and the normality rule

$$\boxed{\text{Drucker-Ilyushin} \iff \text{Hill}}$$

✓ For brittle scalar damage laws

- stress-strain relation

$$\sigma = \frac{\partial \psi}{\partial \varepsilon}(\varepsilon, \alpha), \qquad \alpha \in [0, \alpha_M)$$

- yield criterion : damage grows only when the strains (or the stresses) reach some yield surface which is damage dependent
- Théorem (JJM, '89)

$$\boxed{\text{Drucker-Ilyushin} \iff \text{Standard Law}}$$

yield criterion : $\boxed{-\frac{\partial \psi}{\partial \alpha}(\varepsilon, \alpha) \leq w'(\alpha)}$

Damage without plasticity

General form of standard non regularized damage laws

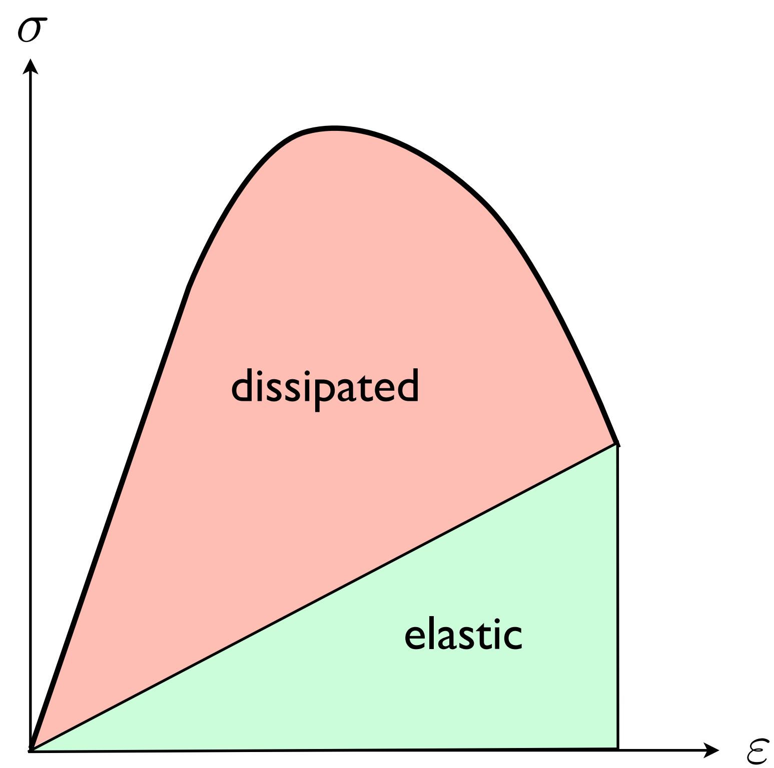
✓ constitutive relations

σ–ε relation : $\sigma = \dfrac{\partial \psi}{\partial \varepsilon}(\varepsilon, \alpha)$

irreversibility : $\dot{\alpha} \geq 0$

yield criterion : $-\dfrac{\partial \psi}{\partial \alpha}(\varepsilon, \alpha) \leq w'(\alpha)$

energy balance : $\left(\dfrac{\partial \psi}{\partial \alpha}(\varepsilon, \alpha) + w'(\alpha)\right) \dot{\alpha} = 0$



✓ *energetic interpretation*

the strain work is a state function equal to the sum of the elastic energy and the dissipated energy

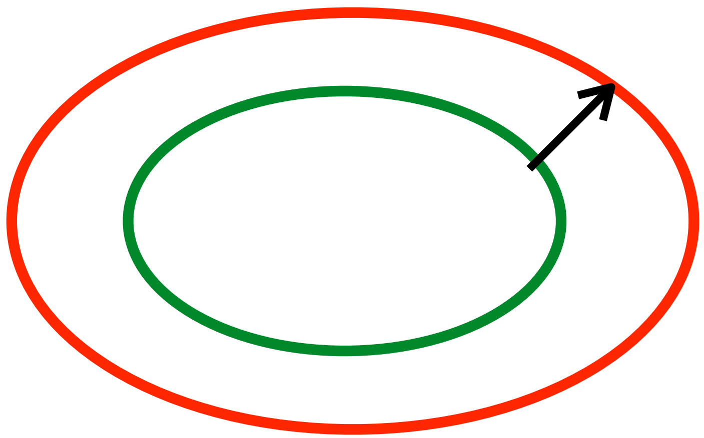
$$\boxed{\int_{\overrightarrow{0\varepsilon}} \sigma \cdot d\varepsilon = W(\varepsilon, \alpha) = \psi(\varepsilon, \alpha) + w(\alpha)}$$

$$\boxed{\psi(\varepsilon, \alpha) = \tfrac{1}{2} \mathsf{E}(\alpha)_{ijkl} \varepsilon_{ij} \varepsilon_{kl}}$$

"linear" case

Hardening and softening conditions

✓ Strain hardening



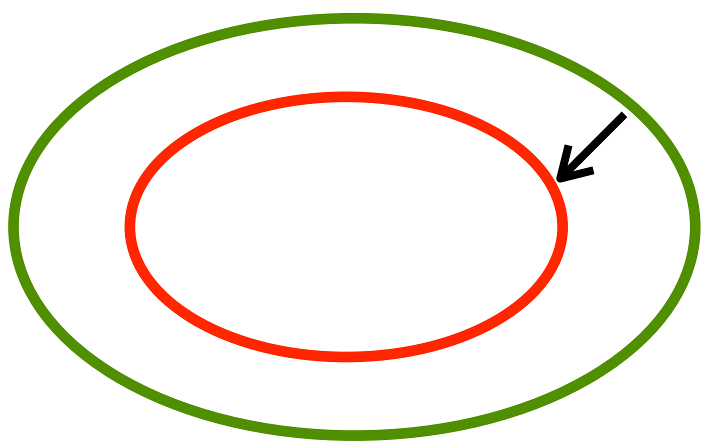
strain space

$$-\frac{1}{2}\,\mathsf{E}'(\alpha)\varepsilon\cdot\varepsilon \le \mathsf{w}'(\alpha)$$

$$\boxed{\alpha \mapsto \mathsf{E}'(\alpha)/\mathsf{w}'(\alpha)\ \text{increasing}}$$

$$\varepsilon = \mathsf{S}(\alpha)\sigma$$

✓ Stress softening



stress space

$$\frac{1}{2}\,\mathsf{S}'(\alpha)\sigma\cdot\sigma \le \mathsf{w}'(\alpha)$$

$$\boxed{\alpha \mapsto \mathsf{S}'(\alpha)/\mathsf{w}'(\alpha)\ \text{increasing}}$$

Construction of the gradient damage models

✓Definition of the strain work density function

$$W(\varepsilon, \alpha, \nabla\alpha) = \frac{1}{2}\,\mathsf{E}(\alpha)(\varepsilon - \varepsilon^{th}) \cdot (\varepsilon - \varepsilon^{th}) + \mathsf{w}(\alpha) + \boxed{\frac{1}{2}\,\mathsf{w}_1 \ell(\alpha)^2 \nabla\alpha \cdot \nabla\alpha}$$

$$\ell(\alpha) = \text{material characteristic length}$$

✓ Choice of the damage parameter

$$\boxed{W(\varepsilon, \alpha, \nabla\alpha) = \mathsf{w}(\alpha) + \tfrac{1}{2}\mathsf{w}_1 \ell^2 \nabla\alpha \cdot \nabla\alpha + \tfrac{1}{2}\mathsf{E}(\alpha)(\varepsilon - \varepsilon^{th}) \cdot (\varepsilon - \varepsilon^{th})} \qquad \boxed{\alpha \in [0, 1]}$$

✓ Constitutive inequalities

$$\mathsf{E}(0) = \mathsf{E}_0 > 0, \qquad \mathsf{E}(1) = 0 \qquad \mathsf{E}(\alpha) > 0, \qquad \mathsf{E}'(\alpha) < 0$$

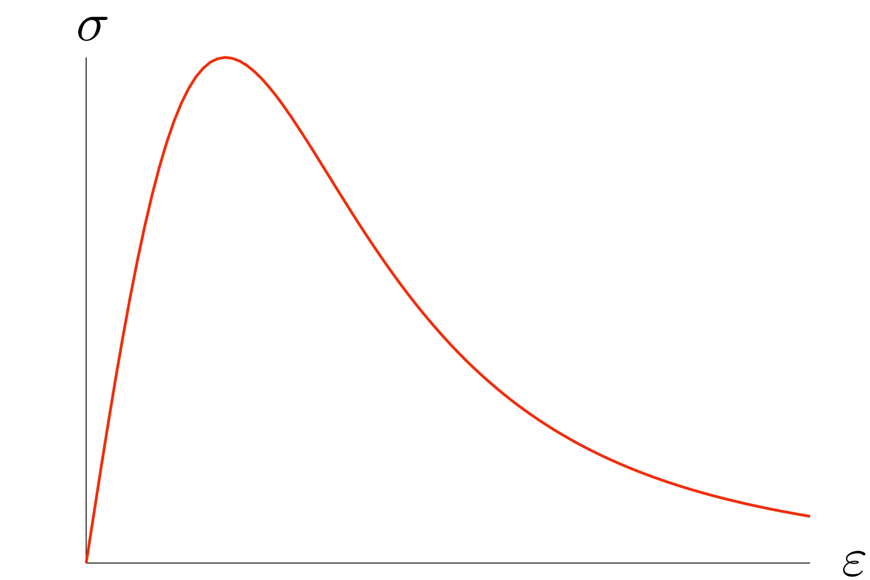
$$\mathsf{w}(0) = 0 \qquad \mathsf{w}'(\alpha) > 0 \qquad \mathsf{w}_1 = \mathsf{w}(1) < +\infty$$

$$\text{stress softening} = \boxed{\alpha \mapsto \mathsf{S}'(\alpha)/\mathsf{w}'(\alpha) \text{ increasing}}$$

$$\mathsf{S}(\alpha) = \mathsf{E}(\alpha)^{-1} = \text{compliance tensor}$$

✓Examples

- Ambrosio-Tortorelli model

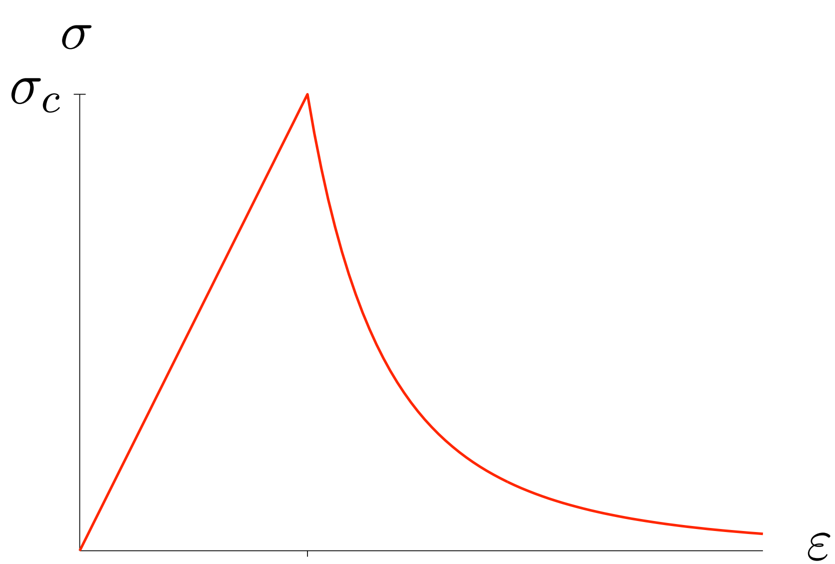


$$\mathsf{E}(\alpha) = (1-\alpha)^2 \mathsf{E}_0$$

$$\mathsf{w}(\alpha) = \mathsf{w}_1\, \alpha^2$$

no elastic range

- A model with finite critical stress and stress softening



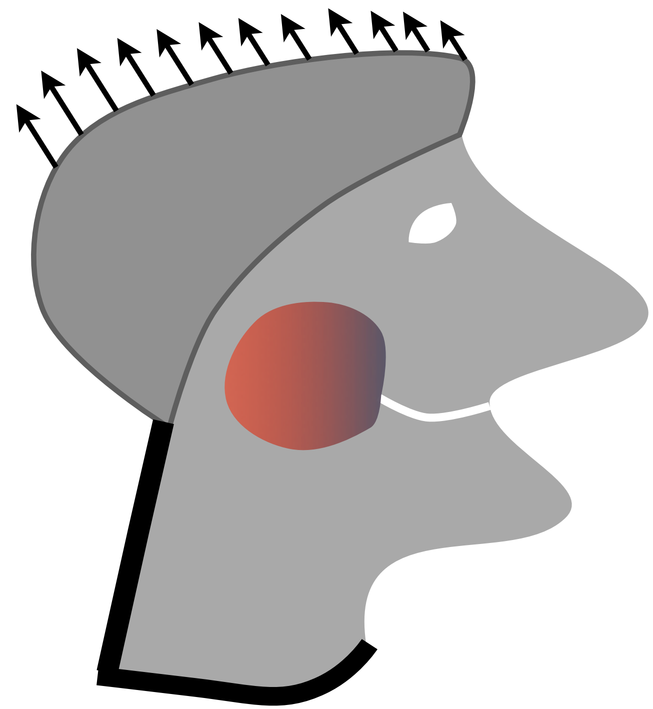
$$\mathsf{E}(\alpha) = (1-\alpha)^2 \mathsf{E}_0$$

$$\mathsf{w}(\alpha) = \frac{\sigma_c^2}{\mathsf{E}_0}\alpha$$

✓the global evolution problem

▸the global total energy

$$\mathcal{E}_t(u,\alpha) = \int_\Omega W_t(\varepsilon(u),\alpha,\nabla\alpha)dV - f_t(u)$$



▸the evolution problem in its variational form

1. Irreversibility

$$\dot{\alpha}_t \geq 0$$

2. First order stability condition

$$\mathcal{E}_t'(u_t,\alpha_t)(v-u_t,\beta-\alpha_t) \geq 0, \quad \forall v \in \mathcal{C}_t, \quad \forall \beta \ : \ \alpha_t \leq \beta \leq 1$$

2'. Complete stability condition

$$\forall(v,\beta) \text{ admissible and } h \text{ small enough}, \quad \mathcal{E}_t(u_t,\alpha_t) \leq \mathcal{E}(u_t+hv,\alpha_t+h\beta)$$

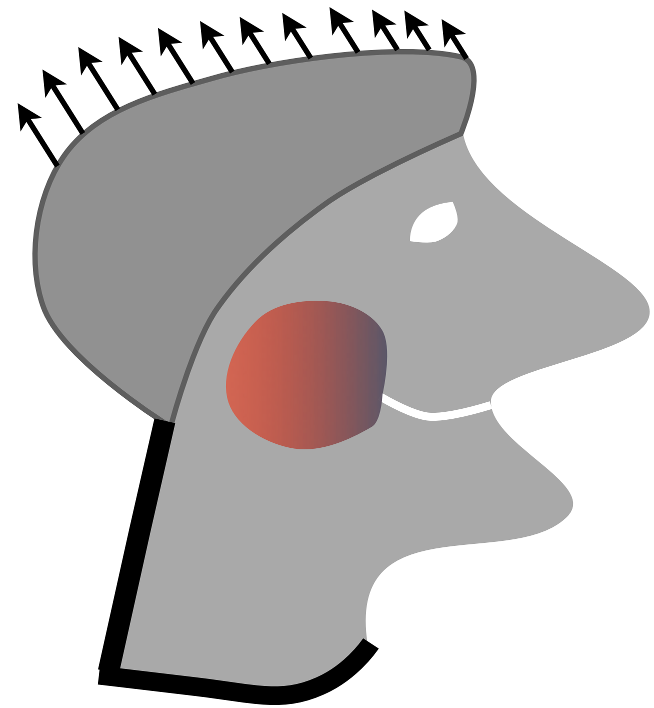
$$\boxed{\beta \geq 0}$$

3. Global energy balance

$$\frac{d}{dt}\mathcal{E}_t(u_t,\alpha_t) = \frac{\partial \mathcal{E}_t}{\partial t}(u_t,\alpha_t)$$

‣the evolution problem in its local form

$$\begin{cases} \operatorname{div} \sigma_t + f_t = 0 & \text{in} \quad \Omega \\ \sigma_t n = F_t & \text{on} \quad \partial_F \Omega \\ u_t = U_t & \text{on} \quad \partial_D \Omega \end{cases}$$



Stress-strain relation : $\sigma_t = \mathsf{E}(\alpha_t)(\varepsilon_t - \varepsilon_t^{th})$ in Ω

Irreversibility : $\dot{\alpha}_t \geq 0$ in Ω

Damage condition : $\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t \cdot \sigma_t - w'(\alpha_t) + \boxed{w_1 \ell^2 \Delta \alpha_t} \leq 0$ in Ω

Consistency condition : $\left(\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t \cdot \sigma_t - w'(\alpha_t) + \boxed{w_1 \ell^2 \Delta \alpha_t}\right) \dot{\alpha}_t = 0$ in Ω

Boundary condition : $\boxed{\dfrac{\partial \alpha_t}{\partial n} \geq 0}$, $\boxed{\dfrac{\partial \alpha_t}{\partial n} \dot{\alpha}_t = 0}$ on $\partial \Omega$

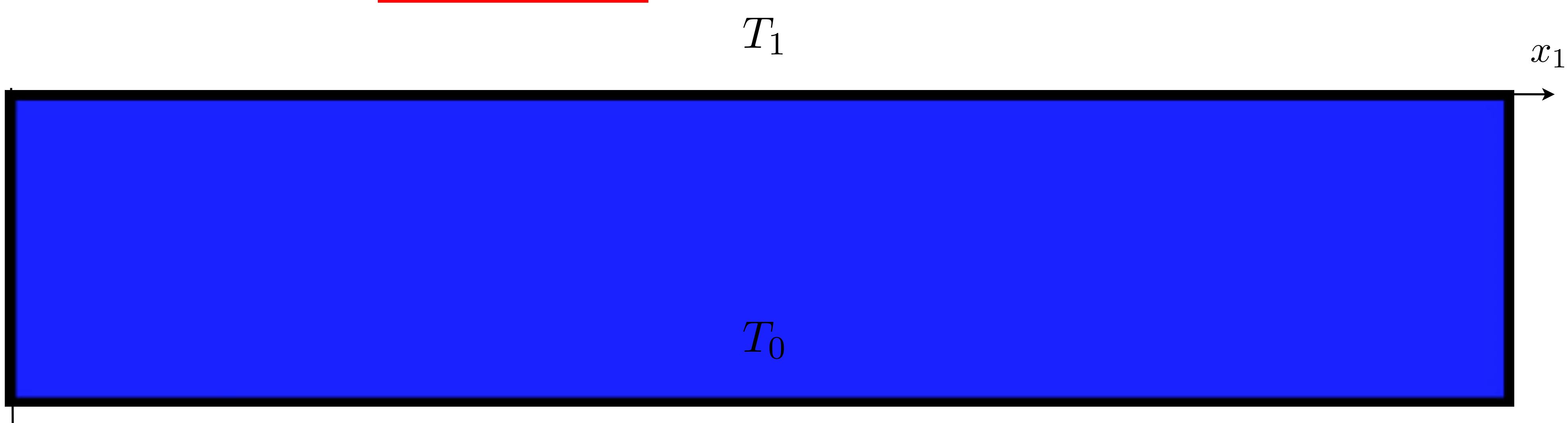
✓ numerical method

- time discretization $\quad u_i^n = \mathrm{argmin}_u \mathcal{E}_i(u, \alpha_i^n)$
- alternate minimization algorithm: $\quad \alpha_i^{n+1} = \mathrm{argmin}_{\alpha \geq \alpha_{i-1}} \mathcal{E}_i(u_i^n, \alpha)$

✓ Example of a thermal shock

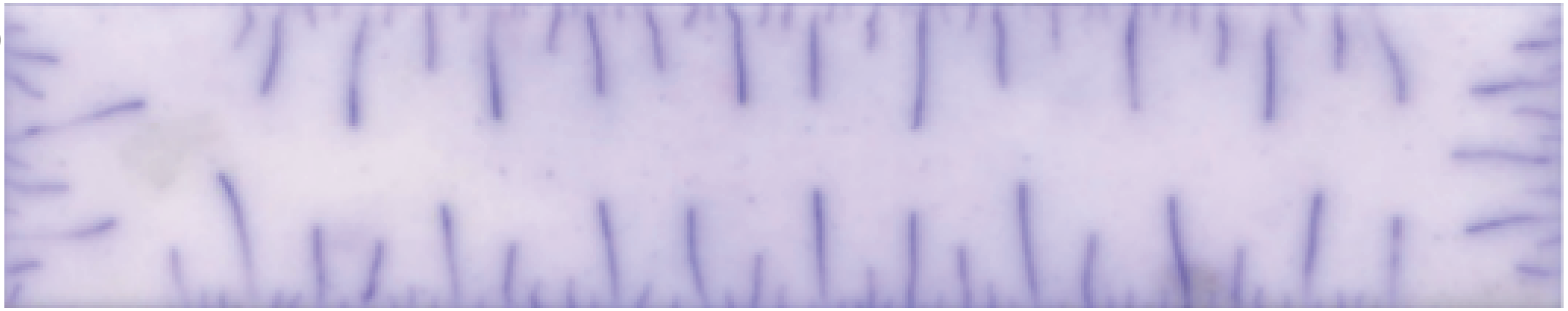
model $\quad \mathsf{E}(\alpha) = (1-\alpha)^2 \mathsf{E}_0 \quad \mathsf{w}(\alpha) = \mathsf{w}_1 \alpha \quad \ell$

$$\boxed{\sigma_c = \sqrt{\mathsf{w}_1 \mathsf{E}_0}}$$

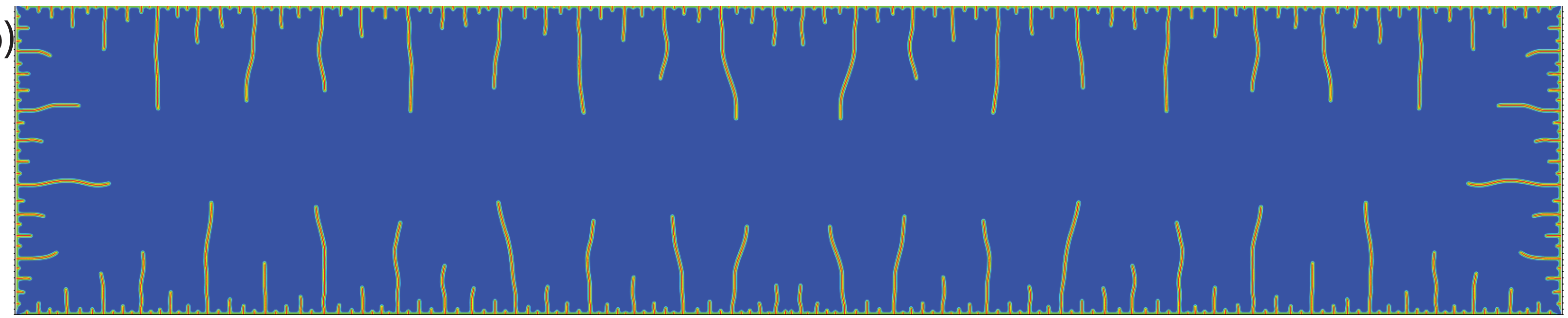


heat diffusion $\quad \varepsilon_t^{th}(x_2) = -\mathsf{a}(T_0 - T_1)\, \mathsf{f}_c\left(\frac{x_2}{2\sqrt{\mathsf{k}t}}\right) \mathbf{I}$

a)



b)



Ceramic parameters: E_0=340 GPa, G_c =42 J.m^{-2}, σ_c =340 MPa, ν = .22
(from G_c and σ_c one deduces ℓ = .05 mm)
Temperature gradient $\mathbf{T_0\text{-}T_1}$ = 380°.

(a) Experimental crack pattern in a slab (10 mm × 50 mm × 1mm) after a thermal shock (from Jiang et al. [2012]).

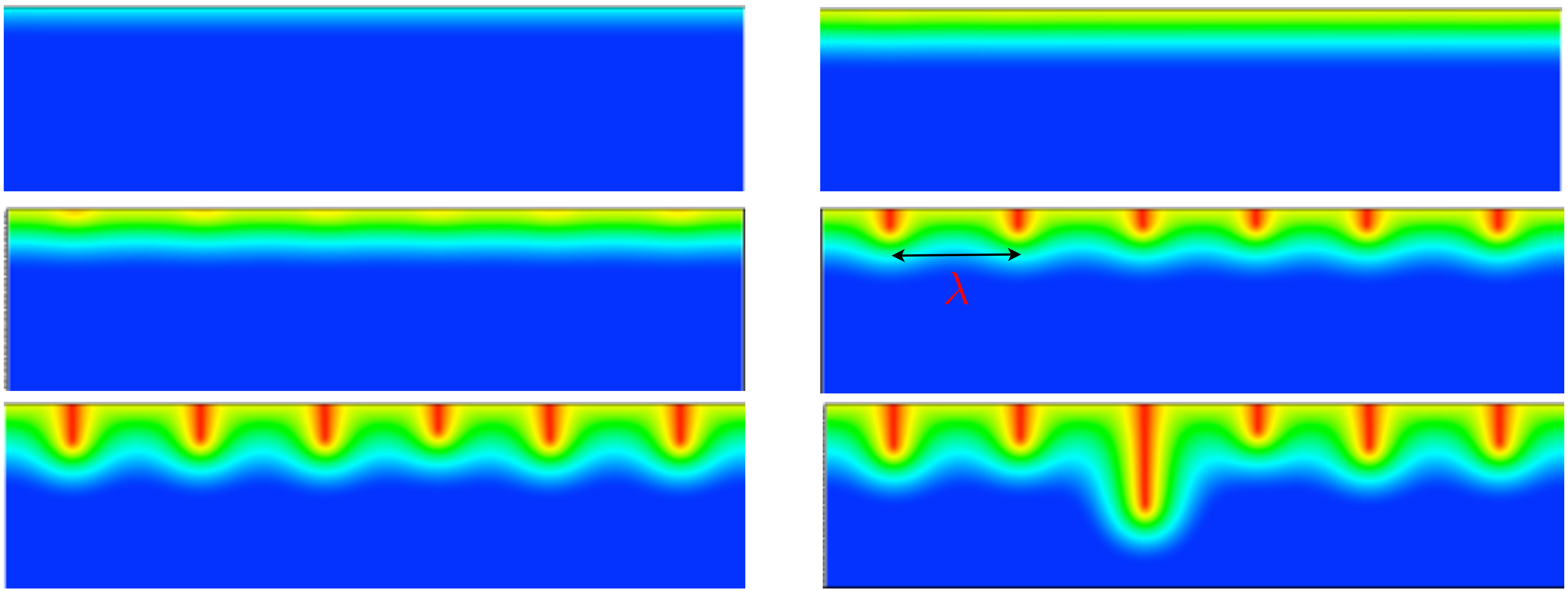
(b) Value of the computed damage field.
Numerical simulation: 20 × 10^6 d.o.f., mesh size h = .01mm

Case $\boxed{T_0 - T_1 \le \frac{\sigma_c}{\mathsf{a}\mathsf{E}_0}}$: no damage, no crack

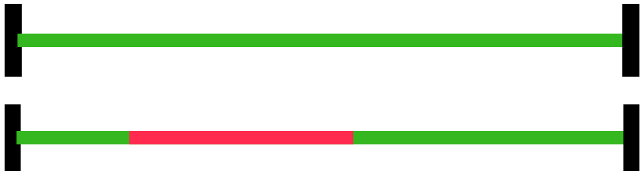
$$\boxed{\sigma_c = \sqrt{\mathsf{w}_1 \mathsf{E}_0}}$$

Case $\boxed{T_0 - T_1 > \frac{\sigma_c}{\mathsf{a}\mathsf{E}_0}}$

$$\boxed{\lambda \sim \frac{\sigma_c}{\mathsf{E}_0 \mathsf{a} (T_0 - T_1)} \ell}$$



Construction of a solution with damage localization in 1D

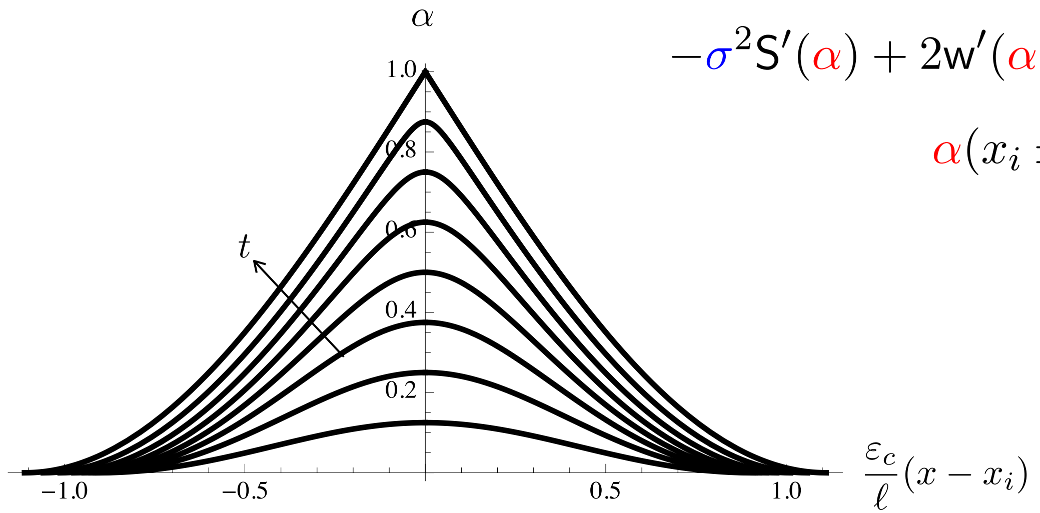


At $t = 0$, $\sigma = \sigma_c$, $\alpha(x) = 0$

At $t > 0$

1. σ decreases from σ_c to 0,
2. damage localization in $(x_i - D, x_i + D)$

- damage localization



$$-\sigma^2 \mathsf{S}'(\alpha) + 2\mathsf{w}'(\alpha) - 2\mathsf{w}_1 \ell^2 \alpha'' = 0 \quad \text{in} \quad (x_i - D, x_i + D)$$

$$\alpha(x_i \pm D) = \alpha'(x_i \pm D) = 0$$

first integral

$$\boxed{\mathsf{w}_1 \ell^2 \alpha'^2 = 2\mathsf{w}(\alpha) - \sigma^2(\mathsf{S}(\alpha) - \mathsf{S}(0))}$$

- until rupture

$$\boxed{\alpha(x_i) = 1}$$

damage profile

$$\boxed{|x - x_i| = \ell \int_\alpha^1 \sqrt{\frac{\mathsf{w}_1}{2\mathsf{w}(\beta)}} d\beta, \qquad D = \ell \int_0^1 \sqrt{\frac{\mathsf{w}_1}{2\mathsf{w}(\alpha)}} d\alpha}$$

dissipated energy

$$\boxed{G_c = 2\ell \int_0^1 \sqrt{2\mathsf{w}_1 \mathsf{w}(\alpha)} d\alpha}$$

Damage with plasticity

Damage alone

$$\boxed{\mathsf{W}_D = \frac{1}{2}\,\mathsf{E}(\alpha)\varepsilon\cdot\varepsilon + \mathsf{w}(\alpha) + \mathsf{w}_1\ell^2\nabla\alpha\cdot\nabla\alpha}$$

Plasticity alone

$$\boxed{\mathsf{W}_P = \frac{1}{2}\,\mathsf{E}(\varepsilon-\varepsilon^p)\cdot(\varepsilon-\varepsilon^p) + \sigma_Y\ p}$$

$$\dot p = \sqrt{\frac{2}{3}\dot\varepsilon^p\cdot\dot\varepsilon^p}$$

Damage with Plasticity

$$\boxed{\mathsf{W} = \frac{1}{2}\,\mathsf{E}(\alpha)(\varepsilon-\varepsilon^p)\cdot(\varepsilon-\varepsilon^p) + \mathsf{w}(\alpha) + \boxed{\sigma_Y(\alpha)p} + \mathsf{w}_1\ell^2\nabla\alpha\cdot\nabla\alpha}$$

$$\boxed{\sigma_Y(\alpha)\text{ decreasing from }\sigma_Y^0\text{ to }0}$$

Evolution law (variational approach)

✓ Stress-strain relation

$$\boxed{\sigma = \mathsf{E}(\alpha)(\varepsilon - \varepsilon^p)}$$

✓ Plasticity criterion

$$\boxed{\sqrt{\frac{3}{2}\sigma^D \cdot \sigma^D} \leq \boxed{\sigma_Y(\alpha)}}$$

$$\boxed{\text{Flow rule : } \dot{\varepsilon}^p = \dot{p}\,\frac{\sigma^D}{\sigma_Y(\alpha)}}$$

✓ Damage criterion

$$\boxed{\tfrac{1}{2}\,\mathsf{S}'(\alpha)\sigma \cdot \sigma + 2\mathsf{w}_1\ell^2\Delta\alpha \leq \mathsf{w}'(\alpha) + \boxed{\sigma_Y'(\alpha)p}}$$

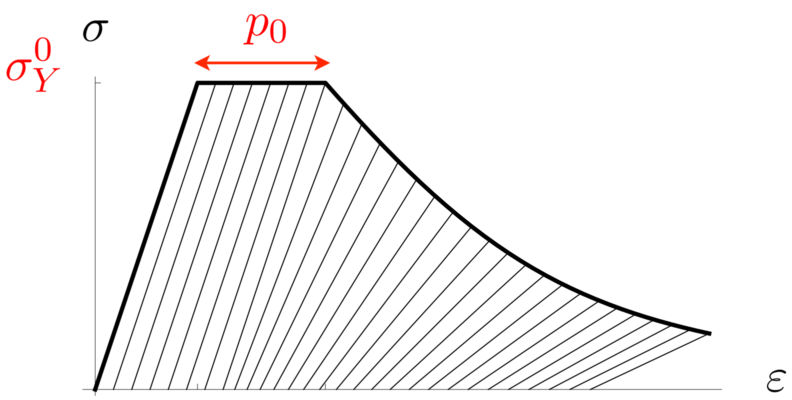
2 critical stress

$$\boxed{\sigma_Y^0 := \sigma_Y(0)}$$

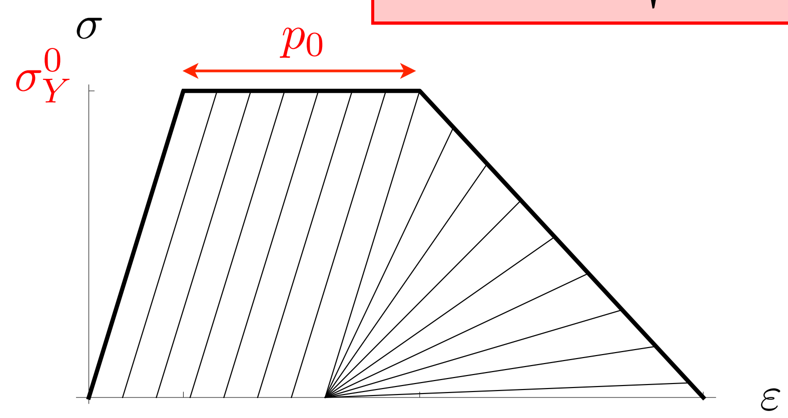
$$\boxed{\sigma_c := \sqrt{\frac{2\mathsf{w}'(0)}{\mathsf{S}'(0)}}}$$

Uniaxial local response

✓ Case where plasticity occurs before damage $\boxed{\sigma_Y^0 < \sigma_c := \sqrt{\dfrac{2\mathsf{w}'(0)}{\mathsf{S}'(0)}}}$



E–P–DP



E–P–D–R

Evolution of the damage criterion during the P stage

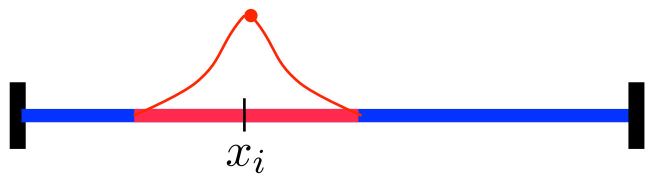
$$\frac{1}{2}\,\mathsf{S}'(0){\sigma_Y^0}^2 \leq \mathsf{w}'(0) - \boxed{\left|\sigma_Y'(0)\right| p}$$

Onset of damage : $\boxed{p_0 = \dfrac{\mathsf{S}'(0)}{2\left|\sigma_Y'(0)\right|}(\sigma_c^2 - {\sigma_Y^0}^2)}$

Then damage alone or damage with plasticity according to $\mathsf{w}(\alpha)$, $\mathsf{S}(\alpha)$, $\sigma_Y(\alpha)$ properties

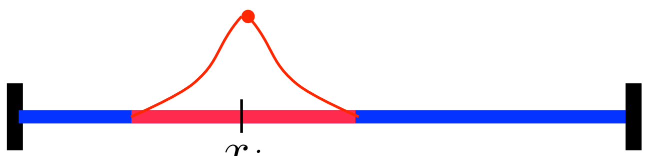
Response with damage localization

At $t = 0$, $\sigma = \sigma_Y^0$, $\alpha(x) = 0$, $\varepsilon^p(x) = p(x) = p_0$



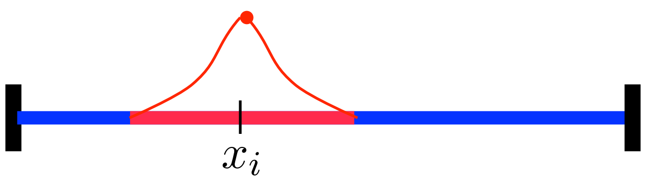
At $t > 0$

1. σ decreases from σ_Y^0 to 0,
2. damage localization in $(x_i - D, x_i + D)$
3. $\alpha(x)$ maximal at x_i



In the damage zone except at x_i :

$$\begin{cases} \boxed{\sigma < \sigma_Y(\alpha(x))} \quad \Longrightarrow \quad \boxed{p(x) = p_0} \\ \boxed{\frac{1}{2}\,\mathsf{S}'(\alpha)\sigma^2 + 2\mathsf{w}_1\ell^2\alpha'' = \mathsf{w}'(\alpha) + \sigma_Y'(\alpha)p_0} \quad \Longrightarrow \quad \boxed{\text{first integral}} \end{cases}$$



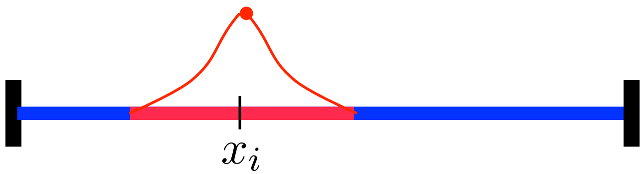
In the damage zone except at x_i :

$$\begin{cases} \sigma < \sigma_Y(\alpha(x)) \quad \Longrightarrow \quad p(x) = p_0 \\ \\ \frac{1}{2}\,\mathsf{S}'(\alpha)\sigma^2 + 2\mathsf{w}_1\ell^2\alpha'' = \mathsf{w}'(\alpha) + \sigma_Y'(\alpha)p_0 \quad \Longrightarrow \quad \text{first integral} \end{cases}$$

At x_i : Nucleation of a cohesive crack at the center of the damage zone

$$\begin{cases} \alpha(x_i) = \alpha^* \\ \\ \boxed{\sigma = \sigma_Y(\alpha^*)} \end{cases} \Rightarrow \boxed{\text{jump of } \alpha'} \Rightarrow \boxed{\text{plasticity concentration}} \Rightarrow \boxed{\text{displacement jump}}$$

$$\frac{1}{2}\,\mathsf{S}'(\alpha)\sigma^2 + 2\mathsf{w}_1\ell^2\alpha'' = \mathsf{w}'(\alpha) + \sigma_Y'(\alpha)p$$



In the damage zone except at x_i :

$$\begin{cases} \sigma < \sigma_Y(\alpha(x)) \quad \Longrightarrow \quad p(x) = p_0 \\ \\ \frac{1}{2}\,\mathsf{S}'(\alpha)\sigma^2 + 2\mathsf{w}_1\ell^2\alpha'' = \mathsf{w}'(\alpha) + \sigma_Y'(\alpha)p_0 \quad \Longrightarrow \quad \text{first integral} \end{cases}$$

At x_i :

$$\begin{cases} \alpha(x_i) = \alpha^* \\ \\ \sigma = \sigma_Y(\alpha^*) \end{cases} \quad \Rightarrow \text{ jump of } \alpha' \ \Rightarrow \text{ plasticity concentration } \Rightarrow \text{ displacement jump}$$

$$\frac{1}{2}\,\mathsf{S}'(\alpha)\sigma^2 + 2\mathsf{w}_1\ell^2\alpha'' = \mathsf{w}'(\alpha) + \sigma_Y'(\alpha)p$$

$$\begin{cases} \text{damage criterion : } \quad \boxed{2\mathsf{w}_1\ell^2[\![\alpha']\!] = \sigma_Y'(\alpha^*)[\![u]\!]} \\ \\ \text{first integral : } \quad \boxed{\sqrt{\mathsf{w}_1}\ell[\![\alpha']\!] = -2\sqrt{\mathsf{w}(\alpha^*) - (\sigma_Y^0 - \sigma_Y(\alpha^*))p_0 - \frac{1}{2}\left(\mathsf{S}(\alpha^*) - \mathsf{S}_0\right)\sigma^2}} \end{cases} \Rightarrow \boxed{\text{cohesive law}}$$

E_0 ν σ_Y^0

Example

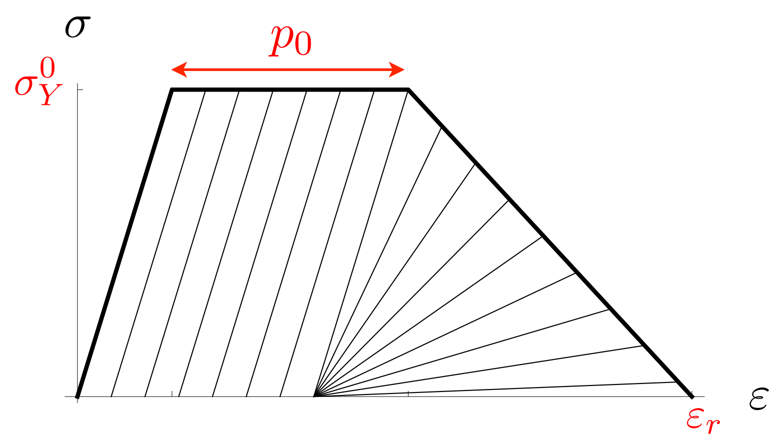
$\mathsf{k} > 1$ $\theta = \dfrac{\sigma_Y^0}{\sigma_c} < 1$

$$\boxed{\mathsf{E}(\alpha) = \frac{(1-\alpha)^2}{\mathsf{k} - (\mathsf{k}-1)(1-\alpha)^2}\mathsf{E}_0}$$

$$\boxed{\mathsf{w}(\alpha) = \frac{\mathsf{k}\sigma_c^2}{2\mathsf{E}_0}(1-(1-\alpha)^2)}$$

$$\boxed{\sigma_Y(\alpha) = (1-\alpha)^2\sigma_Y^0}$$

Homogeneous response



$$p_0 = \frac{\mathsf{k}(1-\theta^2)\sigma_Y^0}{2\theta\mathsf{E}_0}$$

$$\varepsilon_r = \frac{\mathsf{k}(1+\theta^2)\sigma_Y^0}{2\theta\mathsf{E}_0}$$

E_0 ν σ_Y^0

Example

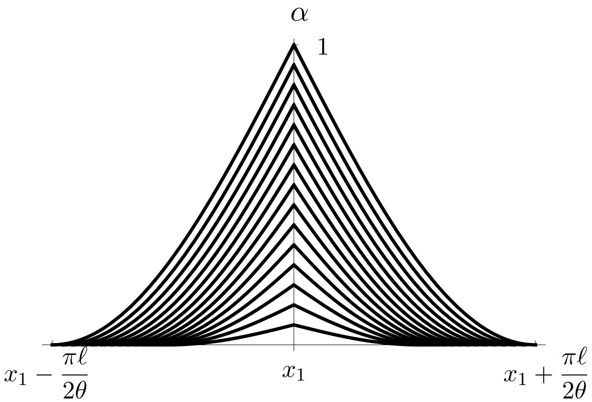
$\mathsf{k} > 1$ $\theta = \dfrac{\sigma_Y^0}{\sigma_c} < 1$

$$\boxed{\mathsf{E}(\alpha) = \frac{(1-\alpha)^2}{\mathsf{k} - (\mathsf{k}-1)(1-\alpha)^2}\mathsf{E}_0}$$

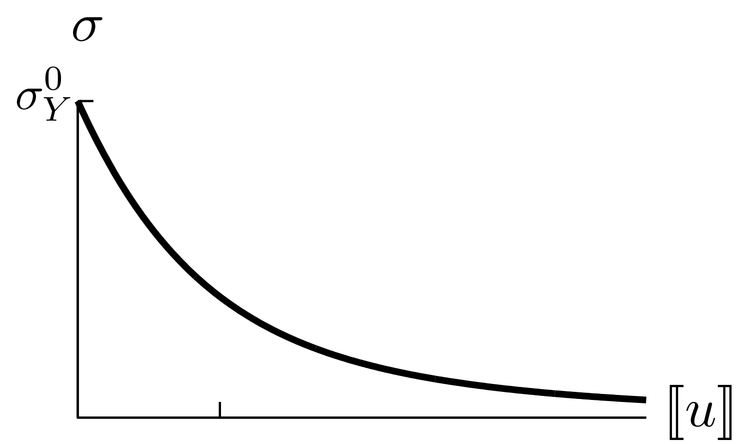
$$\boxed{\mathsf{w}(\alpha) = \frac{\mathsf{k}\sigma_c^2}{2\mathsf{E}_0}(1-(1-\alpha)^2)}$$

$$\boxed{\sigma_Y(\alpha) = (1-\alpha)^2\sigma_Y^0}$$

Response with damage localization



damage profile



$$\boxed{\frac{[\![u]\!]}{\ell} = \frac{\mathsf{k}\sigma_c}{\mathsf{E}_0}\left(\sqrt{\frac{\sigma_Y^0}{\sigma}} - \sqrt{\frac{\sigma}{\sigma_Y^0}}\right)}$$

cohesive law

$$\boxed{G_c = \frac{\pi\mathsf{k}\theta}{2}\frac{\sigma_c^2\ell}{\mathsf{E}_0}}$$

dissipated energy to create a crack

✓ numerical method

– time discretization

$$\mathcal{E}_i(u, \alpha, \varepsilon^p) = \int_\Omega \Big(\tfrac{1}{2}\, \mathsf{E}(\alpha)(\varepsilon(u) - \varepsilon^p) \cdot (\varepsilon(u) - \varepsilon^p + \mathsf{w}(\alpha) + \mathsf{w}_1 \ell^2 \nabla\alpha \cdot \nabla\alpha \Big) dx + \int_\Omega \sigma_Y(\alpha) \Big(p_{i-1} + \|\varepsilon^p - \varepsilon^p_{i-1}\| \Big) dx - f_i(u)$$

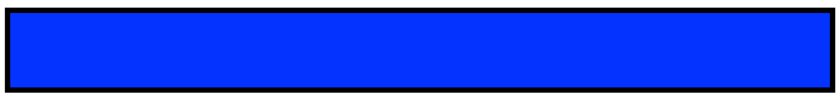
– alternate minimization algorithm:

$$u_i^n = \mathrm{argmin}_u \mathcal{E}_i(u, \alpha_i^n, (\varepsilon^p)_i^n)$$

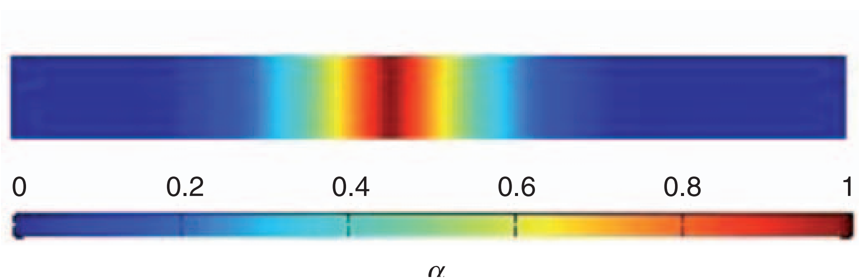
$$\alpha_i^{n+1} = \mathrm{argmin}_{\alpha \geq \alpha_{i-1}} \mathcal{E}_i(u_i^n, \alpha, (\varepsilon^p)_i^n)$$

$$(\varepsilon^p)_i^{n+1} = \mathrm{argmin}_{\varepsilon^p} \mathcal{E}_i(u_i^n, \alpha_i^{n+1}, \varepsilon^p)$$ local problem=projection

without plasticity

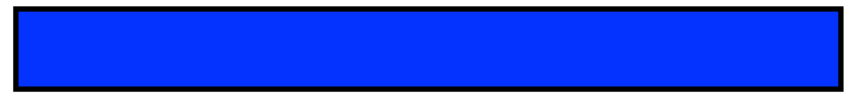


(i) elastic phase

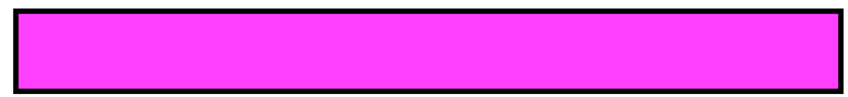


(ii) fracture

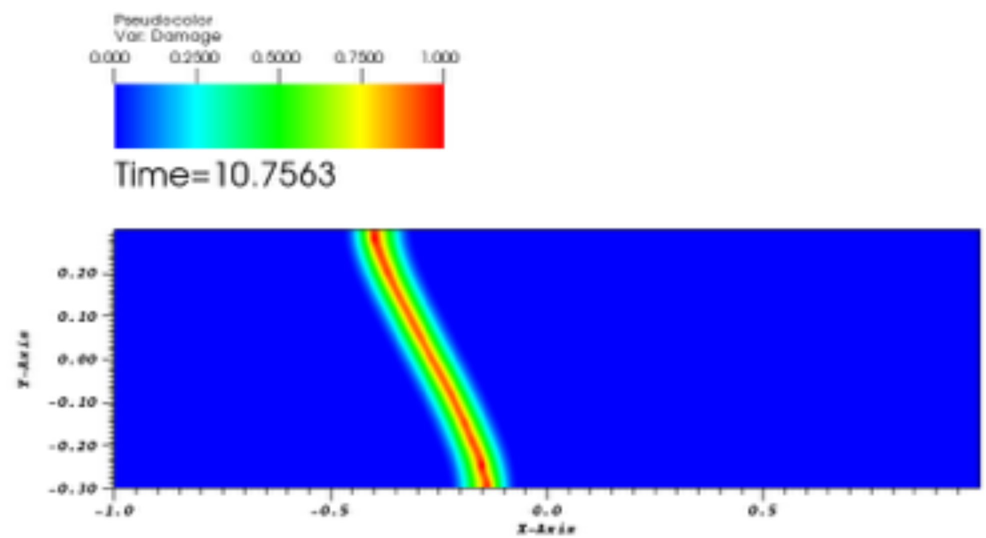
with plasticity



(i) elastic phase



(ii) plastic phase



(iii) fracture

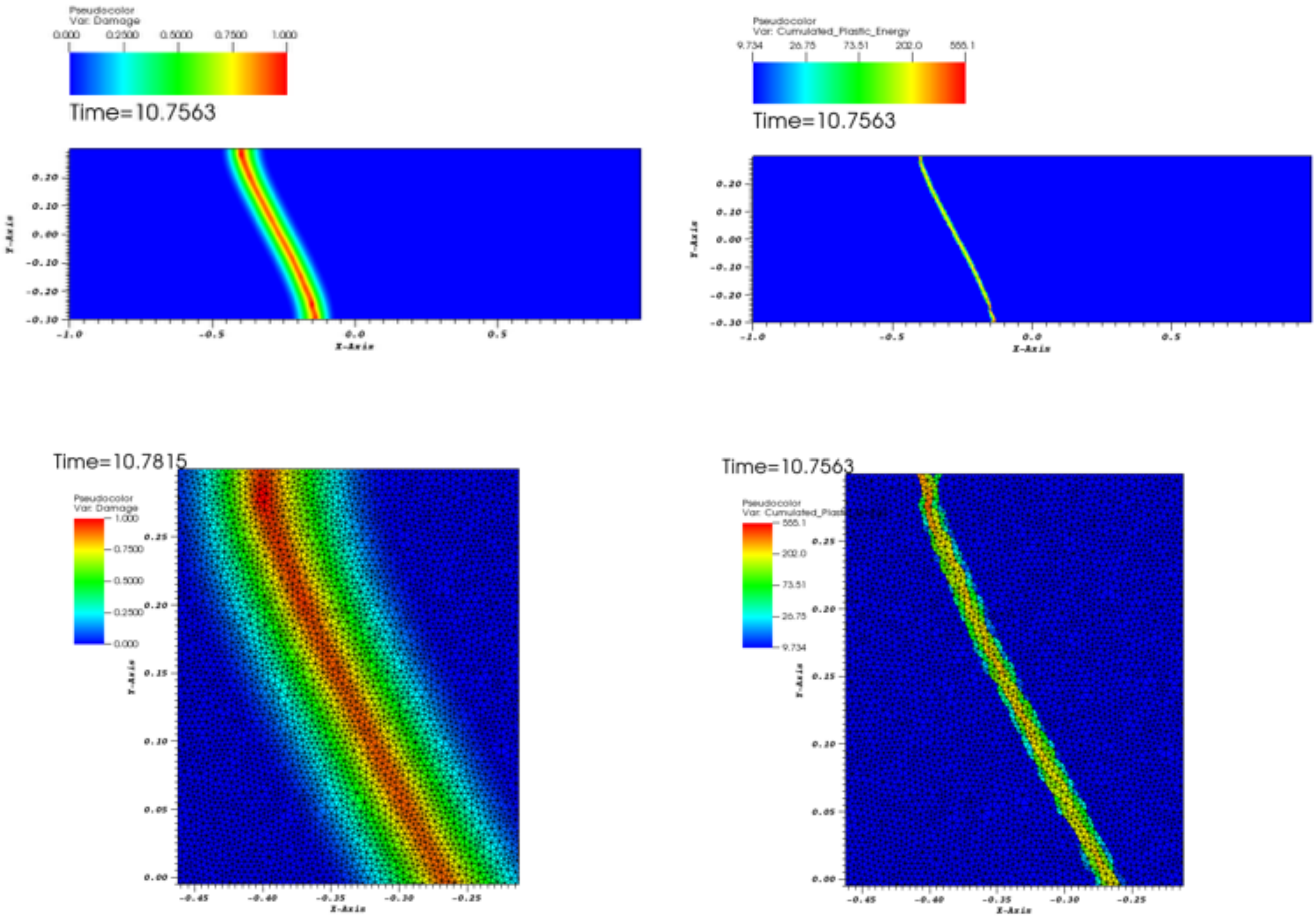
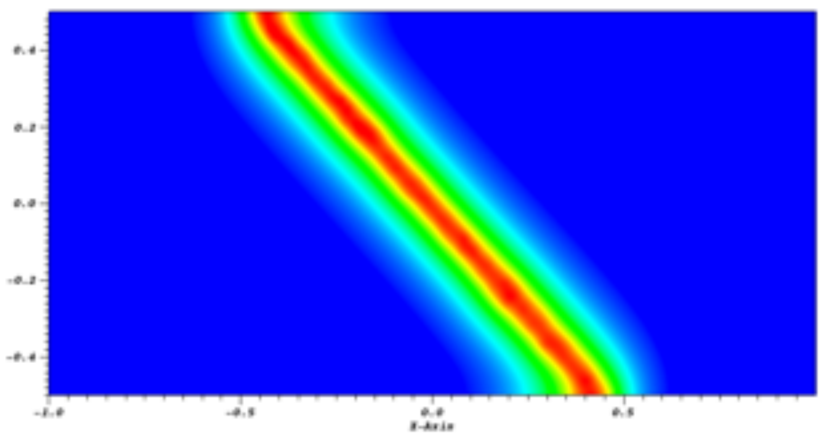
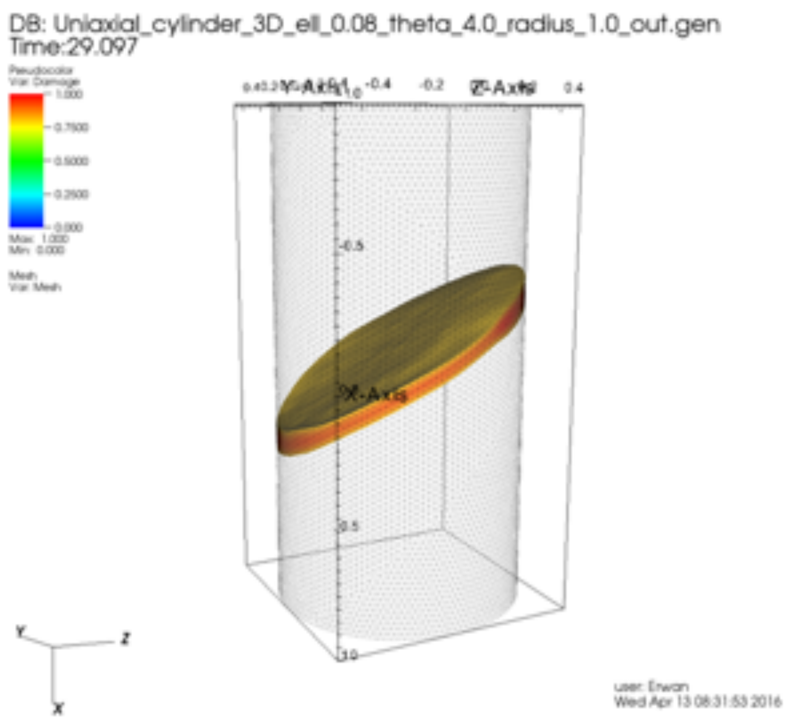

Pseudocolor
Var: Damage
0.000 0.2500 0.5000 0.7500 1.000
Time=10.7563
Y-Axis
X-Axis
Pseudocolor
Var: Cumulated_Plastic_Energy
9.734 26.75 73.51 202.0 555.1
Time=10.7563
Y-Axis
X-Axis
Time=10.7815
Pseudocolor
Var: Damage
1.000 0.7500 0.5000 0.2500 0.000
Y-Axis
X-Axis
Time=10.7563
Pseudocolor
555.1 202.0 73.51 26.75 9.734
Y-Axis
X-Axis

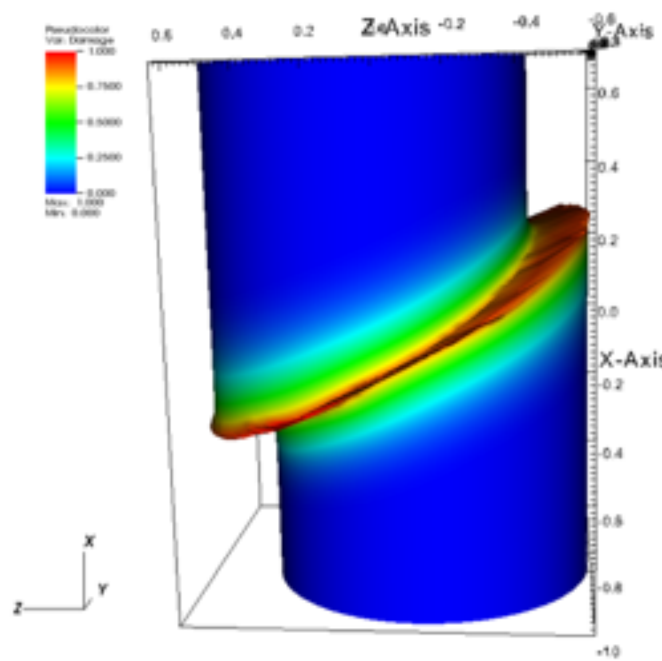
Illustration of ductile cracks:



Ductile response (slant crack 45°)
in 2D plane strain theory for VM plasticity.

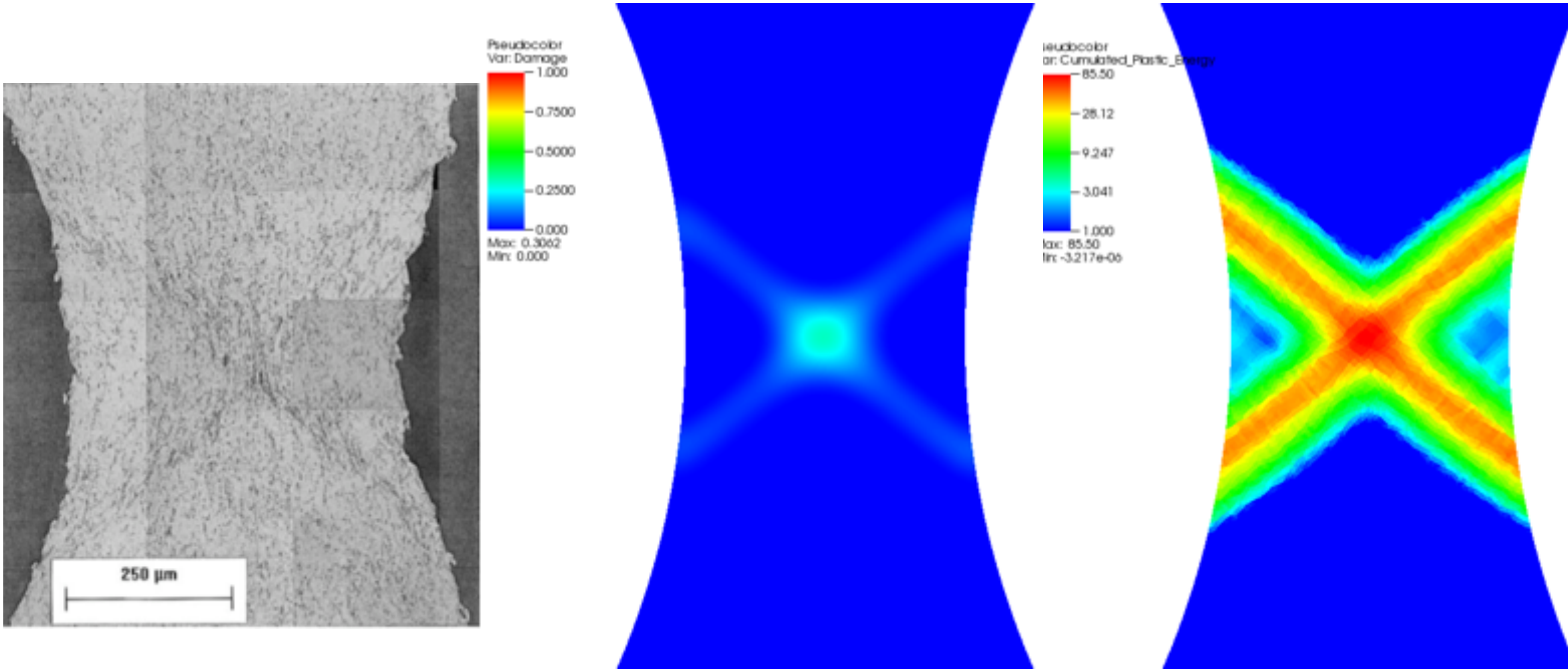
Cylinder in compression 3D





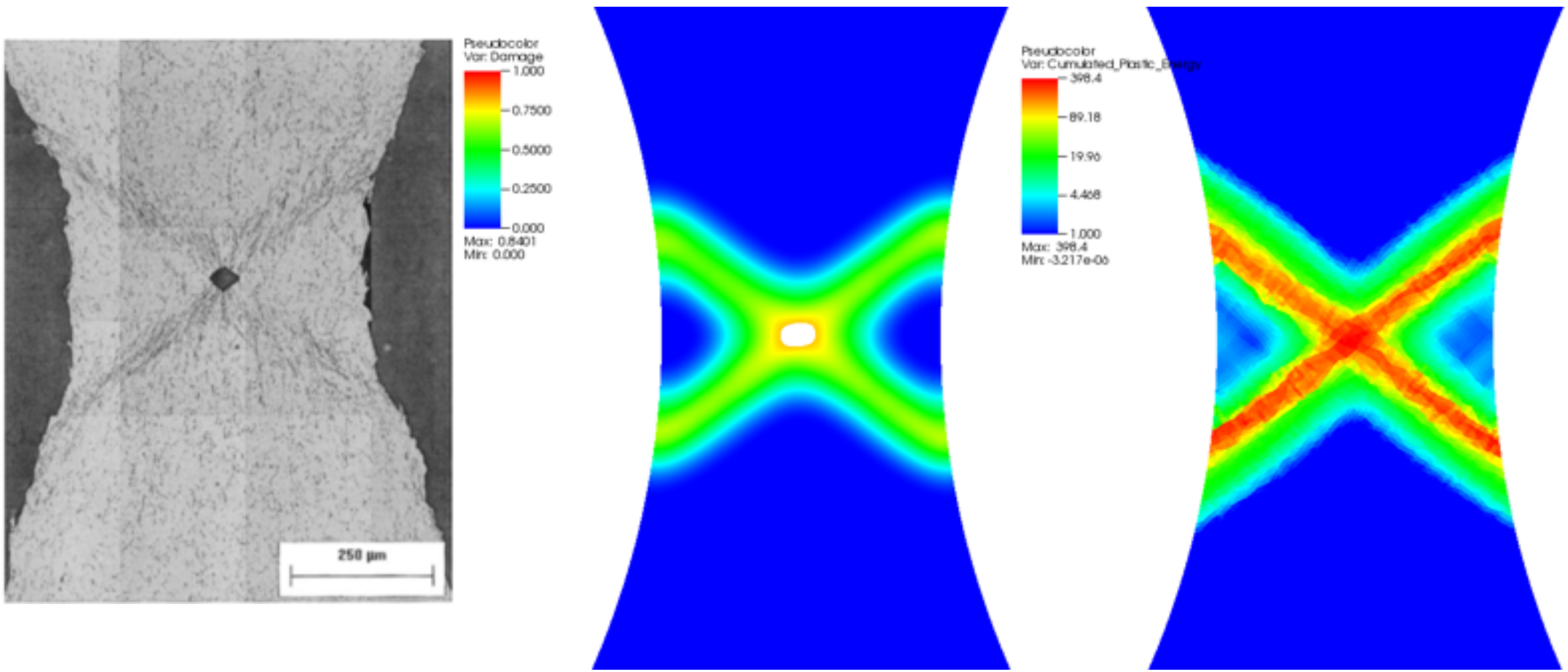
$$\mathcal{E}(\mathrm{u},\alpha,p)=\frac{1}{2}a(\alpha)\mathbb{A}(e(\mathrm{u})-p):(e(\mathrm{u})-p)+\frac{G_c}{4c_w}\Big(\frac{w(\alpha)}{\ell}+\ell|\nabla\alpha|^2\Big)+b(\alpha)\int_0^t\sup_{\substack{||\sigma_D||\leq\sigma_p\\ \mathrm{tr}(p)=0}}\{\sigma:\dot{p}\}\mathrm{d}t,$$

$a(\alpha)=b(\alpha)=(1-\alpha)^2$ $\qquad$ $\sigma_c/\sigma_p=4$ $\qquad$ $\ell/D=0.1$



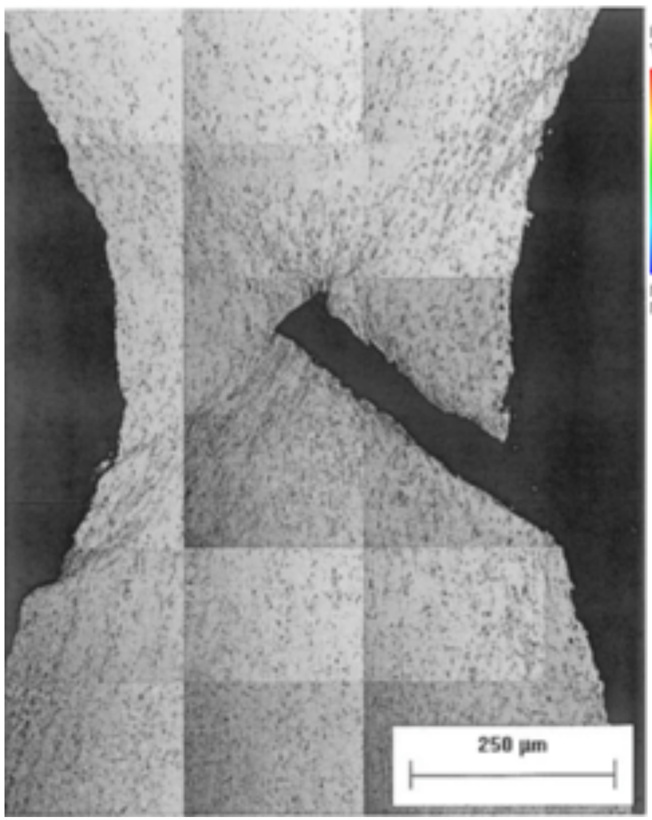
$$\mathcal{E}(\mathrm{u},\alpha,p)=\frac{1}{2}a(\alpha)\mathbb{A}(e(\mathrm{u})-p):(e(\mathrm{u})-p)+\frac{G_c}{4c_w}\Big(\frac{w(\alpha)}{\ell}+\ell|\nabla\alpha|^2\Big)+b(\alpha)\int_0^t\sup_{\substack{||\sigma_D||\leq\sigma_p\\ \mathrm{tr}(p)=0}}\{\sigma:\dot{p}\}\mathrm{d}t,$$

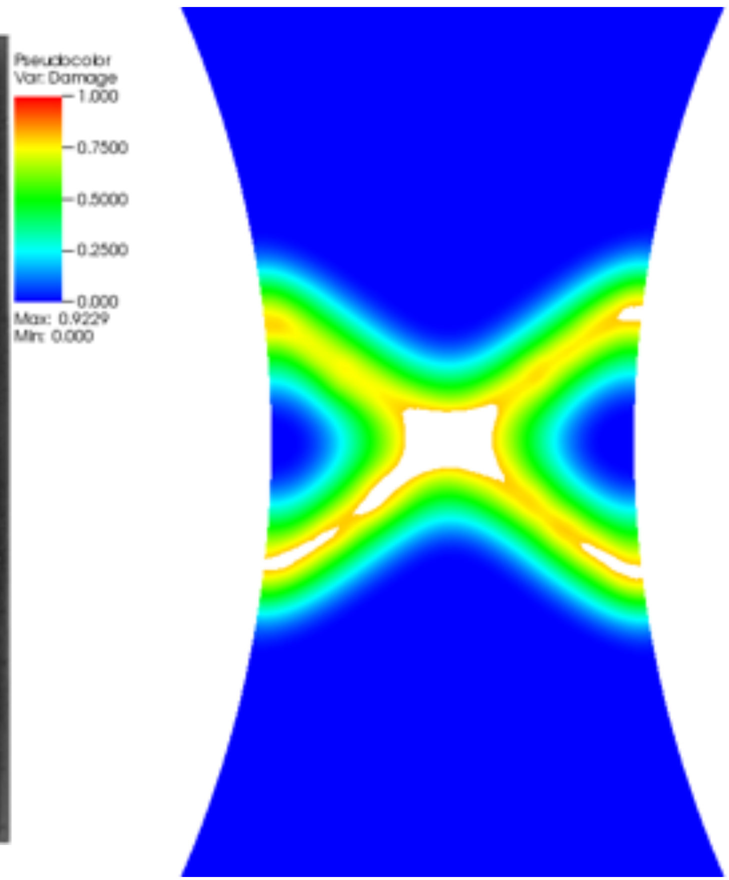
$$a(\alpha)=b(\alpha)=(1-\alpha)^2 \qquad \sigma_c/\sigma_p=4 \qquad \ell/D=0.1$$

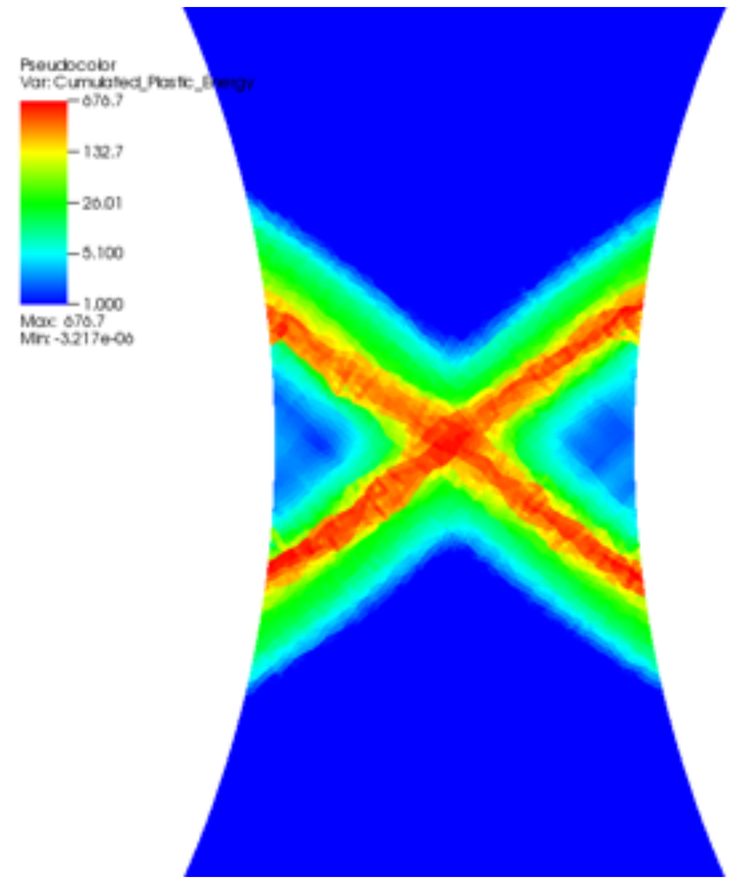


$$\mathcal{E}(\mathrm{u},\alpha,p)=\frac{1}{2}a(\alpha)\mathbb{A}(e(\mathrm{u})-p):(e(\mathrm{u})-p)+\frac{G_c}{4c_w}\Big(\frac{w(\alpha)}{\ell}+\ell|\nabla\alpha|^2\Big)+b(\alpha)\int_0^t\sup_{\substack{||\sigma_D||\leq\sigma_p\\ \mathrm{tr}(p)=0}}\{\sigma:\dot{p}\}\mathrm{d}t,$$

$$a(\alpha)=b(\alpha)=(1-\alpha)^2 \qquad \sigma_c/\sigma_p=4 \qquad \ell/D=0.1$$

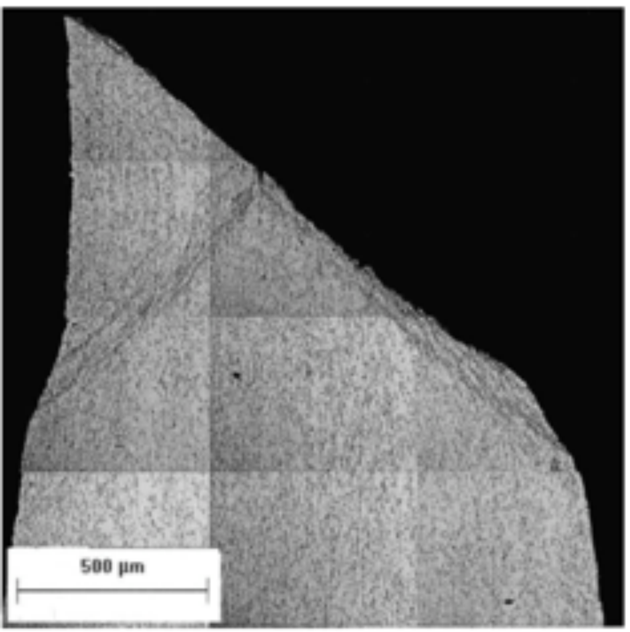


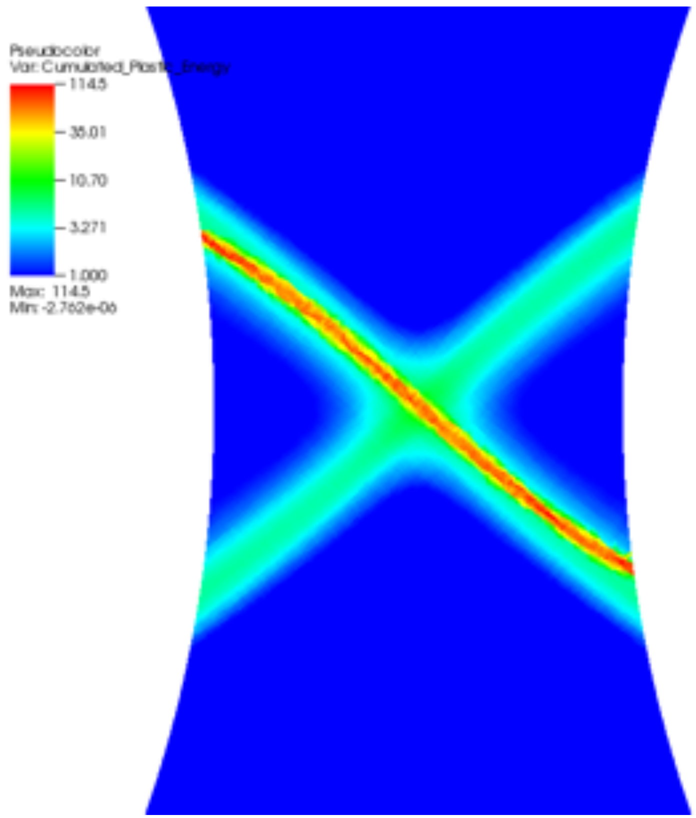


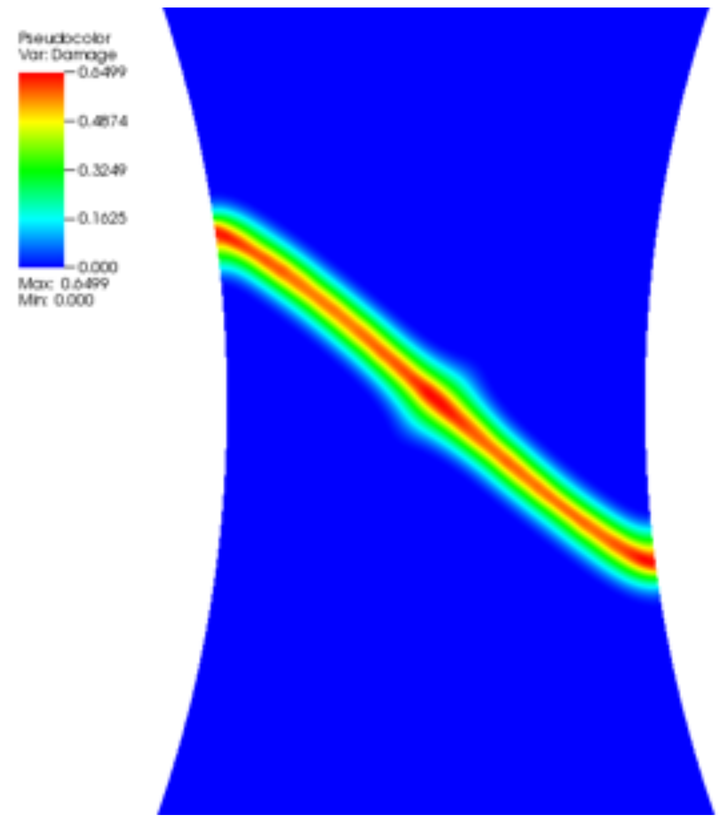


$$\mathcal{E}(\mathrm{u},\alpha,p) = \frac{1}{2}a(\alpha)\mathbb{A}(e(\mathrm{u})-p):(e(\mathrm{u})-p) + \frac{G_c}{4c_w}\left(\frac{w(\alpha)}{\ell} + \ell|\nabla\alpha|^2\right) + b(\alpha)\int_0^t \sup_{\substack{||\sigma_D||\leq\sigma_p \\ \mathrm{tr}(p)=0}} \{\sigma:\dot{p}\}\mathrm{d}t,$$

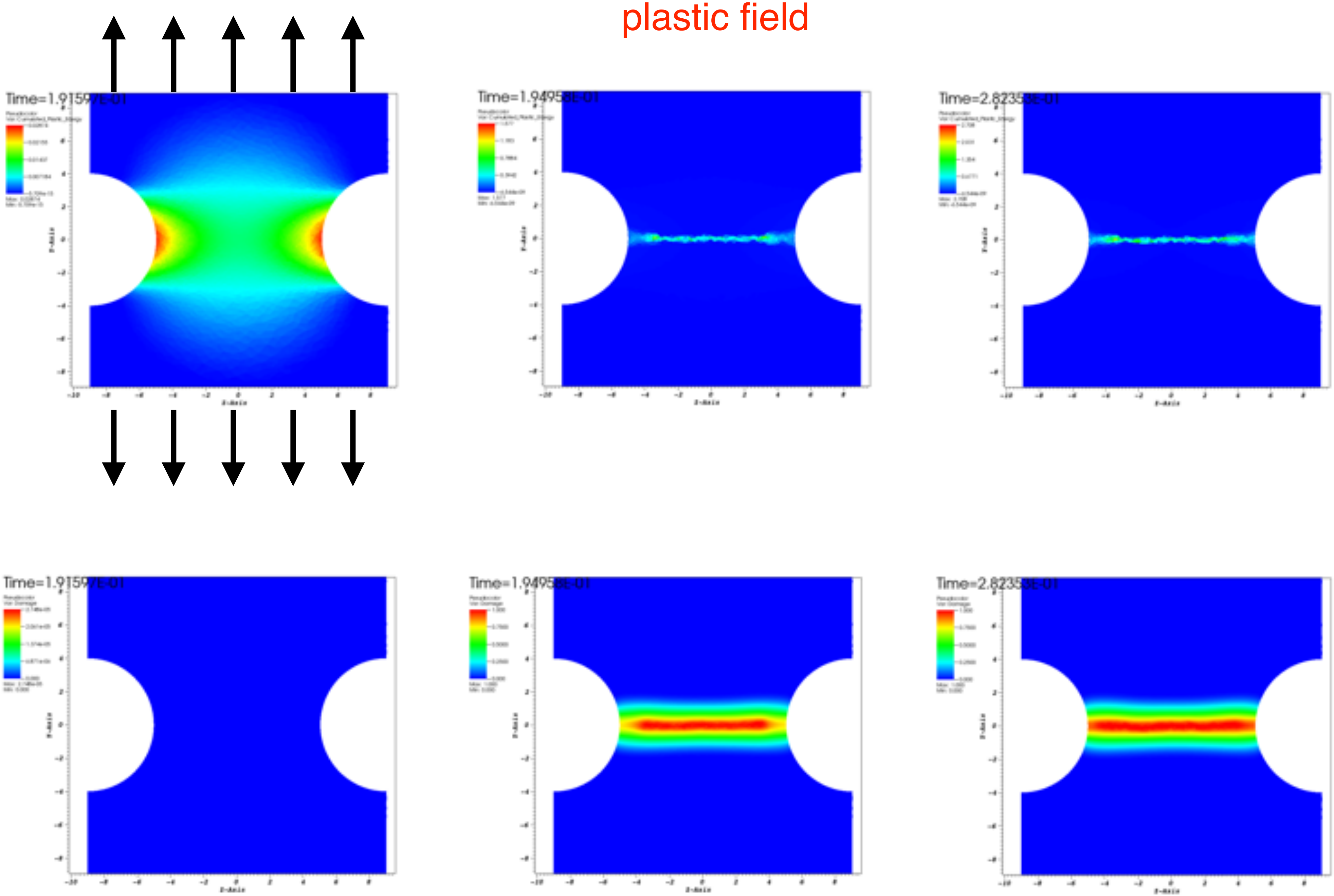
$$a(\alpha) = b(\alpha) = (1-\alpha)^2 \qquad \sigma_c/\sigma_p = 8 \qquad \ell/D = 0.05$$







plastic field



damage field