

# The mod $p$ motivic Steenrod algebra in characteristic $p$

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# Outline

Note: Work in progress.

1. Setup
2. Hopkins–Morel isomorphism
3. Motivic dual Steenrod algebra
4. Reduction step
5. Sketch of ideas

# Setup

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Motivic spaces = localization of simplicial presheaves on  $\mathrm{Sm}_S$  at  $\mathbb{A}^1$ -equivalences and Nisnevich hypercovers.

Motivic spectra = stabilization of pointed motivic spaces with respect to  $S^1 \wedge -$  and  $\mathbb{G}_m \wedge -$ .

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Let  $\mathrm{SH}(S)$  denote the motivic stable homotopy category over  $S$ . It is a compactly generated tensor triangulated category.

# Bigraded spheres

Motivic spheres:

$$S^{p,q} = (S^1)^{\wedge(p-q)} \wedge_S (\mathbb{G}_m)^{\wedge q}$$

and corresponding suspension functors:

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**Example.** •  $\mathbb{G}_m = \mathbb{A}^1 - \{0\} = S^{1,1}$ .

•  $\mathbb{A}^n - \{0\} \simeq S^{2n-1,n}$ .

•  $\mathbb{P}^1 \simeq \mathbb{A}^1 / (\mathbb{A}^1 - \{0\}) \simeq S^1 \wedge \mathbb{G}_m = S^{2,1}$ .



# Motivic Eilenberg-MacLane spectra

$H\mathbb{Z}$  is a motivic spectrum representing motivic cohomology in  $\mathrm{SH}(S)$ .  $H\mathbb{Z}$  is an  $E_\infty$  ring spectrum, in an essentially unique way. Likewise for  $H\mathbb{F}_p$ .

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**Notation:** In case of ambiguity, write the dependency on the base scheme as  $H\mathbb{F}_p^S$ .

**Remark.**  $H\mathbb{F}_l$  has complicated homotopy:

$$\begin{aligned}\pi_{p,q}H\mathbb{F}_l &= \mathrm{SH}(S)(S^{p,q}, H\mathbb{F}_l) \\ &= \mathrm{SH}(S)(S^{0,0}, \Sigma^{-p,-q}H\mathbb{F}_l) \\ &= H^{-p,-q}(S; \mathbb{F}_l),\end{aligned}$$

motivic cohomology of the base scheme  $S$ .

# Hopkins–Morel isomorphism

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The orientation map  $MU \rightarrow H\mathbb{Z}$  induces a map

$$MU/(a_1, a_2, \dots) \xrightarrow{\cong} H\mathbb{Z}$$

which is an equivalence.

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**Theorem** (Hopkins–Morel 2004; Hoyois 2015). Let  $S$  be essentially smooth over a field  $\mathbb{k}$ .

1. In the case  $\text{char}(\mathbb{k}) = 0$ , then  $\Phi$  is an equivalence.
2. In the case  $\text{char}(\mathbb{k}) = p$ , then  $\Phi$  becomes an equivalence after inverting  $p$ .

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**Remark.** This theorem has interesting applications to the slice filtration.

## Hopkins–Morel, III

**Key step:** For any prime number  $l \neq p$ ,

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**Key ingredient:** Motivic Steenrod algebra and its dual.

# Motivic dual Steenrod algebra

There are certain classes in  $\pi_{*,*}(H\mathbb{F}_l \wedge H\mathbb{F}_l)$

$$\tau_i, \text{ with } |\tau_i| = (2l^i - 1, l^i - 1), i \geq 0$$

$$\xi_i, \text{ with } |\xi_i| = (2l^i - 2, l^i - 1), i \geq 1.$$

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Consider sequences  $I = (\epsilon_0, r_1, \epsilon_1, r_2, \epsilon_2, \dots)$  with  $\epsilon_i \in \{0, 1\}$ ,  $r_i \geq 0$ , and only finitely many non-zero terms. Consider monomials of the form

$$\tau_0^{\epsilon_0} \xi_1^{r_1} \tau_1^{\epsilon_1} \xi_2^{r_2} \tau_2^{\epsilon_2} \cdots \in \pi_{*,*}(H\mathbb{F}_l \wedge H\mathbb{F}_l).$$

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Consider the induced map of  $H\mathbb{F}_l$ -modules

$$\bigoplus_{\text{such sequences } I} \Sigma^{p_I, q_I} H\mathbb{F}_l \xrightarrow{\psi^S} H\mathbb{F}_l \wedge H\mathbb{F}_l.$$



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Note that the indexing set is the same for any base scheme  $S$ .

## Motivic dual Steenrod algebra, II

**Theorem** (Voevodsky 2003; Hoyois–Kelly–Østvær 2013). Assume  $l$  is invertible on the base scheme  $S$ . Then the map  $\psi^S$  is an equivalence of  $H\mathbb{F}_l$ -modules.

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**Goal.** Show that  $\psi^S$  is an equivalence in the case  $S = \text{Spec}(\mathbb{k})$  where  $\mathbb{k}$  is a field of characteristic  $p$ , and  $l = p$ .

# Strategy

Let  $\mathbb{k}$  be a field of characteristic  $p$ .

Let  $R$  be a discrete valuation ring  $R$  having  $\mathbb{k}$  as residue field, and a fraction field  $Q = \text{Frac}(R)$  of characteristic zero.

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**Example.**  $\mathbb{k} = \mathbb{F}_p$ ,  $R = \mathbb{Z}_p$ , the  $p$ -adic integers, and  $Q = \mathbb{Q}_p$ , the  $p$ -adic rationals.

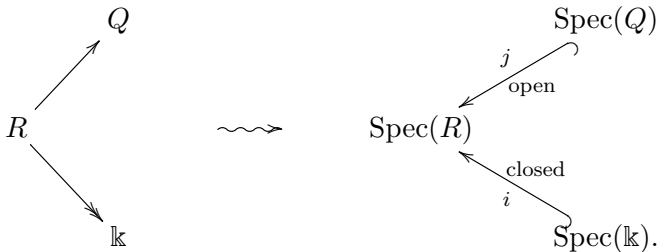
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Consider the ring maps and induced maps of affine schemes:



# The ingredients

What happens on the generic point  $\text{Spec}(Q)$ ?

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What happens on the closed point  $\text{Spec}(\mathbb{k})$ ?

**Proposition** (Spitzweck 2013). 1. There is an equivalence

$$i^* H\mathbb{F}_p^R \simeq H\mathbb{F}_p^{\mathbb{k}}.$$

2. There is an equivalence of  $H\mathbb{F}_p^{\mathbb{k}}$ -module spectra

$$i^! H\mathbb{F}_p^R \simeq \Sigma^{-2, -1} H\mathbb{F}_p^{\mathbb{k}}.$$



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**Proposition** (F.-Spitzweck). There is an equivalence of  $H\mathbb{F}_p^{\mathbb{k}}$ -module spectra

$$i^* j_* H\mathbb{F}_p^Q \simeq H\mathbb{F}_p^{\mathbb{k}} \oplus \Sigma^{-1, -1} H\mathbb{F}_p^{\mathbb{k}}.$$

## Reduction step

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If  $\mathbb{k}$  is a field of characteristic zero, then every object of  $\mathrm{SH}(\mathbb{k})$  is smooth (Röndigs–Østvær 2008).

## Reduction step, II

**Proposition** (F.-Spitzweck). **If**  $H\mathbb{F}_p^R$  is smooth in  $\mathrm{SH}(R)$ , then the map of  $H\mathbb{F}_p^{\mathbb{k}}$ -module spectra

$$\bigoplus_I \Sigma^{p_I, q_I} H\mathbb{F}_p^{\mathbb{k}} \xrightarrow{\psi^{\mathbb{k}}} H\mathbb{F}_p^{\mathbb{k}} \wedge H\mathbb{F}_p^{\mathbb{k}}$$

is an equivalence. In other words, the structure theorem for the dual Steenrod algebra holds for  $S = \mathrm{Spec}(\mathbb{k})$ .

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**Proof idea:** Use the previous ingredients and the six functor formalism.



## Why smoothness?

**Lemma.** Let  $f: S \rightarrow T$  be a map of schemes. For  $Y$  in  $\mathrm{SH}(T)$  and  $X$  in  $\mathrm{SH}(S)$ , consider the natural map in  $\mathrm{SH}(T)$

$$\alpha: (f_*X) \wedge_T Y \rightarrow f_*(X \wedge_S f^*Y).$$

If  $Y$  is a smooth object of  $\mathrm{SH}(T)$ , then  $\alpha$  is an isomorphism.

In other words,  $Y$  and  $f_*$  satisfy the *projection formula*

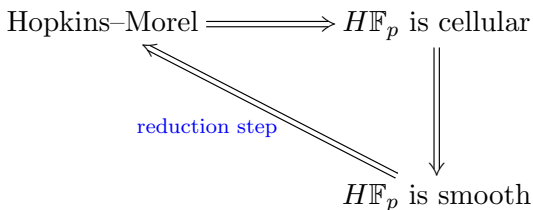
$$(f_*X) \wedge_T Y \cong f_*(X \wedge_S f^*Y).$$

## More on smoothness

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## Sketch of ideas

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$$HZ = \mathrm{hocolim}_n \Sigma^{-2n, -n} \Sigma^\infty K(\mathbb{Z}, 2n, n)$$

Note: The motivic Eilenberg-MacLane space  $K(\mathbb{Z}, p, q)$  is also known as  $K(\mathbb{Z}(q), p)$ .

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$\Rightarrow$  It suffices to show that  $\Sigma^\infty K(\mathbb{Z}, 2n, n)$  is smooth for  $n$  large enough.

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In characteristic zero, we have:

$$\mathrm{Sym}^\infty((\mathbb{P}^1)^{\wedge n}) \simeq \mathrm{Sym}^\infty(S^{2n,n}) \simeq K(\mathbb{Z}, 2n, n).$$

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$$\mathrm{Sym}^\infty(X) = \mathrm{hocolim}_k \mathrm{Sym}^k(X)$$

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# Resolutions of symmetric powers

## Plan of attack:

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- Build an appropriate iterated homotopy pushout  $P_k$  of smooth projective schemes, which implies that  $\Sigma^\infty P_k$  is smooth in  $\mathrm{SH}(R)$ .
- In the case  $X = (\mathbb{P}^1)^{\wedge n}$ , show that  $P_k$  is a good approximation of  $\mathrm{Sym}^k(X)$ .

**Thank you!**