Nonlinear water waves and wave–current interactions

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Existence of solution

- In this talk I will describe a recently derived *exact and explicit* solution to the geophysical water wave problem in the equatorial region which was presented in

Main Result

DH, Equatorially trapped nonlinear water waves in a $\beta-$plane approximation with centripetal forces, *J. Fluid Mech.* **804** (2016), R1.

- This solution is explicit in the Lagrangian formulation, and it represents a three-dimensional equatorially-trapped geophysical water wave with a constant underlying current.

Existence of solution

- A novel aspect of this work is the presence of centripetal force terms in the $\beta-$plane governing equations.
- Remarkably, we will see that centripetal force terms play a central role in facilitating the admission of both following and adverse (constant underlying) currents of any physically plausible magnitude in the exact solution we present below.

Exact (and explicit) solutions

- Exact solutions (particularly if they are explicit) offer an opportunity to generate more general, 'useful' solutions, representing more physically complex flows, by way of employing perturbative or asymptotic considerations.
- Exact solutions play an important role in the study of water waves since many apparently intangible wave motions can often be viewed as perturbations of these solutions.
- By controlling perturbations one can extract and divine qualitative, as well as quantitative, information about the dynamics of more complicated flows.

Gerstner's wave

- In 1802, the Bohemian physicist and engineer Franz Josef von Gerstner constructed an explicit example of a periodic travelling wave in water of infinite depth with a particular nonzero vorticity.
- This solution was later (independently) re-discovered by William John Macquorn Rankine, the Scottish civil engineer, physicist and mathematician, in the 1830s.
- Gerstner's wave is truly remarkable, since it is the only known explicit solution of the two-dimensional governing equations for gravity water waves with a nonflat free surface.

- An interesting aside is that the *Gerstner wave function*, based on Gerstner's explicit solution, is a commonly used method to calculate waves and simulate water in video games and movies and most 3d simulations.
- The solution we presented below generalises Gerstner's solution significantly, since it is three-dimensional in character, it incorporates geophysical effects, and additionally it admits an underlying current.
- Modifications of Gerstner's solution which incorporated a constant underlying current were investigated by Mollo-Christensen in the context of modelling billows between two fluids.

Geophysical water waves

- Geophysical fluid dynamics is the study of large-scale physical phenomena where the effect of the Earth's rotation plays a significant role and therefore must be taken into account through the presence of Coriolis forces in the governing equations.
- Geophysical processes which occur in the equatorial region are of particular interest for a number of reasons.
- Physically, the equator has the remarkable property of acting as a natural wave guide, whereby equatorially trapped zonal waves decay exponentially away from the equator in the oceans.

Equatorial wave-current interactions

- The consideration of wave-current interactions, while a highly compelling subject from a purely mathematical viewpoint, is physically highly important in a variety of contexts.
- This is particularly true in the equatorial region, where for example the presence of strong currents in the equatorial Pacific is well-documented, and they feature significantly in the geophysical dynamics of the Equatorial region.

Geophysical preliminaries

- The reference frame has its origin located at a point on the Earth's surface, as it rotates with the earth.
- The $x-$axis is the longitudinal direction (horizontally due east), the $y-$axis the latitudinal direction (horizontally due north), and the $z-$axis vertically upwards.
- The Earth is assumed to be a perfect sphere of radius $R = 6378km$, with a constant rotational speed of $\Omega = 73 \cdot 10^{-6} rad/s$, and $g = 9.8ms^{-2}$ is the standard gravitational acceleration at the earth's surface.

Governing equations

$\mathbf{U} = (u, v, w)$ velocity field, ϕ latitude, ρ density, P pressure. For small latitudes, $\sin\phi \approx \phi, \cos\phi \approx 1$, leading to:

Governing equations— modified β-plane

$$u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v = -\frac{1}{\rho} P_x \qquad (1a)$$

$$v_t + uv_x + vv_y + wv_z + \beta y u + \Omega^2 y = -\frac{1}{\rho} P_y \qquad (1b)$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u - \Omega^2 R = -\frac{1}{\rho} P_z - g, \qquad (1c)$$

Here $\beta = 2\Omega/R = 2.28 \cdot 10^{-11} m^{-1} s^{-1}$.

- We have the equation for incompressibility

$$u_x + v_y + w_z = 0. \tag{2}$$

- The boundary conditions for the fluid on the free-surface η are given by

$$w = \eta_t + u\eta_x + v\eta_y, \tag{3}$$
$$P = P_0 \qquad \text{on } y = \eta(x, y, t), \tag{4}$$

where P_0 is the (constant) atmospheric pressure.

- We assume the water to be infinitely deep, with

$$(u, v, w) \to (-c_0, 0, 0) \ \text{ as } \ z \to -\infty. \tag{5}$$

- If $c_0 \neq 0$ then there is an underlying constant uniform current.
- The equator acts as a natural wave-guide, and equatorial waves tend to be trapped.
- We model this by ensuring the wave surface profile decays in the latitudinal direction away from the equator.

Centripetal forces

- Typically, in an oceanographic context, centripetal terms are neglected due to their comparably small size.
- Remarkably, centripetal forces play an important role in ensuring that the exact solution we present below can admit all physically plausible ranges of underlying currents.
- This phenomenon whereby terms which, although negligible in general oceanographic considerations, play an important role in specific wave (and wave-current) dynamics, while curious, may be observed in other GFD contexts, e.g Rossby waves.

Balance of momemtum

- We remark that (1) requires a very specific pressure distribution in the absence of motion ($u = v = w = 0$).
- If the free surface is a surface of constant atmospheric pressure ($P_{atm} = 1$ atm $= 1.01325$ bar), then

$$P(x, y, z, t) = P_{atm} - \frac{1}{2}\rho\Omega^2 y^2 + \rho(\Omega^2 R - g)\, z$$

throughout the fluid.

Balance of momemtum

- Therefore the free surface is given by the geoid

$$z = \frac{P_{atm}}{\rho(g - \Omega^2 R)} - \frac{\Omega^2}{2(g - \Omega^2 R)} y^2 \approx \frac{P_{atm}}{\rho g} - \frac{\Omega^2}{2g} y^2$$

  since $\Omega^2 R \approx 3 \times 10^{-2}$ m/s$^2 \ll g \approx 9.8$ m/s^2.
- The above distortion from a constant value of z corresponds to a free surface following the curvature of Earth away from the equator, as the curved surface of the Earth drops below the tangent plane at the Equator.
- This is consistent with, and indeed a consequence of, the β-plane approximation.

Eulerian vs Lagrangian formulations

- In the Eulerian formalism, the flow is described by the determination of the fluid velocity at fixed points in space as a function of time.
- In the Lagrangian framework one specifies the motion of the individual particles.
- Accordingly, Lagrangian solutions can be advantageous in the sense that the fluid kinematics may be described explicitly.

Exact and explicit solution

Define Eulerian coordinates (x, y, z) in terms of Lagrangian labelling variables (q, r, s) as follows:

Exact, explicit solution

$$x = q - c_0 t - \frac{1}{k} e^{k[r-f(s)]} \sin[k(q - ct)], \tag{6a}$$

$$y = s, \tag{6b}$$

$$z = r + \frac{1}{k} e^{k[r-f(s)]} \cos[k(q - ct)]. \tag{6c}$$

Solution details

- Here $q \in \mathbb{R}$, $r \in (-\infty, r_0]$ for $r_0 < 0$, $s \in [-s_0, s_0]$, where $s_0 = \sqrt{\tilde{c}/\beta} \approx 250$km is a typical value for the equatorial radius of deformation.
- The constant $c > 0$ is the wave phase-speed, the constant k is the wavenumber and the function $f(s)$ takes the form

$$f(s) = \frac{c\beta}{2\mathfrak{g}} s^2 \qquad \text{for } \mathfrak{g} = g + 2\Omega c_0 - \Omega^2 R > 0$$

for all physically plausible values of the current c_0.

Main Result

DH, *J. Fluid Mech.* 804 (2016) R1.

The fluid motion prescribed by (6) represents an exact solution of the governing equations if the underlying current c_0 satisfies

$$c_0 < \frac{\Omega R}{2} \approx 2.33 \times 10^2 \text{m/s}. \tag{7}$$

The free-surface $z = \eta(x, y, t)$ is implicitly prescribed at the equator $(y = s = 0)$ by setting $r = r_0$ in (6). For any other fixed latitude $s \in [-s_0, s_0]$, whenever (7) holds, there exists a unique value $r(s) < r_0$ which implicitly prescribes the free-surface $z = \eta(x, s, t)$ by way of setting $r = r(s)$ in (6).

Solution properties

- The solution (6) of the $\beta-$plane governing equations (1)–(5) is three-dimensional, and very much *nonlinear*.
- Indeed, at any given latitude the free-surface profile in Eulerian coordinates $\eta(x - ct, s)$ is troichoidal.
- The steepness of the wave-profile (half amplitude $\times$ wavenumber) is given by

$$\tau(s) = e^{k(r-f(s))}.$$

- At fixed latitudes, the wave-like part of the solution (ignoring the current c_0) describes circles, and represents an equatorially trapped, travelling wave propagating in zonally due east.

Solution properties

- Furthermore, the flow prescribed by the above solution is *rotational*.
- We can calculate the vorticity explicitly to get

$$\begin{aligned}\omega &= (w_y - v_z, u_z - w_x, v_x - u_y) \\ &= \left(-s\frac{kc^2\beta}{\mathfrak{g}}\frac{e^{\xi}\sin\theta}{1-e^{2\xi}}, -\frac{2kce^{2\xi}}{1-e^{2\xi}}, s\frac{kc^2\beta}{\mathfrak{g}}\frac{e^{\xi}\cos\theta - e^{2\xi}}{1-e^{2\xi}} \right),\end{aligned}$$

where $\xi = k\,(r - f(s))\,,\ \theta = k(q - ct)$. As the underlying current is constant, it does not impact on the vorticity.

Wave-current interactions prescribed by (6)

- The underlying current we introduce into our exact solution (6) assumes an apparently simple form in the Lagrangian setting, yet it leads to significant complications, both mathematically and physically, in the resulting fluid motion.
- This is perhaps not surprising since the nonlinear passage from Lagrangian to Eulerian coordinates is a delicate issue.
- An examination of the mean flow velocities, and related mass transport, for a similar exact solution is performed in

Mean flow properties

DH and Silvia Sastre-Gómez, *J. Math. Fluid. Mech.* **18** (2016).

Dispersion Relations

- As part of the derivation process for the solution (6) we obtain the dispersion relation

$$kc^2 + 2\Omega c - 2\Omega c_0 - g + \Omega^2 R = 0. \tag{8}$$

- If $c_0 = c$ then (8) implies that $c = \sqrt{(g - \Omega^2 R)/k}$: for sufficiently large wavenumbers k this relation may, in principle, be physically attainable.
- This dispersion relation is a perturbation of the standard Gerstner wave (and deep-water gravity water wave) dispersion relation $c = \sqrt{g/k}$.

Dispersion Relations

- More generally, for $c_0 \neq c$, (8) gives the dispersion relation

$$c = \frac{\sqrt{\Omega^2 + k(g + 2\Omega c_0 - \Omega^2 R)} - \Omega}{k}.$$

- This relation features contributions from the Coriolis force, the centripetal force and the underlying current.
- Letting $\Omega \to 0$ we recover the standard expression for the deep-water gravity water (and Gerstner) wave.
- The speed predicted by the dispersion relation is quite accurate— surface waves with wavelengths$\approx$ 300m, propagating at speed$\approx$ 22m/s, are common in the Pacific.

Stratification

- In the absence of a constant underlying current ($c_0 = 0$) we may incorporate stratification through introducing a conservation of mass condition

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0.$$

- A compatible density function is defined by

$$\rho(r,s) = F\left(\frac{e^{2\xi}}{2k} - r - \frac{\Omega^2 s^2}{2\tilde{\mathfrak{g}}}\right),$$

where $F : (0,\infty) \to (0,\infty)$ is a continuously differentiable and non-decreasing function, which is otherwise arbitrary.

Stratification

- The pressure distribution then takes the form

$$P = \tilde{\mathfrak{g}}\mathcal{F}\left(\frac{e^{2\xi}}{2k} - r - \frac{\Omega^2 s^2}{2\tilde{\mathfrak{g}}}\right) + P_{atm} - \tilde{\mathfrak{g}}\mathcal{F}\left(\frac{e^{2kr_0}}{2k} - r_0\right).$$

  where $\mathcal{F}' = F$ and $\mathcal{F}(0) = 0$.
- It can be shown that this pressure distribution does indeed prescribe a solution to the modified $\beta-$plane equatorial governing equations we consider.

Go raibh maith agaibh!

Thank you for your attention!