Ship generated tsunamis: linearity vs. nonlinearity

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Outline

1. observations of a new phenomenon
2. linear dispersive analysis
3. asymptotics
4. full calculations

(5. potential effect of nonlinearity)

6. conclusions
7. movie!

Photo by Tore Henning Larsen

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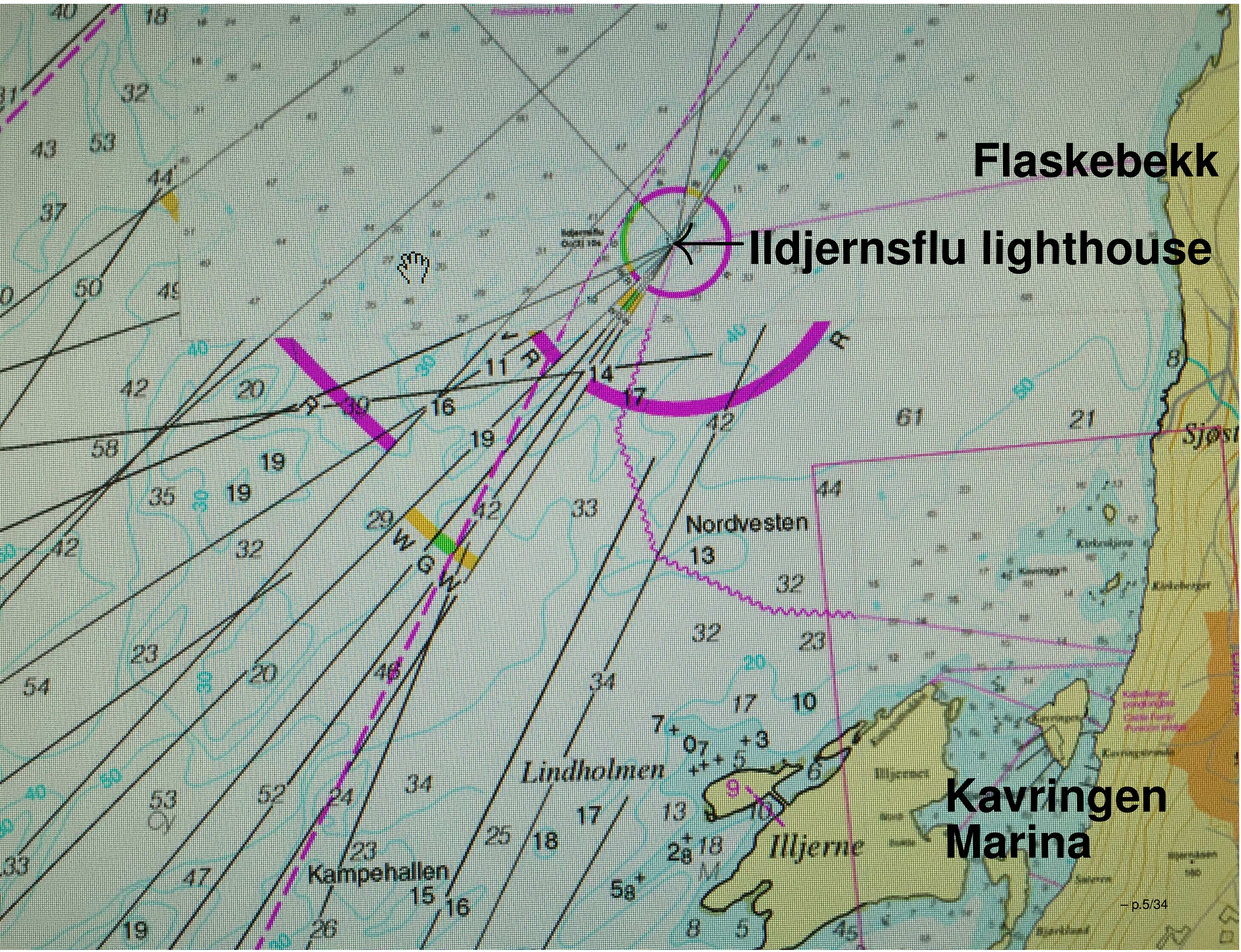
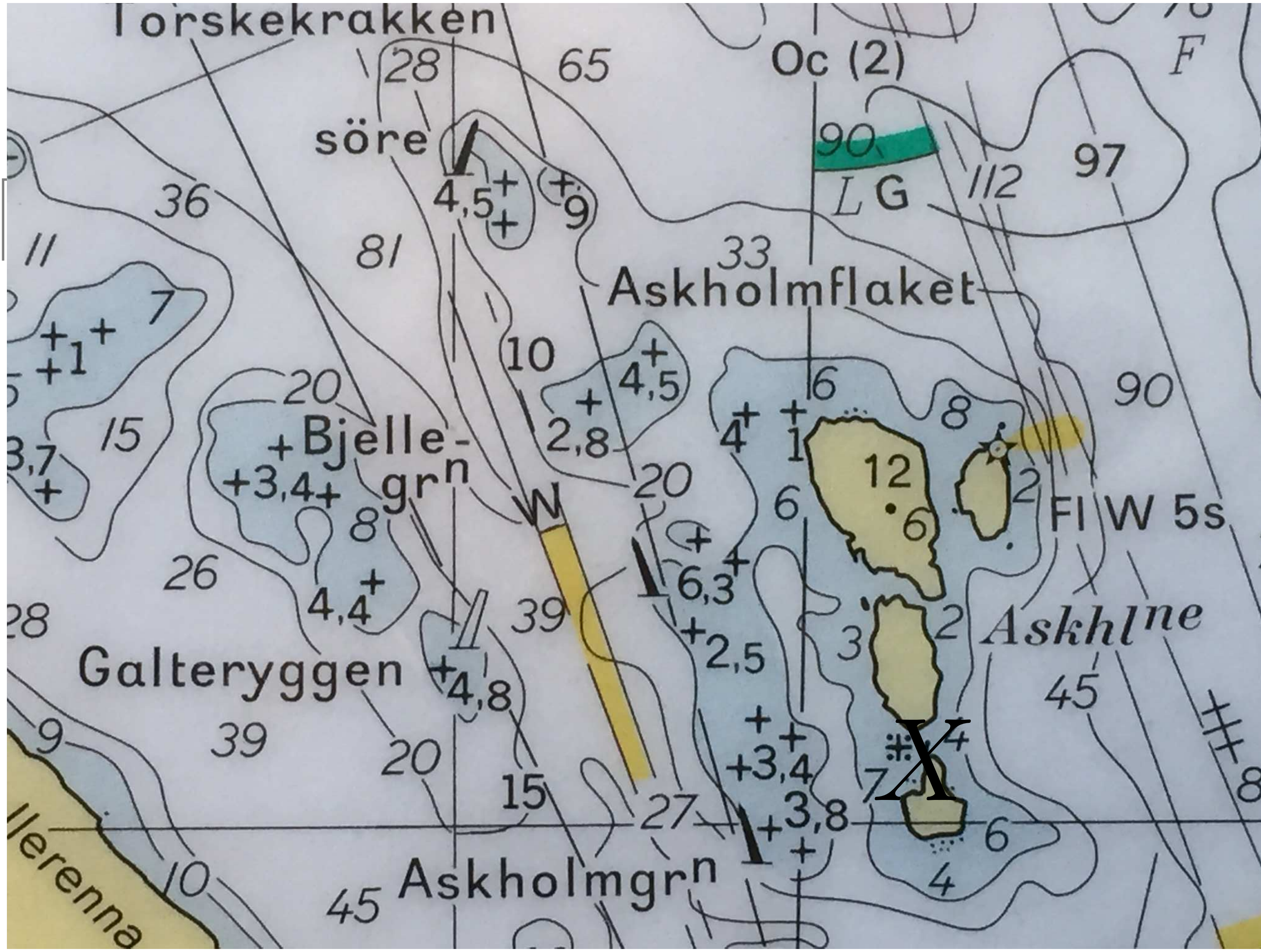
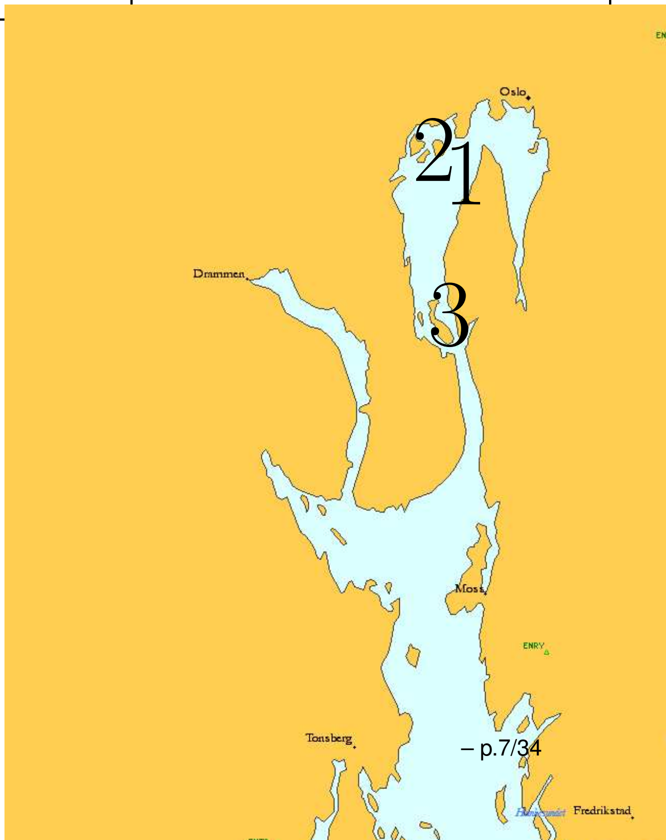

Flaskebekk
Ildjernsflu lighthouse
Nordvesten
Lindholmen
Kampehallen
Illjerne
Kavringen
Marina



The Oslofjord stretches 100 km north-south

Wave periods between two consecutive inflows (in-in) and outflows (out-out) in harbors or small bays due to three different ships observed at three different positions.

	Fantasy, 10.3 ms^{-1}		DFDS, 8.6 ms^{-1}		Magic, $U = 7.8$ ms^{-1}		Magic, $U = 10$ ms^{-1}	
	Position 1		Position 1		Position 2		Position 3	
wave no.	in-in	out-out	in-in	out-out	in-in	out-out	in-in	out-out
1	34 s	36 s	26 s	34 s	58 s	57 s	35 s	30 s
2	32 s	-	34 s	-	49 s	40 s	29 s	34 s
3	13 s	-	-	-	64 s	-	46 s	36 s



1. Very large cruise ferries moving along the Oslo fjord generate very long upstream waves: Color Line (36 100 m^3, since 2004) vs. DFDS (24 000 m^3, older).

2. Periods $\sim 30-40$ / $\sim 50-60$ sec.; shorter waves follow. .

3. Crests extend across the 2-3 km wide fjord.

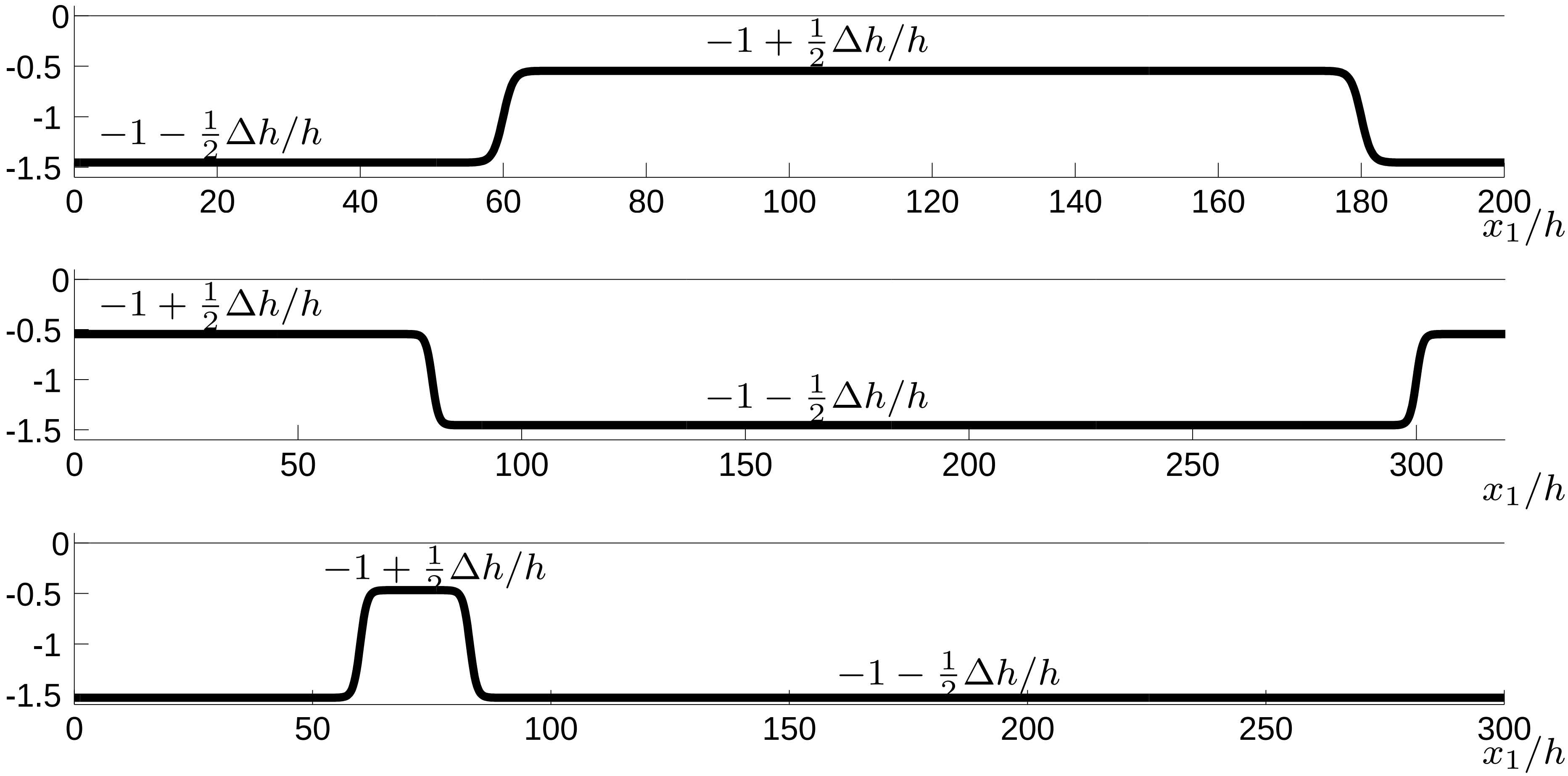
4. Wave height of 1 m (1.4 m) / currents of 1 ms^{-1} documented in small harbors / bays.

5. Generation: ship running over a depth change of Δh which is comparable to the average depth h, i.e. $\Delta h/h \sim 1$.

6. Depth Froude number $U/\sqrt{gh} \sim 0.5-0.7$. Not related to nonlinear transcritical generation of upstream solitons on flat bottom (Ertekin et al. 1986, Wu, 1987; Pedersen, 1988).

7. Problem: Considerable erosion along the shoreline.

Bottom shapes $y = -h + \beta(x)$



Fully dispersive linear theory. Kinematic and dynamic free surface condition (Fourier transformed):

$$\frac{\partial \hat{\eta}}{\partial t} - \hat{V} = 0, \qquad \frac{\partial \hat{\phi}_F}{\partial t} + g\hat{\eta} = -\frac{\hat{p}}{\rho},$$

A set of integral equations connect the vertical surface velocity V, surface potential ϕ_F and the bottom potential $\phi_B = \phi(y = -h + \beta(x_1, x_2))$ (Clamond and Grue, 2001; Fructus and Grue, 2007; Grue, 2015):

$$\int_F V\Big(\frac{1}{r} + \frac{1}{r_1}\Big)dS$$
$$= 2\pi\phi_F' + \int_F V\Big(\frac{1}{r} + \frac{1}{r_1}\Big)\phi_F dS + \int_B \phi_B \frac{\partial}{\partial n}\Big(\frac{1}{r} + \frac{1}{r_1}\Big)dS$$

$$2\pi\phi_B' = \int_F V\Big(\frac{1}{r} + \frac{1}{r_{1B}}\Big)dS - \int_B \phi_B \frac{\partial}{\partial n}\Big(\frac{1}{r} + \frac{1}{r_{1B}}\Big)dS$$

Green fu. expanded in the bottom variation β obtaining

$$\hat{V} = k\hat{\phi}_F T_1 + \frac{\mathsf{i}\mathbf{k}}{C_1} \cdot [\mathcal{F}(\beta \nabla_1 \phi_B) + \frac{k^2}{3!}\mathcal{F}(\beta^3 \nabla_1 \phi_B) + ...]$$

$$\begin{aligned} k\hat{\phi}_B = {} & \frac{k\hat{\phi}_F}{C_1} - \mathsf{i}T_1\mathbf{k} \cdot \mathcal{F}(\beta \nabla_1 \phi_B) + \frac{k}{2-2e_1}\mathcal{F}(\beta^2 \mathcal{F}^{-1}(k\hat{\phi}_B)) \\ & + \frac{k}{1-e_1}\mathcal{F}(\beta \mathcal{F}^{-1}[\mathsf{i}\mathbf{k} \cdot \mathcal{F}(\beta \nabla_1 \phi_B)]) - \frac{ke_1}{2-2e_1}\mathsf{i}\mathbf{k} \cdot \mathcal{F}(\beta^2 \nabla_1 \phi_B) \\ & + ... \end{aligned}$$

$$C_1 = \cosh kh, T_1 = \tanh kh, e_1 = e^{-2kh}.$$

The elevation:

$$\hat{\eta} = \int_{t_0}^{t} \cos\omega(s-t)\,\hat{h}_1(s)ds + \int_{t_0}^{t} \sin\omega(s-t)\,\frac{\omega\,\hat{p}}{\rho g}ds,$$

$$h_1 = \frac{\mathrm{i}\mathbf{k}}{C_1}\cdot[\mathcal{F}(\beta\nabla_1\phi_B) + \frac{k^2}{3!}\mathcal{F}(\beta^3\nabla_1\phi_B) + ...]$$

Term 1: effect of the bottom variation;
Term 2: effect of the moving pressure distribution.

Asymptotic analysis of the waves due to a small bottom step at $x_1 = 0$, where $y = -h \ (x_1 < 0)$ and $y = -h + \Delta h \ (x_1 > 0)$ with $\Delta h/h << 1$; waves are assumed long (small kh):

$$\hat{\eta}_0 = \int_{t_0}^{t} \cos\omega(s-t)\,\hat{h}_1(s)ds$$

$$\hat{h}_1 = \frac{\mathsf{i}\mathbf{k}}{C_1}\cdot[\mathcal{F}(\beta\nabla_1\phi_B) + ...] = \mathsf{i}\mathbf{k}\cdot\mathcal{F}(\beta\nabla_1\phi_F) + O(k^2)$$

$$\hat{h}_1 = \mathsf{i}k_1\Delta h\int_0^\infty\int_{-\infty}^\infty \frac{\partial\phi_F}{\partial x_1}e^{-\mathsf{i}\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} + O(k^2)$$

$$= -\mathsf{i}k_1\Delta h\int_{-\infty}^\infty \phi_F(x_1 = 0)dx_2 + O(k^2)$$

The time derivative:

$$\frac{\partial \hat{h}_1}{\partial t} = -\mathsf{i} k_1 \Delta h \int_{-\infty}^{\infty} \frac{\partial \phi_F(x_1 = 0)}{\partial t} dx_2 \simeq \frac{\mathsf{i} k_1 \Delta h}{\rho} \int_{-\infty}^{\infty} p(x_1 = 0) dx_2$$

Obtaining the elevation:

$$\begin{aligned} \hat{\eta}_0 &= -\int_{t_0}^{t} \frac{\sin \omega (s-t)}{\omega} \frac{\partial \hat{h}_1}{\partial s} ds \\ &\simeq -\frac{\mathsf{i} k_1 \Delta h}{\rho} \int_{t_0}^{t} \frac{\sin \omega (s-t)}{\omega} \int_{-\infty}^{\infty} p(x_1 = 0) dx_2 ds, \\ \omega &= \sqrt{g k T_1} \end{aligned}$$

Pressure distribution: $p(x_1 - Ut, x_2, t) = \rho g V_0 \delta(x_1 - Ut) \delta(x_2)$:

$$\hat{\eta}_0 = \frac{\mathsf{i} k_1 \Delta h \, V_0}{U \omega / g} \sin \omega t,$$

Method of stationary phase gives upstream waves, in 2D, $(t \to \infty)$

$$\eta_0(x_1, t) = -\frac{k_{1,0}\Delta h\, V_0}{(U\omega_0/g)(2\pi|\omega''|t)^{1/2}} \cos(k_{1,0}x_1 - \omega_0 t + \frac{\pi}{4}) + O(t^{-1}).$$

$k_{1,0}$: $\partial\omega/\partial k_1 = x_1/t < \sqrt{gh} = c_0$, $\omega_0 = \omega(k_{1,0})$, double prime means double derivative. Near the wave front ($x_1/t >\sim \sqrt{gh}$)

$$\eta_0(x_1, t) \sim -\frac{\Delta h\, V_0}{(Uc_0/g)(4c_0h^2t)^{1/3}}\,\mathrm{Ai}(Z),$$

$\mathrm{Ai}(Z)$ – the Airy function; $Z = [2(x_1 - c_0 t)^3/c_0 h^2 t]^{1/3}$.

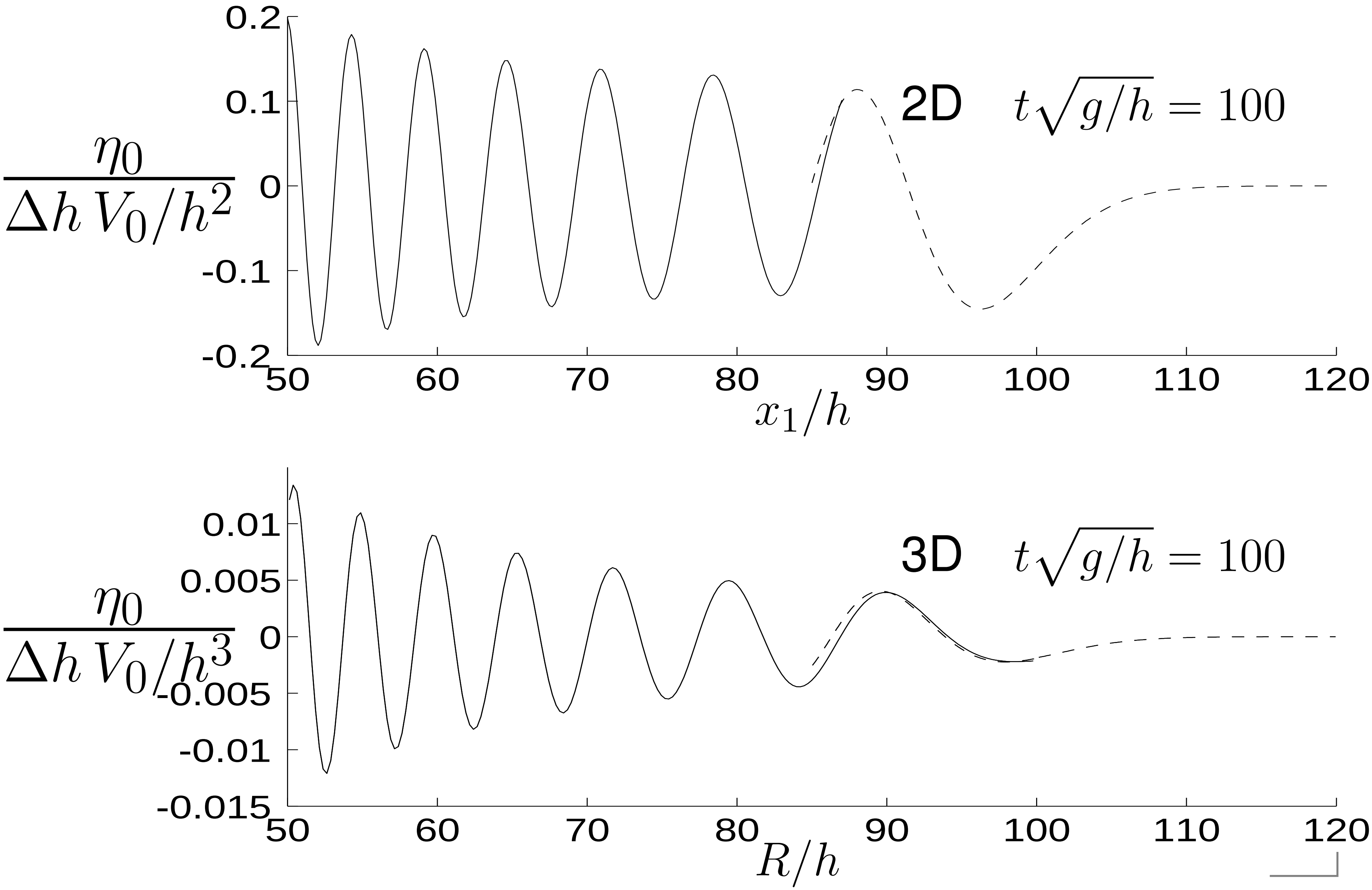
Inverse transform in the three-dimensional case gives:

$$\eta_0(R,t) \sim -\frac{\Delta h\, V_0 k_0}{2\pi\, U\omega_0/g}\,\frac{\partial}{\partial x_1}\,\frac{\sin(k_0 R - \omega_0 t)}{(k_0 R|\omega_0''|t)^{1/2}},$$

where k_0 is determined by $R/t = \partial\omega/\partial k < \sqrt{gh}$, $\omega_0 = \omega(k_0)$. Near the wave front, following Clarisse, Newman and Ursell (1995)

$$\eta_0(x_1,x_2,t) \sim \frac{\Delta h\, V_0/h^2}{2\pi\, U/\sqrt{gh}}\frac{\partial}{\partial x_1}\frac{2^{1/3}\pi}{(R^*/t^*)^{1/2} t^{*2/3}} \times$$

$$\times[E_0 \mathrm{Ai}^2 - t^{*-2/3} C_0 \mathrm{Ai}'^2 + 2t^{*-4/3} A_1 \mathrm{Ai}\mathrm{Ai}'].$$



0.2
0.1
0
-0.1
-0.2
$\frac{\eta_0}{\Delta h\, V_0/h^2}$
2D $t\sqrt{g/h} = 100$
50
60
70
80
90
100
110
120
x_1/h
0.01
0.005
0
-0.005
-0.01
-0.015
$\frac{\eta_0}{\Delta h\, V_0/h^3}$
3D $t\sqrt{g/h} = 100$
50
60
70
80
90
100
110
120
R/h

Numerical wave tank:

$(L_1, L_2) = (200h_0, 13h)$, $h = 55$ m (location 1).
$(L_1, L_2) = (300h_0, 80h)$, $h = 30$ m (location 2).

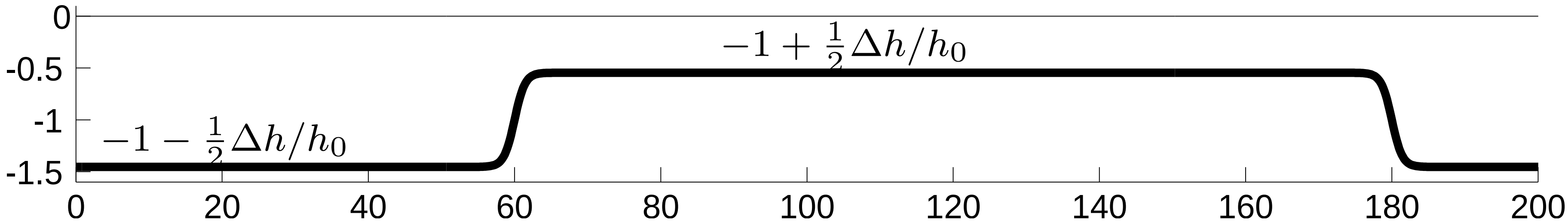
$(N_1, N_2) = (1200, 432)$ or $(2090, 248)$.

Ship replaced by a hovercraft/pressure distribution of (length, width, draught)$= (l, w, d)$,

$l/h_0 = 210$ m / $h \sim 4 - 7$,
$w/h_0 = 30$ m / $h \sim \frac{1}{2} - 1$,

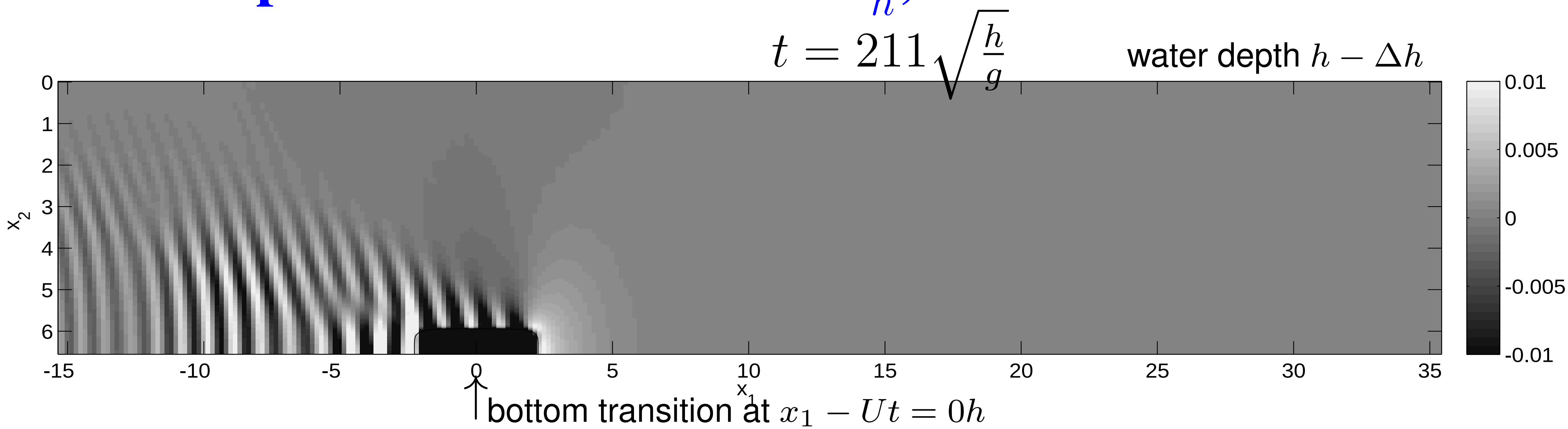
draught such that $\int p/(\rho g)\, dx_1 dx_2 = V_{ship}$.

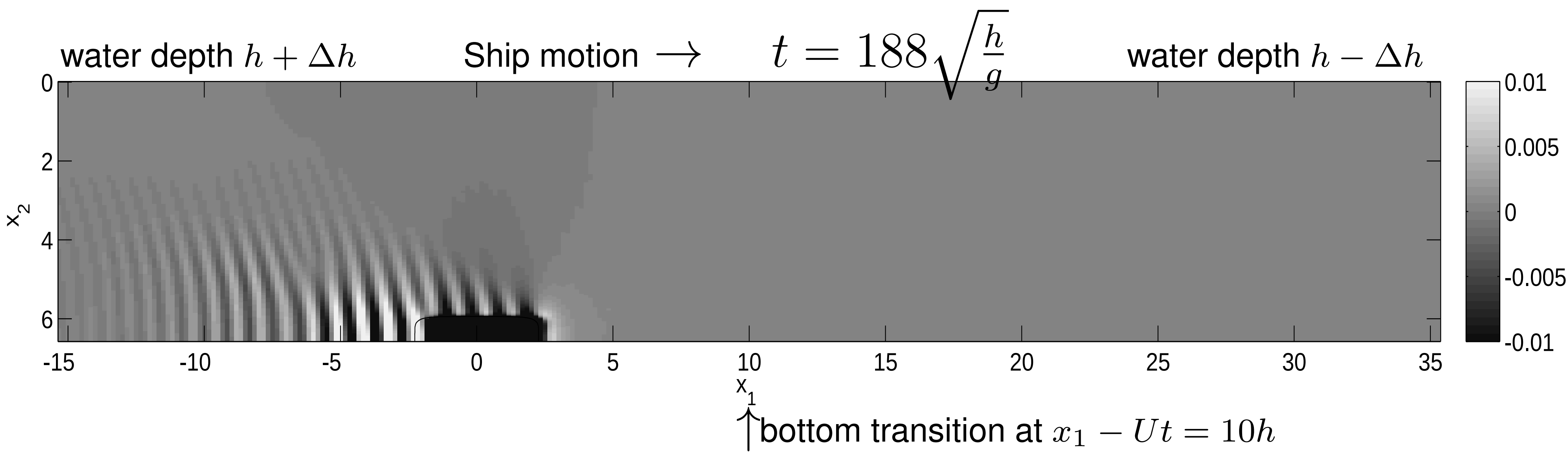
deep2shallow



$\Delta h/h = 0.909$, $h = 55$ m, $l_{ship} = 210$ m

Upstream wave elevation $\frac{\eta}{h}$, $Fr = .43$

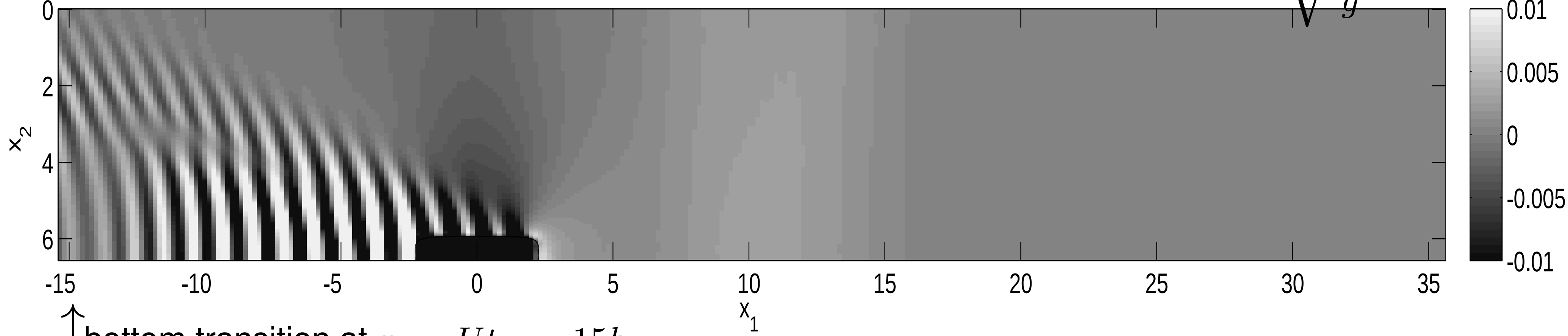




Frame of reference fixed to ship, $\frac{\Delta h}{h} = 0.909$, $h = 55$ m.

Upstream wave elevation $\frac{\eta}{h}$, $Fr = .43$

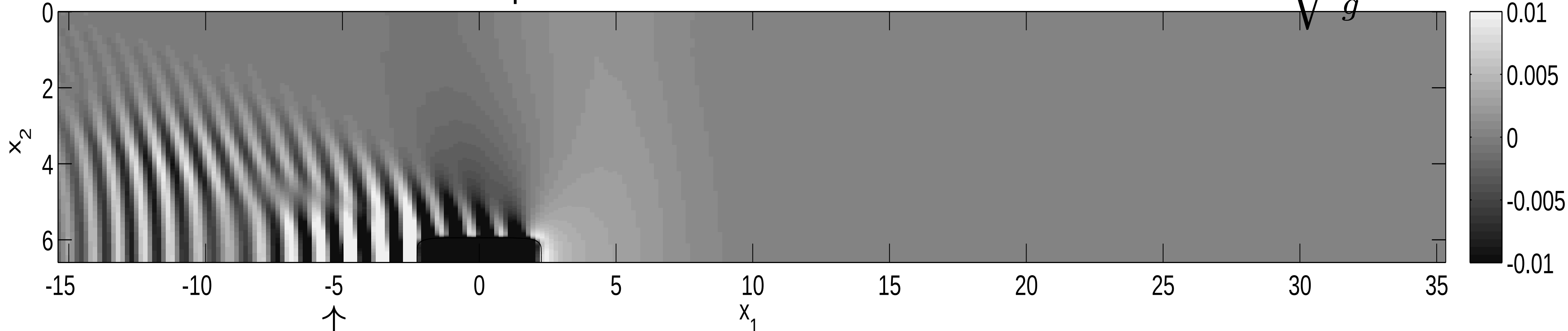
$t = 246\sqrt{\frac{h}{g}}$



bottom transition at $x_1 - Ut = -15h$

Ship motion $\longrightarrow$

$t = 223\sqrt{\frac{h}{g}}$

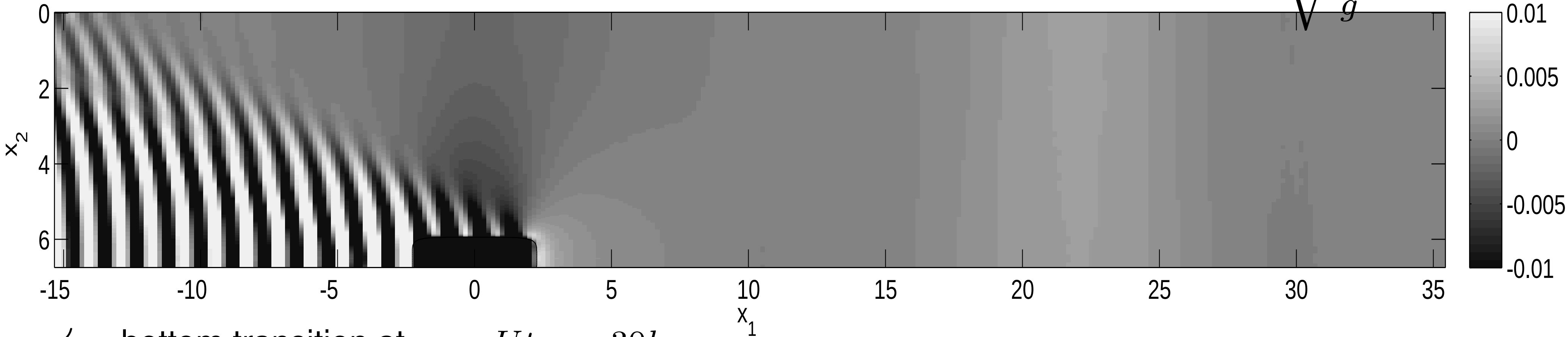


bottom transition at $x_1 - Ut = -5h$

Frame of reference fixed to ship, $\frac{\Delta h}{h} = 0.909$.

Upstream wave elevation $\frac{\eta}{h}$, $\quad Fr = .43$

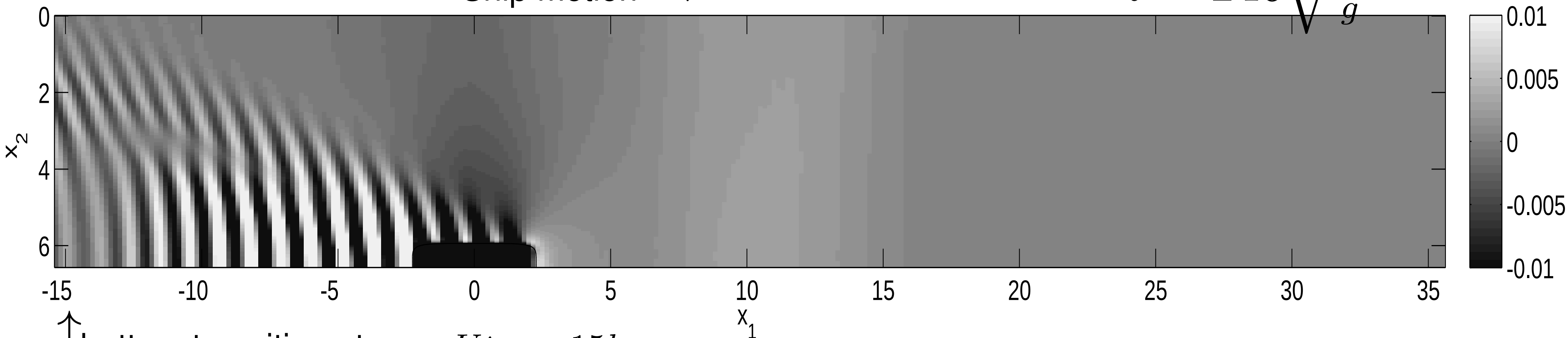
$t = 281\sqrt{\frac{h}{g}}$



$\leftarrow$ bottom transition at $x_1 - Ut = -30h$

Ship motion $\longrightarrow$

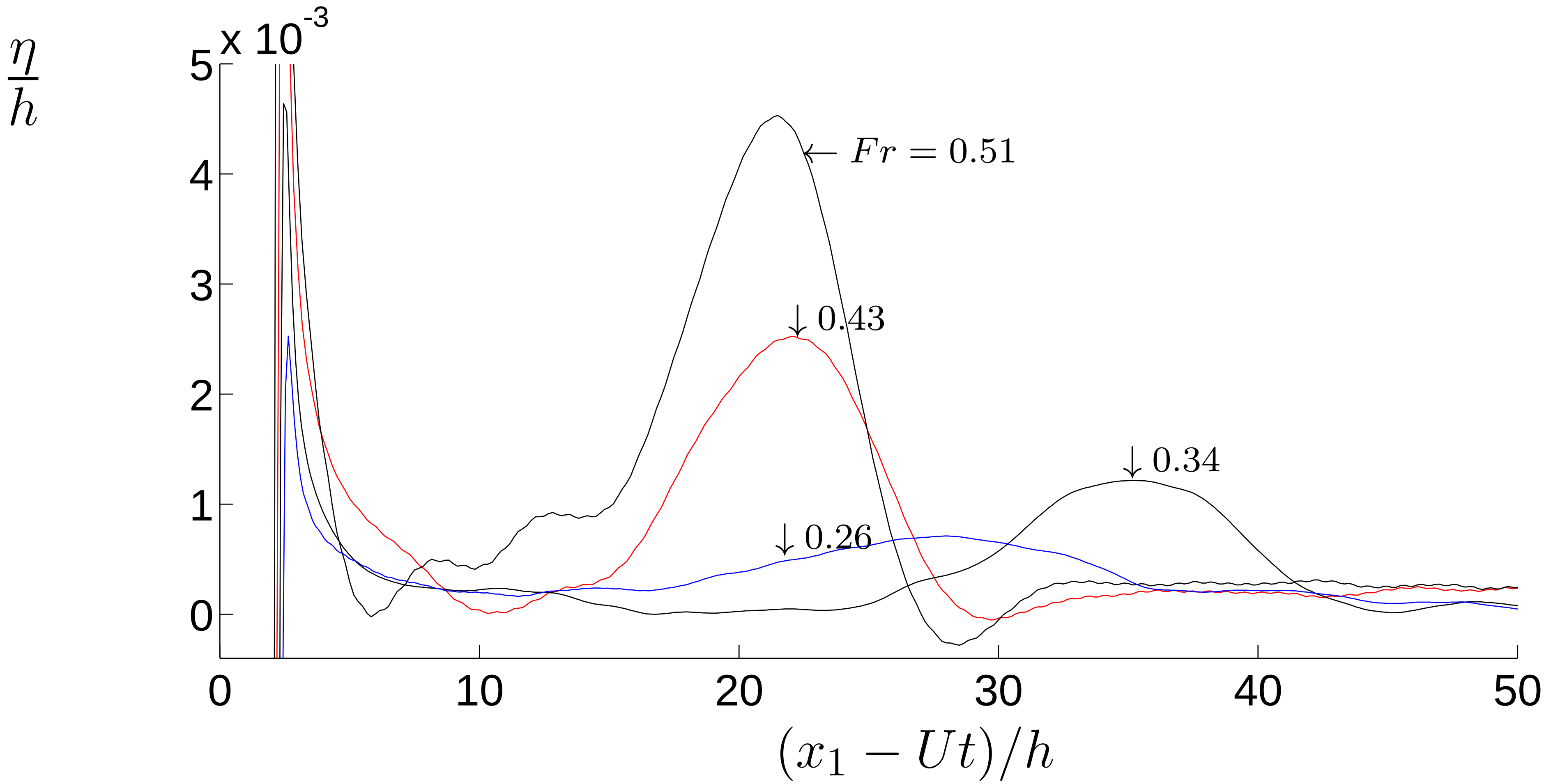
$t = 246\sqrt{\frac{h}{g}}$



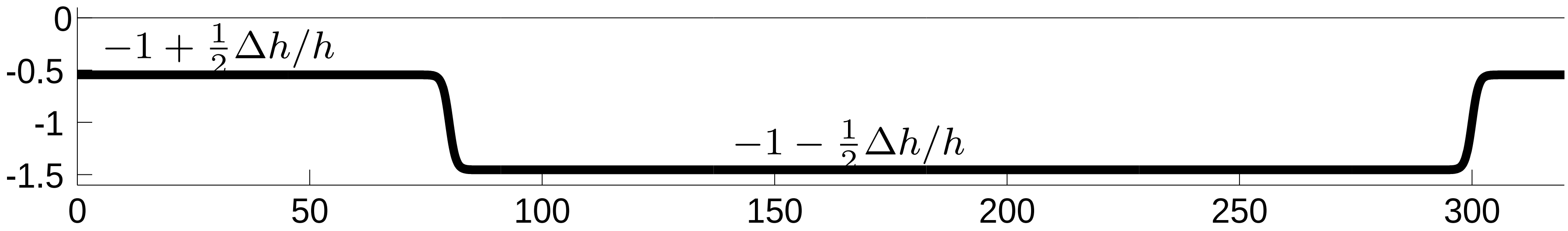
$\uparrow$bottom transition at $x_1 - Ut = -15h$

Frame of reference fixed to ship, $\frac{\Delta h}{h} = 0.909$.

Elevation ahead of ship, $\frac{\Delta h}{h} = 0.909$.

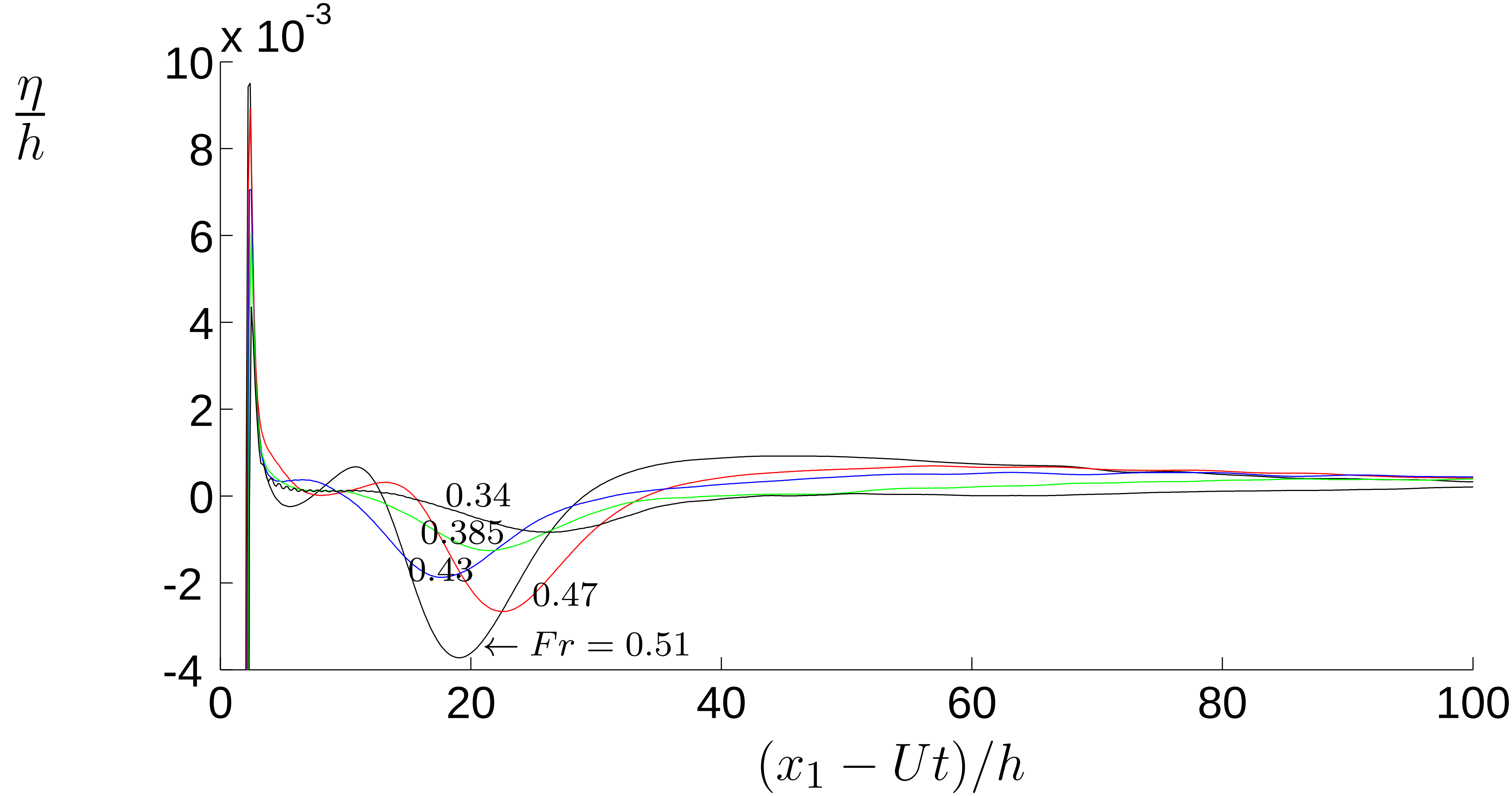


shallow2deep



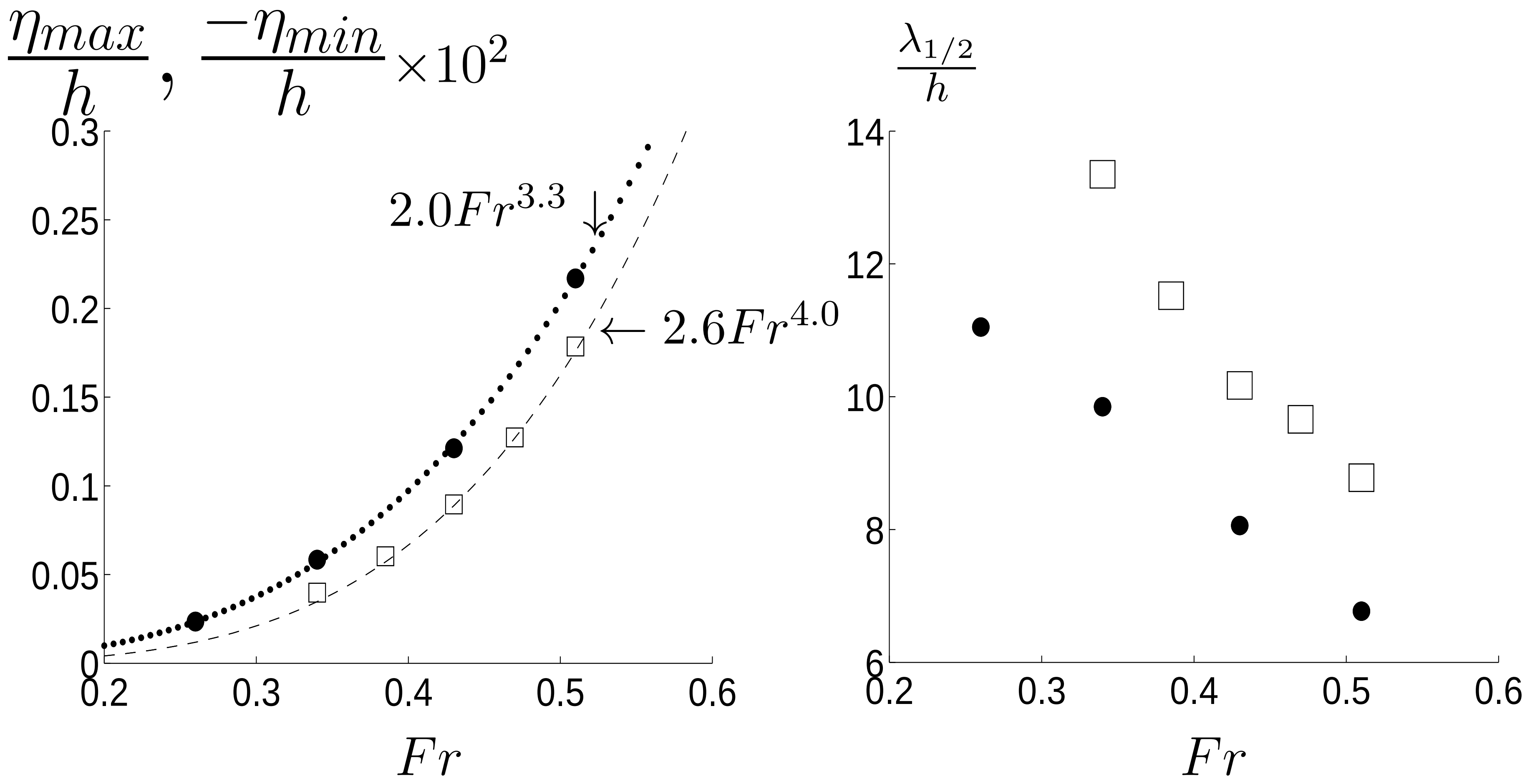
$\frac{1}{2}\Delta h/h = 0.909$, $h = 55$ m, $l_{ship} = 210$ m

Depression ahead of ship, $\frac{\Delta h}{h} = 0.909$.



Ship moving from depth $h - \frac{1}{2}\Delta h$ to depth $h + \frac{1}{2}\Delta h$.

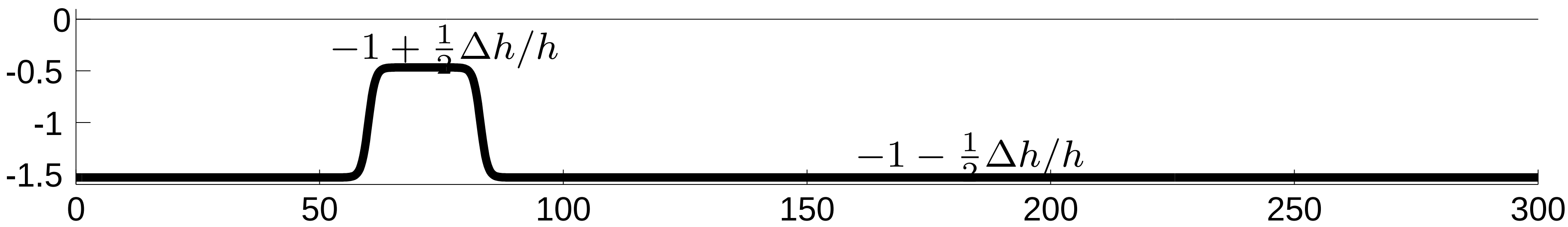
deep2shallow (•); shallow2deep (□)



Elevation/depression (left); wavelength at half height (right) vs. Fr.

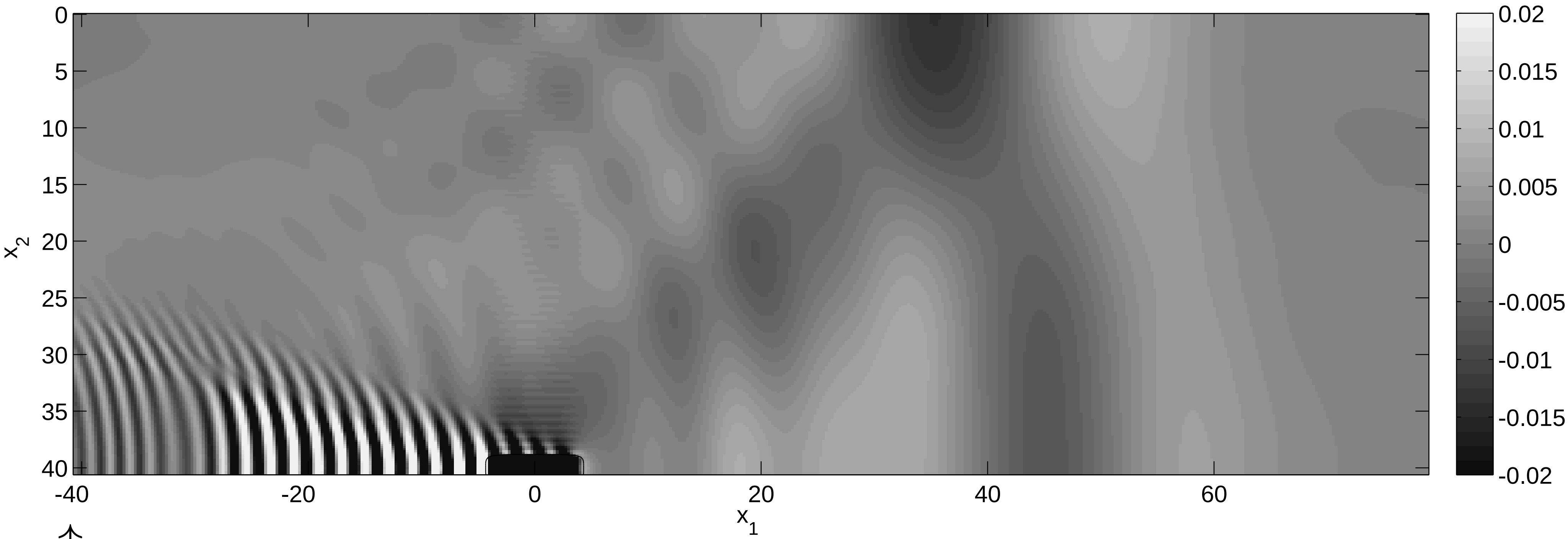
$yy = 0.042 * fr0.^{3.3}$,

deep2shallow2deep



$\Delta h/h = 1.067$, $h = 30$ m, $l_{ship} = 210$ m

Elevation $\frac{\eta}{h}$, $Fr = 0.58$.

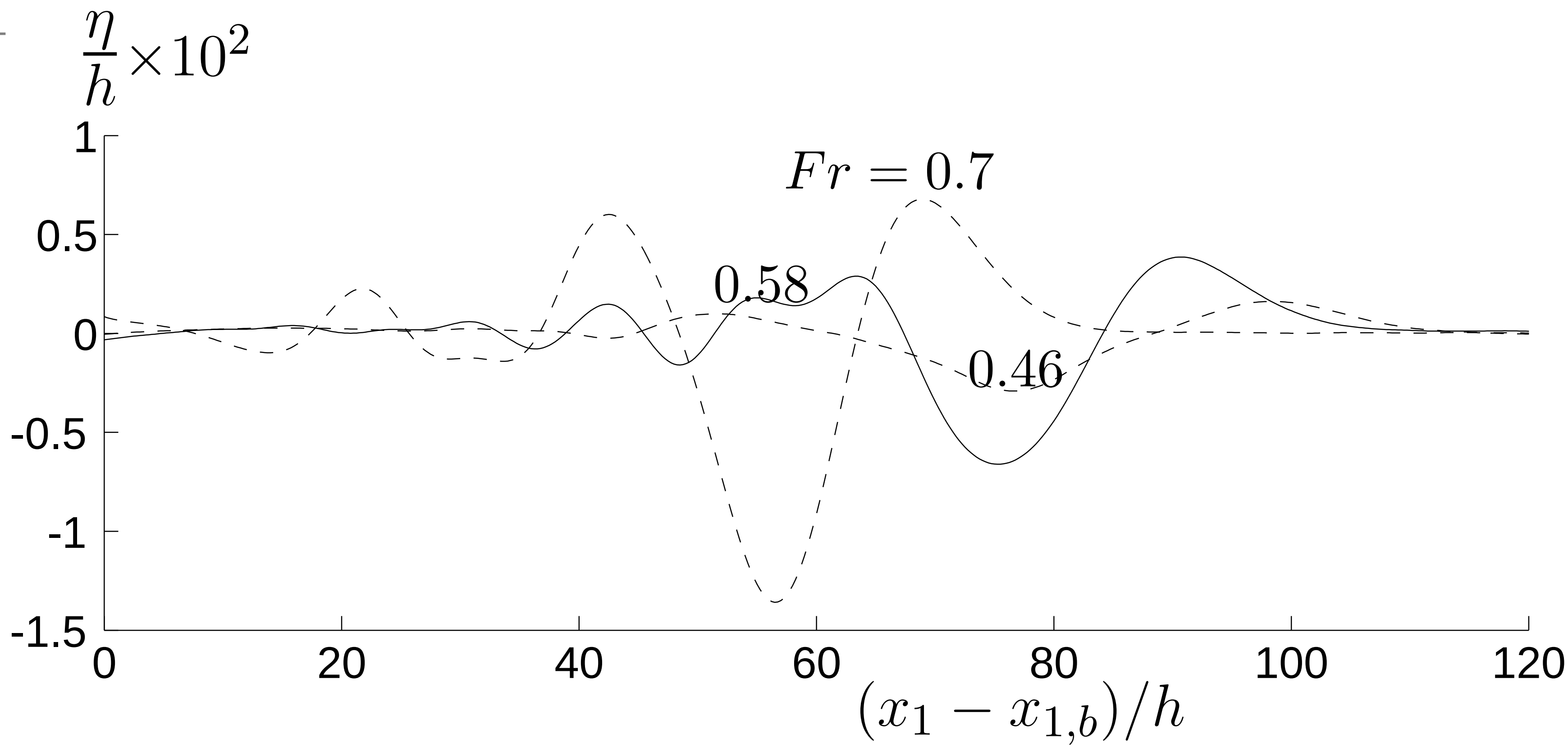


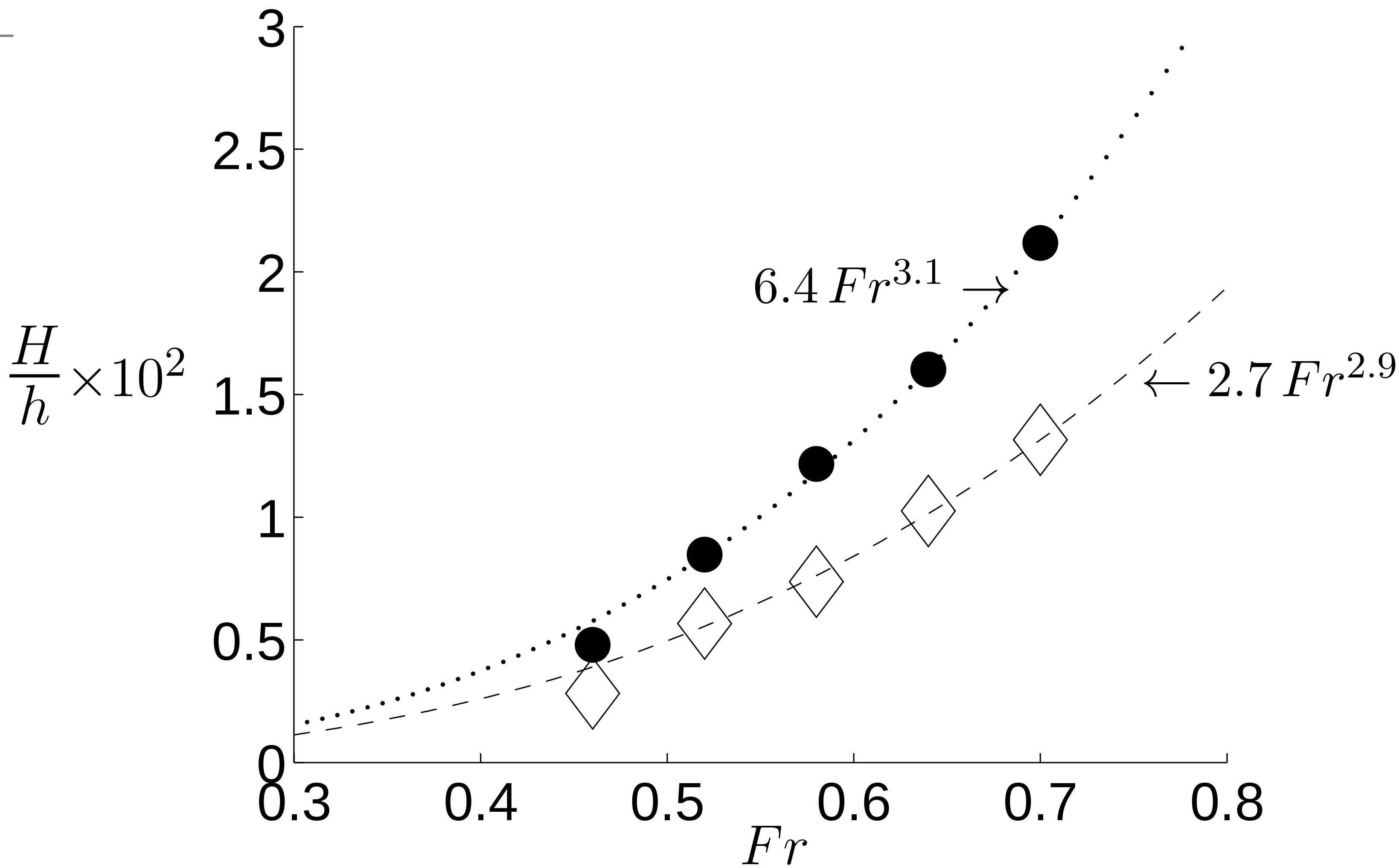
↑end of bottom at $x_1 - Ut = -40h$

Position of ship: $x_b + 40h$.

$\Delta h/h = 1.067$, $x_a = 60h$, $x_b = 83h$, $L_2 = 80h$, $h = 30$ m.

Elevation at the shore





Max. wave height at channel wall ($\bullet$) and ahead of ship ($\Diamond$).

Conclusions

1. A ship at subcritical speed moving across a depth change $\Delta h/h_0 \sim 1$ generates upstream long waves in the form of a small tsunami.

2. An asymptotic analysis provides a qualitative explanation where the upstream waves are expressed in terms of a pressure impulse acting at the depth change, obtaining a leading depression if the depth is reduced, in accordance with observations.

3. A ship moving from deep to shallow to deep water ($\Delta h/h_0 \sim 1$) produces a short upstream wave train: a crest one, **a deep trough**, a second crest, a second trough, and then a train of shorter waves. Reflections at the channel walls magnify the elevation.

4. The wave height ($\sim \frac{1}{2}$ m) grows according to Fr^n, $n \simeq 3$.

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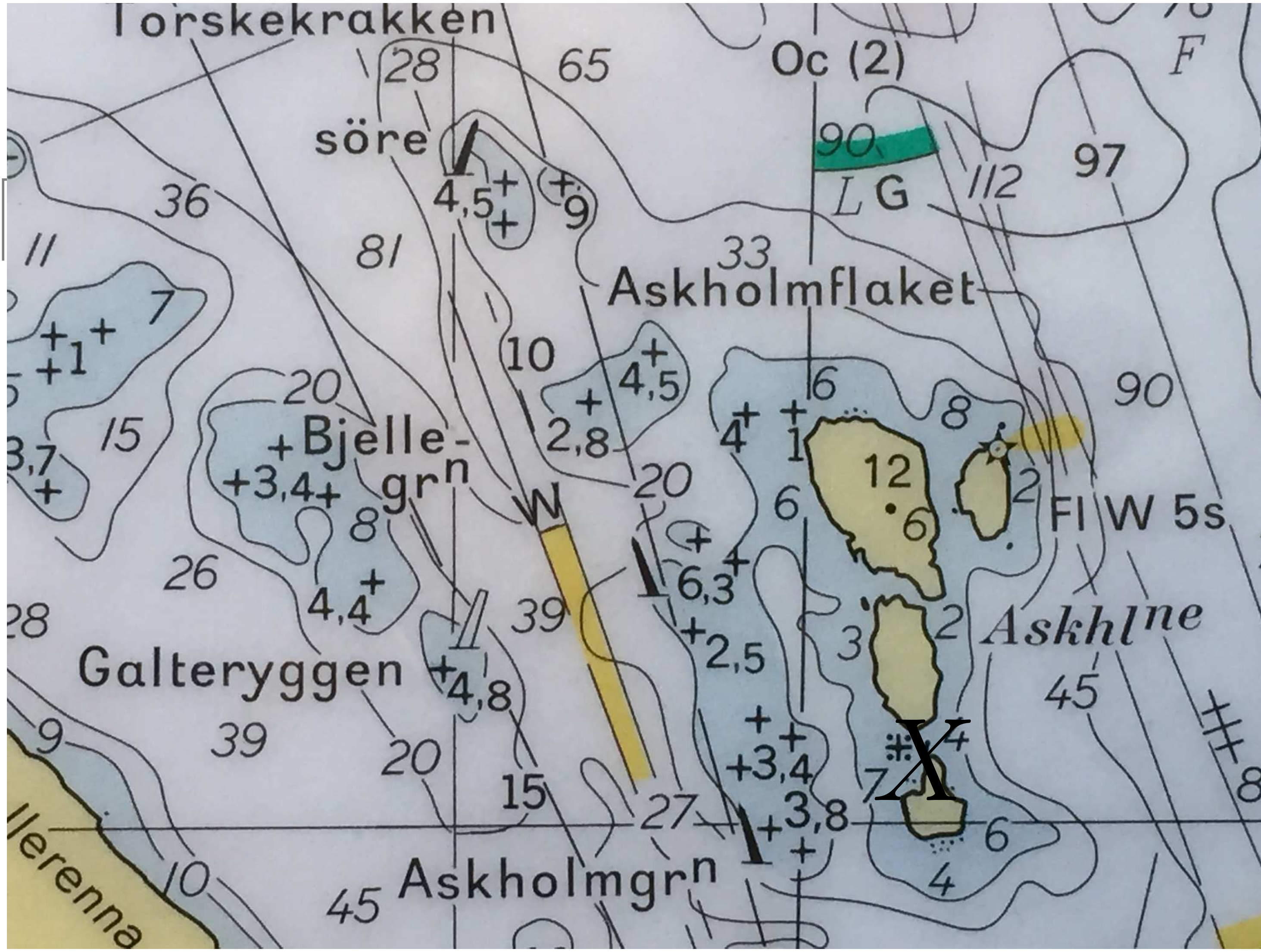
5. More details: Grue (JFM, 2016, in revision).

MOVIE!



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