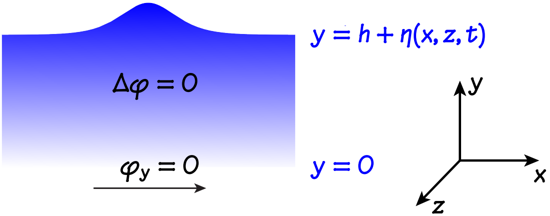
THE WATER-WAVE PROBLEM



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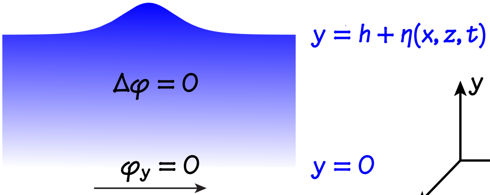

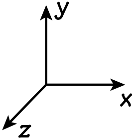

Kinematic boundary condition:

$$\eta_t = \varphi_y - \eta_x \varphi_x - \eta_z \varphi_z$$

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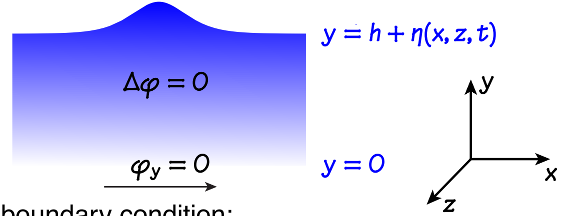

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$$\eta_t = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z$$

Dynamical boundary condition:

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + g\eta - \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \sigma\left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0$$

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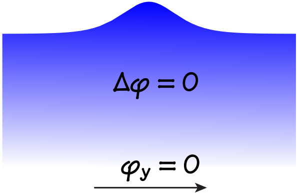
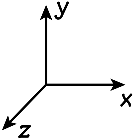


$y = h + \eta(x,z,t)$

$y = 0$



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Difficulties:

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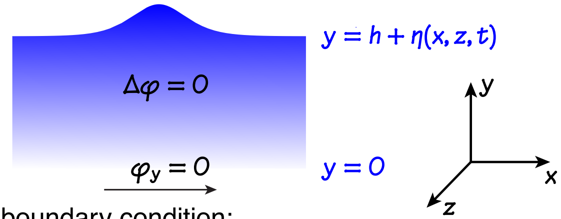

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Difficulties:

- A free-boundary value problem

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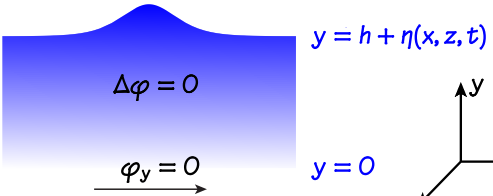

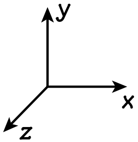

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- A free-boundary value problem
- Nonlinear boundary conditions

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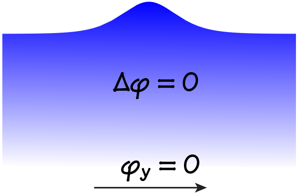

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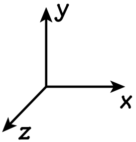

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Difficulties:

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Parameter: $\beta = \sigma/gh^2$

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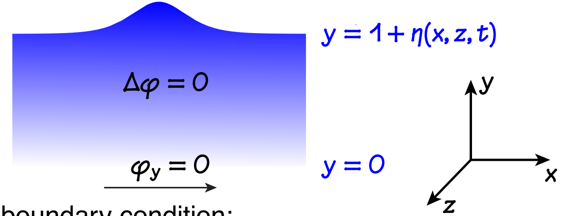

Kinematic boundary condition:

$$-c\eta_x = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z$$

Dynamical boundary condition:

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Difficulties:

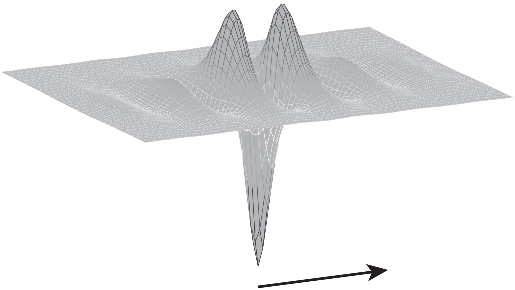
- A free-boundary value problem
- Nonlinear boundary conditions

Parameter: $\beta = \sigma/gh^2$

Solitary waves: $\eta(x, z, t) = \eta(x - ct, z)$, $\eta(x - ct, z) \to 0$ as $|(x - ct, z)| \to \infty$

FULLY LOCALISED SOLITARY WAVES

- Weak surface tension ($\beta < 1/3$):



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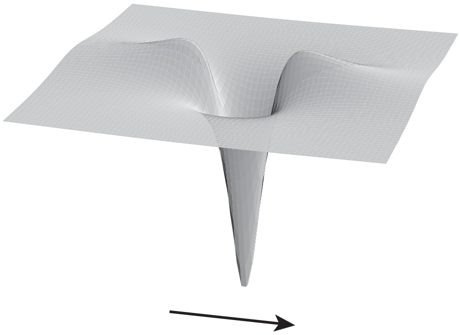

MODELLING

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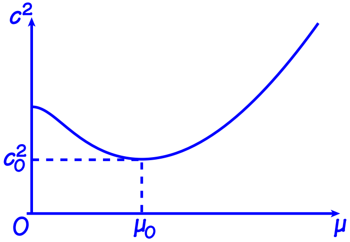
- Dispersion relation for linear wave trains $\eta \sim \cos\mu(x - ct)$:



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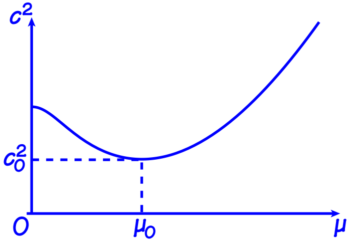

- The Ansatz

$$c^2 = c_0^2(1 - \varepsilon^2), \qquad \eta(x, z) = \varepsilon \left(\zeta(\varepsilon x, \varepsilon z) e^{i\mu_0 x} + \overline{\zeta(\varepsilon x, \varepsilon z)} e^{-i\mu_0 x} \right) + O(\varepsilon^2)$$

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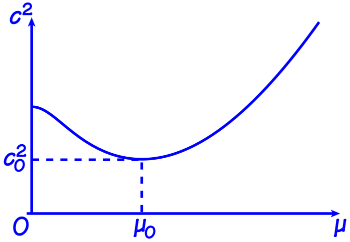

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leads to the Davey-Stewartson equation

$$\zeta - \zeta_{xx} - \zeta_{zz} - |\zeta|^2\zeta - \zeta\Delta^{-1}\partial_x^2|\zeta|^2 = 0$$

THE DS EQUATION

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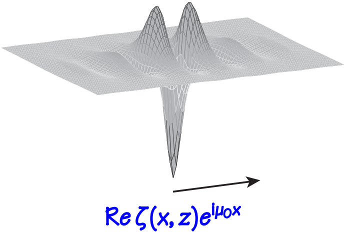

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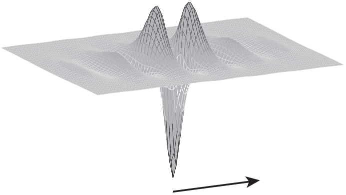

$$\mathrm{Re}\,\zeta(x,z)e^{i\mu_0 x}$$

- This solution is a critical point of the functional

$$\mathcal{I}_0(\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}(|\zeta_x|^2 + |\zeta_z|^2 + |\zeta|^2) - \frac{1}{4}|\zeta|^4 - \frac{1}{4}|\zeta|^2\Delta^{-1}\partial_x^2|\zeta|^2\right\}dx\,dz$$

with function space

$$X = \overline{C_0^\infty(\mathbb{R}^2)} = H^1(\mathbb{R}^2)$$

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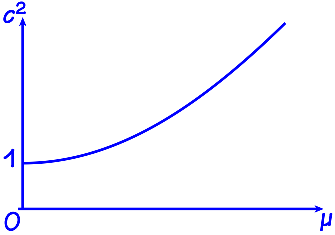

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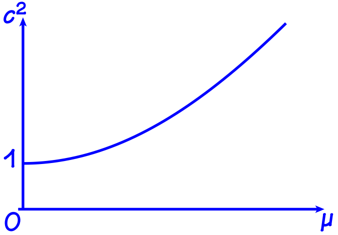

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$$c^2 = 1 - \varepsilon^2, \qquad \eta(x, z) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z) + O(\varepsilon^4)$$

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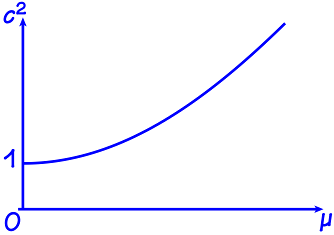

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leads to the Kadomtsev-Petviashvili equation

$$\zeta_{xx} - \zeta - \frac{3}{2}\zeta^2 - \partial_x^{-2}\zeta_{zz} = 0$$

THE KP EQUATION

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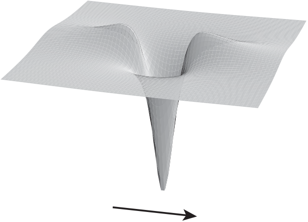
- An explicit solitary-wave solution

$$\zeta(x,z) = -8\frac{3 - x^2 + z^2}{(3 + x^2 + z^2)^2}$$



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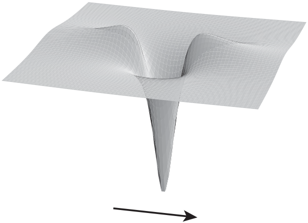

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$$\mathcal{I}_0(\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}(\zeta^2 + (\partial_x^{-1}\zeta_z)^2 + \zeta_x^2) - \frac{1}{3}\zeta^3\right\} dx\, dz$$

with function space

$$X = \overline{\partial_x C_0^\infty(\mathbb{R}^2)}$$

VARIATIONAL PRINCIPLE

- Luke's variational principle

$$\delta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{0}^{1+\eta}\left(-c\varphi_x+\frac{1}{2}(\varphi_x^2+\varphi_y^2+\varphi_z^2)\right)dy + \frac{1}{2}\eta^2+\beta\left(\sqrt{1+\eta_x^2+\eta_z^2}-1\right)\right\}dx\,dz=0$$

recovers the hydrodynamic equations

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$$\left. + \frac{1}{2}\eta^2 + \beta\left(\sqrt{1+\eta_x^2+\eta_z^2} - 1\right) \right\} dx\, dz = 0$$

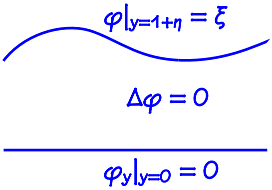
recovers the hydrodynamic equations

- Use a Dirichlet-Neumann operator:

$$\delta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -c\eta_x\xi + \frac{1}{2}\xi G(\eta)\xi + \frac{1}{2}\eta^2 + \beta\left(\sqrt{1+\eta_x^2+\eta_z^2} - 1\right) \right\} dx\, dz = 0$$

where $\xi = \varphi|_{y=1+\eta}$ and

$$G(\eta)\xi = \sqrt{1+\eta_x^2+\eta_z^2}\,\varphi_n|_{y=1+\eta}$$



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$$J(\eta) = \mathcal{F}(\eta, \xi(\eta)) = \mathcal{K}(\eta) - c^2\mathcal{L}(\eta),$$

where

$$\mathcal{K}(\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}\eta^2 + \beta\sqrt{1+\eta_x^2+\eta_z^2} - \beta\right\} dx\,dz,$$

$$\mathcal{L}(\eta) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta K(\eta)\eta\,dx\,dz, \qquad K(\eta)\xi := -\partial_x(G(\eta)^{-1}\xi_x)$$

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- $K: H^3(\mathbb{R}^2) \to \mathcal{B}(H^{5/2}(\mathbb{R}^2), H^{3/2}(\mathbb{R}^2))$ is analytic at the origin

REDUCTION $(\beta < 1/3)$

- Find critical points of $\mathcal{J}(\eta) = \mathcal{K}(\eta) + c^2\mathcal{L}(\eta)$

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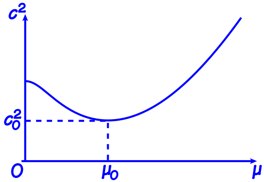

$$c^2 = c_0^2(1-\varepsilon^2),$$

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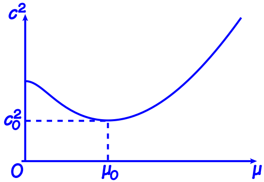

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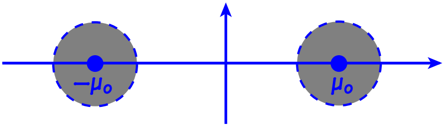
- Write

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1-\chi(D))\eta,$$

where χ is the characteristic function of this set:



$$|\mu - \mu_0| < \delta$$

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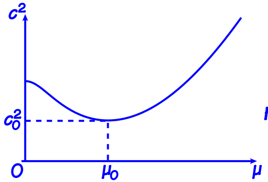

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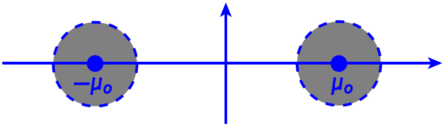


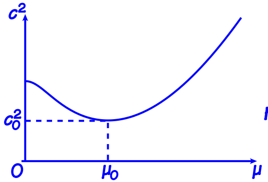
$$|\mu - \mu_0| < \delta$$

- $J'(\eta) = 0 \quad \Rightarrow$

$$\chi(D)J'(\eta_1+\eta_2) = 0,$$
$$(1-\chi(D))J'(\eta_1+\eta_2) = 0$$

REDUCTION ($\beta < 1/3$)

- Find critical points of $J(\eta) = \mathcal{K}(\eta) + c^2\mathcal{L}(\eta)$
- Modelling:



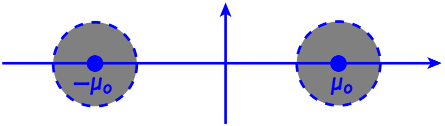
$$c^2 = c_0^2(1-\varepsilon^2),$$

$$\eta(x,z) = \varepsilon\left(\zeta(\varepsilon x, \varepsilon z)e^{i\mu_0 x} + \overline{\zeta(\varepsilon x, \varepsilon z)}e^{-i\mu_0 x}\right) + O(\varepsilon^2)$$

- Write

$$\eta_1 = \chi(D)\eta, \qquad \eta_2 = (1-\chi(D))\eta,$$

where χ is the characteristic function of this set:



$$|\mu - \mu_0| < \delta$$

- $J'(\eta) = 0 \qquad \Rightarrow$

$$\chi(D)J'(\eta_1 + \eta_2) = 0,$$
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Solve for $\eta_2 = \eta_2(\eta_1)$, set $\mathfrak{J}(\eta_1) = J(\eta_1 + \eta_2(\eta_1))$, consider $\mathfrak{J}'(\eta_1) = 0$

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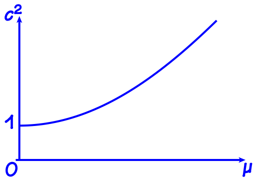

$$c^2 = 1 - \varepsilon,$$

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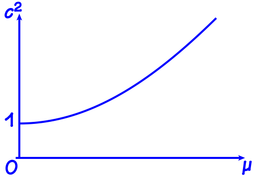

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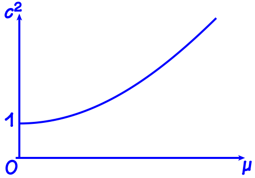

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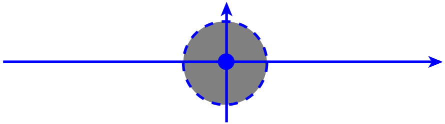

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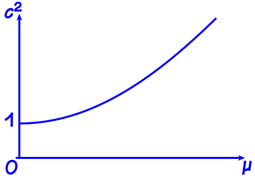

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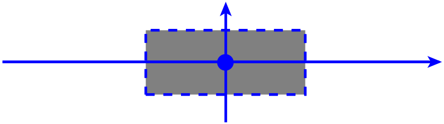

$$|\mu_1| < \delta, \ |\mu_2| < \delta^2$$

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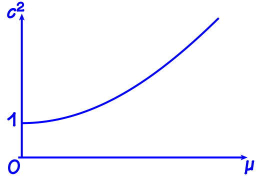

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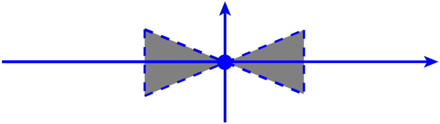

$$|\mu_1| < \delta, \left|\frac{\mu_2}{\mu_1}\right| < \delta$$

- $\mathcal{J}'(\eta) = 0 \qquad \Rightarrow \qquad \begin{array}{l} \chi(D)\mathcal{J}'(\eta_1 + \eta_2) = 0, \\ (1 - \chi(D))\mathcal{J}'(\eta_1 + \eta_2) = 0 \end{array}$

Solve for $\eta_2 = \eta_2(\eta_1)$, set $\mathfrak{J}(\eta_1) = \mathcal{J}(\eta_1 + \eta_2(\eta_1))$, consider $\mathfrak{J}'(\eta_1) = 0$

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- Arrive at the reduced variational functional

$$\tilde{\mathcal{I}}_\varepsilon(\zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{1}{2}(\zeta^2 + (\partial_x^{-1}\zeta_z)^2 + \zeta_x^2) - \frac{1}{3}\zeta^3 \right\} dx\, dz + O(\varepsilon^{1/2}\|\zeta\|^2)$$

or

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- Study this functional in

$$B_R(0) \subseteq X_\varepsilon := \chi(\varepsilon D_1, \varepsilon^2 D_2) X, \qquad X = \overline{\partial_x C_0^\infty(\mathbb{R}^2)}$$

or

$$B_R(0) \subseteq X_\varepsilon := \chi(\varepsilon D) X, \qquad X = \overline{C_0^\infty(\mathbb{R}^2)}$$

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- Look for minimisers of $\tilde{I}_0$ over N

GEOMETRICAL INTERPRETATION

$$N = \{\zeta \neq 0 : \langle \tilde{\Pi}'_0(\zeta), \zeta \rangle = 0\}$$

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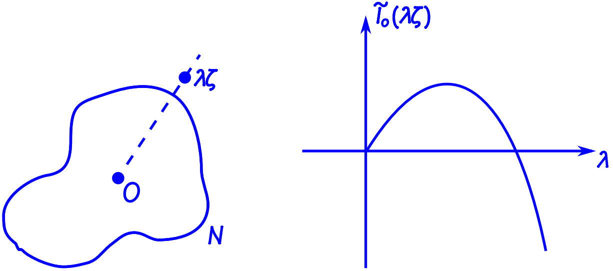
$$N = \{\zeta \neq 0 : \langle \tilde{I}_0'(\zeta), \zeta \rangle = 0\}$$

Any ray

$$\{\lambda\zeta : K(\zeta) > 0, \lambda > 0\}$$

intersects N in precisely one point and the value of $\tilde{I}_0$ along such a ray attains a strict maximum at this point



EXISTENCE THEORY

How to find a minimiser for $\tilde{I}_0(\zeta) = \frac{1}{2}\|\zeta\|^2 - K(\eta)$ over

$$N = \{\zeta \neq 0 : \underbrace{\langle \tilde{I}_0'(\zeta), \zeta \rangle}_{:= F(\zeta)} = 0\}?$$

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  – Key: Weak convergence of $\{\zeta_n\}$ implies convergence of $\{\mathcal{K}(\zeta_n)\}$