Breaking and Modulational Instability

in full-dispersion shallow water models

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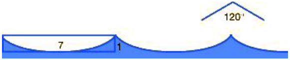
In the 1880s, Stokes made many contributions about periodic traveling waves at the surface of water, for instance, observing that the crest of the wave of greatest possible height is a stagnation point with a 120° corner.



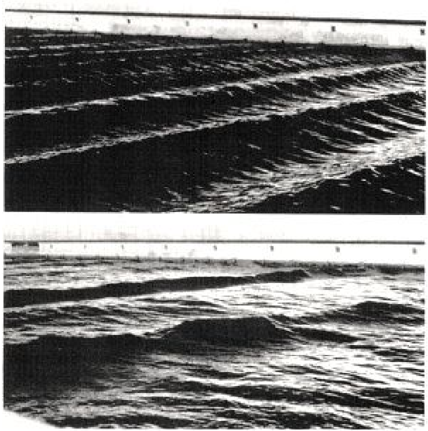
In the 1920s, Levi-Civita and Nekrasov proved the existence of Stokes waves when the amplitude is sufficiently small.

In the early 1960s, Krasov'skii proved the existence, subject to that the maximum slope $< 30^\circ$.
In the early 1980s, Amick and Toland were finally able to prove rigorously Stokes' 120° conjecture at the crest.

Therefore, "no doubt has remained that Stokes waves be theoretically possible as states of dynamic equilibrium."

But, in the mid 1960s, Benjamin had trouble producing Stokes waves in the laboratory and believed that they might be unstable....

Benjamin's experiments



An oscillating plunger generates a train of waves, of wavelength 2.3m in water 7.6m deep, traveling away from the observer in a large towing basin at the Ship Division of the National Physical Laboratory. The upper photograph shows, close to the wave maker, a regular pattern of plane waves except for small-scale roughness. In the lower photograph, some 60m (28 wavelengths) farther along the tank, the same wave train has suffered drastic distortion. The instability was triggered by imposing on the motion of the wave maker a slight modulation at the unstable side-band frequencies; but the same disintegration occurs naturally over a somewhat longer distance.

(photos and caption from Van Dyke, *An Album of Fluid Motion*)

In the mid 1960s, Benjamin and Feir and Whitham formally argued that Stokes waves would be unstable to *sideband perturbations*, if

(carrier wave number)·(undisturbed water depth) $>$ 1.363...

— the Benjamin-Feir or, modulational, instability.

Corroborating results arrived about the same time, but independently, by Lighthill, Benney and Newell, Ostrovsky, Zakharov,... "The idea was emerging when the time was ripe."

[Bridges and Mielke; 1995] studied this in a rigorous fashion. Original arguments, while correct, are hard to justify in an appropriate functional setting.

But the proof breaks down in the infinite depth case. Moreover, it leaves open some important issues, e.g. the full structure of the spectrum of the associated linearized operator.

In the 2000s, huge effort has aimed at translating formal modulation theories to rigorous mathematical theorems.
The arguments make strong use of the Evans function, or other ODE techniques.

But they are not directly applicable, because the water wave problem is inherently nonlocal. In fact, the phase speed in the linear theory is

$$\boxed{c_{ww}^2(k) = \frac{g\tanh(kh)}{k}}$$

$g =$the gravity constant, $h =$the undisturbed fluid depth.

One may resort to simple approximate models to gain insights....

People working in the direction

My collaborators are

Jared Bronski (Illinois)
Mat Johnson (Kansas)
Ashish Pandey (Illinois)
Leeds Tao (UC Riverside)

Other people and groups (incomplete list)

Mats Ernström
John Carter
Henrik Kalisch
Dave Nicholls
Erik Wahlen

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The birth of the Whitham equation

Whitham said, "the breaking phenomenon is one of the most intriguing long-standing problems of water wave theory."

The nonlinear shallow water equations:

$$\boxed{\eta_t + u_x + (u(h+\epsilon\eta))_x = 0, \qquad u_t + g\eta_x + \epsilon u u_x = 0}$$

explain breaking. But it goes too far — all solutions carrying an increase of elevation break.

Neglected dispersion effects inhibit breaking. Recall $c_{ww}(k) = \sqrt{\frac{g\tanh(kh)}{k}} = \sqrt{gh}(1 - \frac{1}{6}(kh)^2) + \cdots$ for $kh \ll 1$.

But the Korteweg-de Vries equation:

$$\boxed{u_t + \sqrt{gh}(1 + \tfrac{1}{6}h^2\partial_x^2)u_x + \tfrac{3}{2}\sqrt{\tfrac{g}{h}}\epsilon u u_x = 0}$$

in turn, goes too far — no solutions break.

Whitham in 1967 proposed

$$u_t + c_{ww}(|D|)u_x + \frac{3}{2}\sqrt{\frac{g}{h}}\epsilon u u_x = 0,$$

combining the dispersion relation of surface water waves and the nonlinearity of the shallow water equations.
He conjectured wave breaking — bounded solutions with unbounded derivatives.

[Seliger; 1968] and [Constantin and Escher; 1998] proved gradient blowup, under some extra assumptions. But they do not include the Whitham equation.
[Naumkin and Shishmarev; 1994] made an alternative breaking argument.

[H.; 2015] proved wave breaking for the Whitham equation, provided that $-\inf u_x(x, 0) \gg 1$.

The Whitham equation, after normalization of parameters,

$$\boxed{u_t + c_{ww}(|D|)u_x + (u^2)_x = 0}$$

admits a one-parameter family of smooth small-amplitude, $2\pi/k$-periodic solutions:

$$u(x) = a\cos(kx) + \frac{1}{2}a^2\Big(\frac{1}{c_{ww}(k) - c_{ww}(0)} + \frac{\cos(2kx)}{c_{ww}(k) - c_{ww}(2k)}\Big) + \cdots$$

for $|a| \ll 1$, traveling without change of form at the constant speed

$$c = c_{ww}(k) + a^2\Big(\frac{1}{c_{ww}(k) - c_{ww}(0)} + \frac{1}{c_{ww}(k) - c_{ww}(2k)}\Big) + \cdots$$

Consider the growing mode problem for the linearized equation

$$\boxed{\lambda v = \partial_x (c_{ww}(|D|) - 2u - c)v =: \mathcal{L}(u)v.}$$

We say u is spectrally unstable if $\operatorname{spec}_{L^2(\mathbb{R})}(\mathcal{L}) \not\subset i\mathbb{R}$.

NB. v needs *not* be $2\pi/k$-periodic.

From Floquet theory,

$$\operatorname{spec}_{L^2(\mathbb{R})}(\mathcal{L}) = \bigcup_{\xi \in (-1/2k, 1/2k]} \operatorname{spec}_{L^2([0,2\pi/k])}(\mathcal{L}_\xi),\ \mathcal{L}_\xi = e^{-i\xi x}\mathcal{L}e^{i\xi x}.$$

The spectrum of $\mathcal{L}_\xi$ consists merely of discrete eigenvalues.

The strategy is: (1) to study the spectrum of $\mathcal{L}_0$ at *the origin*, and (2) to examine how the spectrum of $\mathcal{L}_\xi$ varies for $|\xi| \ll 1$.

BF instability in the Whitham equation

3. Spectrum of $\mathcal{L}_0 = \mathcal{L}$

For $\xi = 0$, zero is a generalized eigenvalue of $\mathcal{L}_0 = \mathcal{L}$ with algebraic multiplicity three and geometric multiplicity two. Moreover,

$$\begin{aligned}
\phi_1(x) &= \cos(kx) + a\frac{-1/2 + \cos(2kx)}{c_{ww}(k) - c_{ww}(2k)} + \cdots , \\
\phi_2(x) &= \sin(kx) + a\frac{\sin(2kx)}{c_{ww}(k) - c_{ww}(2k)} + \cdots , \\
\phi_3(x) &= 1
\end{aligned}$$

are the generalized eigenfunctions for $|a| \ll 1$.

For $a = 0$, $\mathcal{L}_\xi$ has eigenvalues $i(n+\xi)(c_{ww}(k) - c_{ww}(k(n+\xi))$, $n \in \mathbb{Z}$. For $|n| \geqslant 2$, they come with positive Krein signature and do *not* contribute to instability.

BF instability in the Whitham equation

For $|\xi| \ll 1$, $\mathcal{L}_\xi = \mathcal{L}_0 + i\xi[\mathcal{L}_0, x] - \frac{1}{2}\xi^2[[\mathcal{L}_0, x], x] + \cdots$.
Spectra of $\mathcal{L}_\xi$ near the origin agrees with eigenvalues of the 3×3 matrix

$$\big(\langle (\mathcal{L}_\xi - \lambda I)\phi_m, \phi_n \rangle \big)_{m,n=1,2,3}$$

up to terms of order ξ^2.

[H. and Johnson, 2015] proved that a sufficiently small, $2\pi/k$-periodic traveling wave is modulationally unstable, if

$$\boxed{ind(k) = \frac{(kc_{ww}(k))''((kc_{ww}(k))' - c_{ww}(0))}{c_{ww}(k) - c_{ww}(2k)} i_4(k) < 0,}$$

where $i_4(k) = 2(c_{ww}(k) - c_{ww}(2k)) + (kc_{ww}(k))' - c_{ww}(0)$.

It happens when $k > 1.145...$; otherwise spectrally stable to square integrable perturbations.

A few words about the Whitham equation

The Whitham equation seems to successfully explain some high frequency phenomena of water waves.

But the Whitham equation is a heuristic model, combining the full range of dispersion and a nonlinearity of the shallow water theory. It seems difficult to justify the model beyond the KdV scaling regime.

The tools developed here may be potentially useful to the water wave problem and other related ones.

Revisit

$$\boxed{ind(k) = \frac{(kc_{ww}(k))''((kc_{ww}(k))' - c_{ww}(0))}{c_{ww}(k) - c_{ww}(2k)} i_4(k) =: \left(\frac{i_1 i_2}{i_3} i_4\right)(k),}$$

and $i_4(k) = 2i_3(k) + i_2(k)$.

Note that $c_{ww}(k)$=the phase speed, $(kc_{ww}(k))'$=the group speed.

Modulational stability changes when

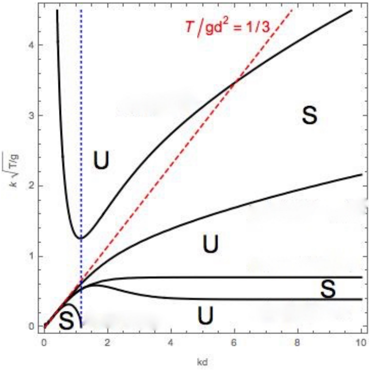
(1) $i_1(k) = 0$; the group speed is an extremum,
(2) $i_2(k) = 0$; the group speed matches the limiting phase speed,
(3) $i_3(k) = 0$; the fundamental harmonic is resonant with the second harmonic.
(4) $i_4(k) = 0$; the dispersion and nonlinear effects are "resonant."

For the Benajmin-Bona-Mahony equation
$u_t - u_{xxt} + u_x + uu_x = 0$, $i_4(k) = 2i_3(k) + c(2k)i_2(k)$.

Effects of surface tension

Let $c(k;T) = \sqrt{\left(g + Tk^2\right)\frac{\tanh(kh)}{k}}$, $T > 0$ is the coefficient of surface tension.

[H. and Johnson, 2015] proved that
If $T/gh^2 < 1/3$, $i_n(k)$, $n = 1, 2, 3, 4$, changes its sign once.
If $T/gh^2 > 1/3$, $i_1(k), i_2(k), i_3(k) > 0$, $i_4(k)$ vanishes once.

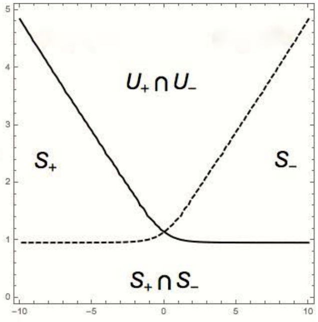


Effects of constant vorticity

Let $c_\gamma(k) = \frac{\gamma \tanh(kh)}{2k} \pm \sqrt{\frac{g \tanh(kh)}{k} + \frac{\gamma^2 \tanh^2(kh)}{4k^2}}$,

γ is the constant vorticity.

For any γ, $i_1(k), i_2(k), i_3(k)$ do not vanish, $i_4(k)$ vanishes once.



The result agrees with Kharif and collaborators' from formal asymptotic expansion for the water wave problem.

Wave breaking for fKdV

Consider $\boxed{u_t + |D|^\alpha u_x + uu_x = 0}$.

$\alpha = 2$ is the KdV equation;
$\alpha = 1$ is the Benjamin-Ono equation;
$\alpha = 0$ is the inviscid Burgers equation;
$\alpha = -1$ is the Burgers-Hilbert equation. (See the works of Hunter, Ifrim, Tataru, and collaborators.)

[Dong, Du and Li; 2009] proved wave breaking for
$u_t - |D|^{\alpha+1}u + uu_x = 0$ for $-1 < \alpha < 0$.
[Castro, Córdoba, Gancedo; 2010], [H.; 2012] proved gradient blowup for $-1 < \alpha < 0$.

[H.; 2015] proved wave breaking for $-1 < \alpha < -1/3$.
[H.; in progress] proved norm inflation in H^s, $5/6 < s < 1/2 - \alpha$ for $-1 < \alpha < -1/3$.

MI of strongly nonlinear waves?

Conserved quantities for are

$$
\begin{aligned}
H &= \int \left(\tfrac{1}{2}u|D|^{\alpha}u - \tfrac{1}{3}u^3\right)\,dx && \text{(Hamiltonian)},\\
&= K + U \\
P &= \int \tfrac{1}{2}u^2\,dx && \text{(momentum)},\\
M &= \int u\,dx && \text{(mass)}.
\end{aligned}
$$

Assume a four-parameter family of smooth, even, and periodic traveling waves $u = u(x - x_0; c, a, T)$ exists, where c is the wave speed, x_0 is the spatial translate, u is T-periodic, and

$$\boxed{|D|^\alpha u - u^2 - cu - a = 0} \text{ for some } a.$$

Under certain nondegeneracy conditions, zero is a generalized eigenvalue of $\mathcal{L}_0 = \mathcal{L}$, and

$$\begin{aligned}
\mathcal{L}v_1 &:= \mathcal{L}u_a = 0, & \mathcal{L}^\dagger w_1 &:= \mathcal{L}^\dagger(M_c u - P_c) = 0,\\
\mathcal{L}v_2 &:= \mathcal{L}u_x = 0, & \mathcal{L}^\dagger w_2 &:= \mathcal{L}^\dagger(\partial_x^{-1}(M_a u_c - M_c u_a)) = w_3,\\
\mathcal{L}v_3 &:= \mathcal{L}u_c = v_2, & \mathcal{L}^\dagger w_3 &:= \mathcal{L}^\dagger(P_a - M_a u) = 0,
\end{aligned}$$

and $\langle w_j, v_k\rangle = (M_c P_a - M_a P_c)\delta_{jk}$.

[Bronski and H.; 2014] proved that u is modulationally unstable, if

$$\mathbf{D} := \begin{pmatrix} \langle w_3, L_1 v_3 \rangle & * & \langle w_3, L_1 v_1 \rangle \\ 1 & \langle w_2, L_1 v_2 \rangle & 0 \\ \langle w_1, L_1 v_3 \rangle & ** & \langle w_1, L_1 v_1 \rangle \end{pmatrix}$$

admits a complex eigenvalue, where $L_1 = [\mathcal{L}, x]$,

$$\begin{aligned}
\langle w_1, L_1 v_1 \rangle &= M_c((1-\alpha)U_a + \alpha c P_a + (\alpha+1)aM_a) + P_c(2P_a + cM_a), \\
\langle w_1, L_1 v_3 \rangle &= M_c((1-\alpha)U_c + \alpha c P_c + (\alpha+1)aM_c) + P_c(2P_c + cM_c), \\
\langle w_3, L_1 v_1 \rangle &= -M_a((1-\alpha)U_a + \alpha c P_a + (\alpha+1)aM_a) - P_a(2P_a + cM_a), \\
\langle w_3, L_1 v_3 \rangle &= -M_a((1-\alpha)U_c + \alpha c P_c + (\alpha+1)aM_c) - P_a(2P_c + cM_c),
\end{aligned}$$

and

$$\langle w_2, L_1 v_2 \rangle = -\alpha(M_c U_a - M_a U_c - c(M_c P_a - M_a P_c)).$$

In the case of the Benjamin-Ono equation ($\alpha = 1$), using Benjamin's explicit form of the solution

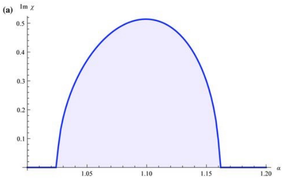
$$u(x; c, a, T) = \frac{\frac{(2\pi/T)^2}{\sqrt{c^2-4a-(2\pi/T)^2}}}{\sqrt{\frac{c^2-4a}{c^2-4a-(2\pi/T)^2}} - \cos(2\pi x/T)} - \frac{1}{2}(\sqrt{c^2 - 4a} + c),$$

where $c < 0$ and $c^2 - 4a - (2\pi/T)^2 > 0$,

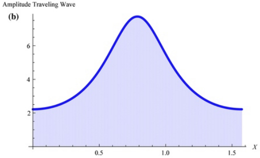
$$\mathbf{D} = \begin{pmatrix} -\pi T & (\pi T)^2(1 - (2\pi/cT)^2) & 0 \\ 1 & \pi T & 0 \\ 2\pi^2 & 0 & \pi T \end{pmatrix}.$$

Eigenvalues are πT and $\pm\pi T\sqrt{2 - (2\pi/cT)^2}$.
Therefore all periodic traveling waves are modulationally stable.

We examined for $c = -5$, $a = 0$, $T = \pi/2$, and for parameter values in the range $1 < \alpha < 2$ (BO-KdV) and, interestingly, found a modulationally unstable wave at $\alpha \approx 1.025$.



the imaginary part of the eigenvalue as a function of α



the wave profile at $\alpha \approx 1.025$, the onset of instability

Compare:

	$u_t + \|D\|^\alpha u_x + uu_x = 0$	$u_t + u_{xxx} + u^p u_x = 0$
L^2 critical	$\alpha = 1/2$	$p = 4$
"L^1" critical	$\alpha = 1$	$p = 2$

Bi-directional or Boussinesq-Whitham equations

In the 1980s, MacKay and Saffman, and others, numerically found other types of instabilities. Explain!

Take the (singular) Boussinesq equation
$\eta_{tt} = gh(1 - \frac{1}{3}\epsilon\partial_x^2)\eta_{xx} + \epsilon(\eta^2)_{xx}$ and "whithamize":

$$\boxed{\eta_{tt} = c_{ww}^2(|D|)\eta_{xx} + \epsilon(\eta^2)_{xx}.}$$

It is linearly ill-posed in the periodic setting. Some nonzero constant solutions are spectrally unstable.
[Deconinck and Trichtchenko; 2015] numerically found high frequency instabilities, though.

Take $(1 + \frac{1}{3}\epsilon\partial_x^2)\eta_{tt} = gh(\eta_{xx} + \epsilon(\eta^2)_{xx})$ and "whithamize":

$$\boxed{c_{ww}^{-2}(|D|)\eta_{tt} = \eta_{xx} + \epsilon(\eta^2)_{xx}.}$$

It is well-posed for short time, but it fails the BF instability.
[H. and Pandey; 2016] proved modulational stability of the (regularized) Boussinesq equation.

I propose the *full-dispersion shallow water equations*:

$$\eta_t + u_x + (u(h+\epsilon\eta))_x = 0, \qquad u_t + c_{ww}(|D|)^2\eta_x + \epsilon u u_x = 0$$

Recalling $c_{ww}^2(k) = gh(1 - \frac{1}{3}(kh)^2) + \cdots$ for $kh \ll 1$, they approximate the shallow water equations:

$$\eta_t + u_x + (u(h+\epsilon\eta))_x = 0, \qquad u_t + g\eta_x + \epsilon u u_x = 0,$$

(all solutions break), and also Boussinesq equations

$$\eta_t + u_x + (u(h+\epsilon\eta))_x = 0, \qquad (1 + \tfrac{1}{3}\epsilon\partial_x^2)u_t + g\eta_x + \epsilon u u_x = 0$$

(no solutions break).

[H. and Tao; 2015] proved breaking for

$$\eta_t + u_x + u(h+\eta_x) = 0, \qquad u_t + c_{ww}(|D|)^2\eta_x + u u_x = 0.$$

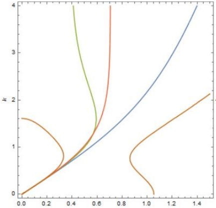
[H. and Pandey; 2016] proved that a sufficiently small, $2\pi/k$-periodic traveling wave of is modulationally unstable, if

$$\boxed{ind(k) = \frac{(kc_{ww}(k))''(((kc_{ww}(k))')^2 - c_{ww}^2(0))}{c_{ww}^2(k) - c_{ww}^2(2k)} i_4(k) < 0,}$$

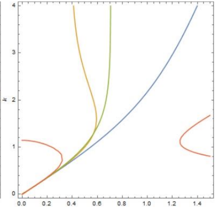
where $i_4(k) = 3c_{ww}(k)(c_{ww}^2(k) + 5c_{ww}^4(k) - 2c_{ww}^2(2k)(c_{ww}^2(k) + 2)) + 18kc_{ww}^4(k)c_{ww}'(k) + k^2c_{ww}(k)(c_{ww}'(k))^2(4c_{ww}^2(2k) + 5c_{ww}^2(k)).$

It happens when $k > 1.610...$; otherwise modulationally stable.

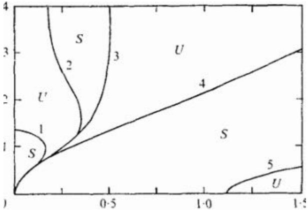
Including the effects of surface tension,



bi-directional Whitham



uni-directional Whitham



the water wave problem

the result agrees with [Kawahara;1975], [Djordjevic and Redekopp; 1977] via formal multi scale expansion.

Around the no-wave solution, infinitely many collisions of pairs of purely imaginary eigenvalues away from the origin.

They do not lead to high frequency instabilities up to $|a|$.

The result agrees with those by Akers and Nicholls, and others, via a formal argument.

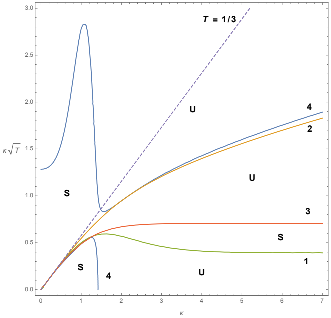
Effects of higher order nonlinearities

[Lannes; 2013] proposed the full-dispersion Camass-Holm equation

$$\eta_t + c_{\mathrm{ww}}(|D|)\eta_x + \frac{3\eta}{1+\sqrt{1+\eta}}\eta_x = -\Big(\frac{5}{12}\eta\eta_{xxx} + \frac{23}{24}\eta_x\eta_{xx}\Big).$$

[H. and Pandey; in progress] proved that a small-amplitude $2\pi/k$ periodic traveling wave is modulationally unstable if $k > 1.42\ldots$.

Including the effects of surface tension,



Effects of three dimensions

[Lannes; 2013] proposed the full-dispersion Kadomtsev-Petviashvili equation

$$\boxed{u_t + \mathcal{H}\sqrt{|D_x|^2 + |D_y|^2}\, c_{ww}(|D_x|^2 + |D_y|^2)u + uu_x = 0}$$

[Pandey; in progress] proved that a small-amplitude $2\pi/k$-periodic traveling wave of the Whitham equation is transversally unstable if

$$\text{ind}(k) = c_{ww}(k) - c_{ww}(2k) < 0.$$

If $T = 0$, stability.
If $0 < T < 1/3$, instability for $k > k_c$ for some k_c.
If $T > 1/3$, instability for all $k > 0$.