Constant vorticity water waves in holomorphic coordinates

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This is joint work with Daniel Tataru

Constant vorticity water waves

The setting:

- inviscid incompressible fluid flow (Euler equations)
- infinite bottom
- free boundary
- constant vorticity
- with gravity
- periodic or nonperiodic setting

Question:

- Study long time solutions: lifespan bounds for small data.

Overview:

Earlier work:

- Local well-posedness in Sobolev spaces:
  - Lindblad & Christodoulou (2000), Lindblad (2003)
  - Lindblad (2005) - higher regularity ($s > n/2 + 3/2$) (all dimensions)
  - Coutand-Shkoller (2006) - regularity of the initial data in H^3 (3D)

Our goals:

- To use the formulation of the equations in holomorphic coordinates (Nalimov '74) in order to provide a simpler approach to the local problem
- To use the quasilinear modified energy method we previously introduced, improving on the normal form method. This method yields an easier route to long time solutions

The standard formulation

Fluid domain: fluid body $\Omega(t)$, free boundary $\Gamma(t)$.
Parameters: velocity field u, pressure p, gravity g, no surface tension

Euler equations in the fluid domain $\Omega(t)$:

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p - gj \\ \text{div } u = 0 \\ \text{curl } u = c \neq 0 \\ u(0,x) = u_0(x) \end{cases}$$

Boundary conditions on $\Gamma(t)$:

$$\begin{cases} \partial_t + u\cdot\nabla \text{ is tangent to } \bigcup \Gamma(t) & \text{(kinematic)} \\ p = p_0 \quad \text{on } \Gamma(t) & \text{(dynamic)} \end{cases}$$

$\Gamma(t)$ is transported along the flow; same happens with the vorticity!

Rotational flows

Velocity potential φ

$$\begin{cases} u_x + y_y = 0 \\ \omega = u_y - v_x = -c, \end{cases}$$

$$\Rightarrow \quad u = (cy + \varphi_x, \varphi_y), \quad \Delta\varphi = 0 \quad \text{in } \Omega(t).$$

(Generalized) velocity potential

Then φ is uniquely determined by its values on the free surface $\Gamma(t)$.

Analogue to the Bernoulli's equation:

Dynamic boundary condition (harmonic conjugate θ):

$$\varphi_t - c\theta + cy\varphi_x + \frac{1}{2}(\varphi_x^2 + \varphi_y^2) + gy = 0 \text{ on } \Gamma(t).$$

Reducing the dimensionality

Equations reduced to the boundary in Eulerian formulation in (η, ψ), where $\psi(t,x) = \varphi(t, \eta(t,x)))$ (Zakharov 1968):

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta + \frac{1}{2}|\nabla \psi|^2 - \frac{1}{2}\frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = 0. \end{cases}$$

$$\sqrt{1+\eta_x^2}(\triangledown \varphi \cdot N) = G(\eta) = \text{ Dirichlet to Neuman operator,}$$

where N is the unit normal at the free surface.

Closed system in (η, φ)

Not trivial to see that this is a closed system of equations in $(\eta, \text{“}\varphi\text{”}, \omega)$! Check D. Lannes and A. Castro paper: *Well-posedness and shallow water stability for a new hamiltonian formulation of the water waves equations with vorticity*

Holomorphic (Conformal) coordinates:

Not new: Ovsjannikov, Dyachenko-Zakharov-Kuznetsov, Wu, Choi-Camassa, Li-Hyman-Choi, Hunter-I.-Tataru, I.-Tataru, ...

- Conformal map:

$$Z : \{\Im z \leq 0\} \to \Omega(t), \qquad Z(\alpha + i\beta) - (\alpha + i\beta) \to 0 \ \text{ at } \ \text{infinity}$$

- Boundary condition at infinity:

$$Z(\alpha) - \alpha \to 0 \ \ (nonperiodic) \qquad Z(\alpha) - \alpha, Q \ periodic \ \ (periodic)$$

- Taking φ to be the velocity potential we define $\psi = \varphi \circ z$ and take its harmonic conjugate to be θ
- We then define the holomorphic function (call it holomorphic velocity potential)

$$Q(t, \alpha) = \psi(t, \alpha, 0) + i\theta(t, \alpha, 0)$$

- We may then write the water wave equations as a system for the holomorphic functions (Z, Q). Pert. of steady state: $W = Z - \alpha$.

Water wave equations in holomorphic coords.

- **P** - Projection onto negative wavenumbers

Equations for (W, Q):

$$\begin{cases} W_t + (W_\alpha + 1)\underline{F} + i\frac{c}{2}W = 0 \\ Q_t - igW + \underline{F}Q_\alpha + icQ + \mathbf{P}\left[\frac{|Q_\alpha|^2}{J}\right] - i\frac{c}{2}T_1 = 0, \end{cases}$$

where

$$J := |1 + W_\alpha|^2, \qquad \mathbf{P} = \frac{1}{2}(I - iH),$$

$$F := \mathbf{P}\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \quad F_1 := \mathbf{P}\left[\frac{W}{1 + \bar{W}_\alpha} + \frac{\bar{W}}{1 + W_\alpha}\right],$$

$$\underline{F} := F - i\frac{c}{2}F_1, \qquad T_1 := \mathbf{P}\left[\frac{W\bar{Q}_\alpha}{1 + \bar{W}_\alpha} - \frac{\bar{W}Q_\alpha}{1 + W_\alpha}\right].$$

These equations are considered either in $\mathbb{R} \times \mathbb{R}$ or in $\mathbb{R} \times \mathbb{S}^1$.

Conserved energies:

- The Hamiltonian:

$$\mathcal{E}(W,Q) = \Re \int g|W|^2(1+W_\alpha) - iQ\bar{Q}_\alpha + cQ_\alpha(\Im W)^2 - \frac{c^3}{2i}|W|^2 W(1+W_\alpha)\, d\alpha$$

- The horizontal momentum:

$$\mathcal{P}(W,Q) = \int \left\{ \frac{1}{i}\left(\bar{Q}W_\alpha - Q\bar{W}_\alpha\right) - c|W|^2 + \frac{c}{2}\left(W^2\bar{W}_\alpha + \bar{W}^2 W_\alpha\right) \right\} d\alpha$$

Symmetries:

- Translations in α and t.
- The space-time scaling (g unchanged , but $c \to \lambda c$)

$$(W(t,\alpha), Q(t,\alpha)) \to \lambda^{-2} W(\lambda t, \lambda^2 x), \lambda^{-3} Q(\lambda t, \lambda^2 x)$$

- The purely spatial scaling (c unaffected, but $g \to \lambda^{-1} g$)

$$(W(t,\alpha), Q(t,\alpha)) \to \lambda^{-2} W(t, \lambda^2 x), \lambda^{-3} Q(t, \lambda^2 x)$$

Physical parameters:

- $R = \dfrac{Q_\alpha}{1+W_\alpha}$ - velocity field on the free boundary
- $\underline{b} := \ b - i\dfrac{c}{2}b_1$ - advection coeff. (high freq. velocity limit)
  - $b = 2\Re \mathbf{P}\left[\dfrac{R}{1+\bar{W}_\alpha}\right]$
  - $b_1 := \mathbf{P}\left[\dfrac{W}{1+\bar{\mathbf{W}}}\right] - \bar{\mathbf{P}}\left[\dfrac{\bar{W}}{1+\mathbf{W}}\right]$
- $\underline{a} := \ a + \dfrac{c}{2}a_1$ - normal derivative of the pressure $= g + \underline{a}$
  - $a := 2\Im \mathbf{P}[R\bar{R}_\alpha] > 0$
  - $a_1 := 2\Re R - N$, where $N := 2\Re \mathbf{P}\left[W\bar{R}_\alpha - \bar{\mathbf{W}}R\right]$

Alternate quasilinear system for diagonal variables $(\mathbf{W} = W_\alpha, R)$

Almost self-contained system in $(\mathbf{W}, R)$:

$$\begin{cases} \mathbf{W}_t + \mathbf{P}[\underline{b}\,\mathbf{W}_\alpha] + \mathbf{P}\left[\dfrac{(1+\mathbf{W})R_\alpha}{1+\bar{\mathbf{W}}}\right] = G \\ R_t + icR + \mathbf{P}[\underline{b}\,R_\alpha] + i\mathbf{P}\left[\dfrac{g+\underline{a}}{1+\mathbf{W}}\right] = K, \end{cases}$$

where

$$G = (1+\mathbf{W})\mathbf{P}\left[\frac{\bar{R}_\alpha}{1+\mathbf{W}} + \frac{R\bar{\mathbf{W}}_\alpha}{(1+\bar{\mathbf{W}})^2}\right] + [\mathbf{P}, \mathbf{W}]\left(\frac{R_\alpha}{1+\mathbf{W}} + \frac{\bar{R}\mathbf{W}_\alpha}{(1+\mathbf{W})^2}\right)$$
$$+ i\frac{c}{2}\mathbf{P}\left[(1+\mathbf{W})M_1\right] + i\frac{c}{2}\mathbf{P}\left[\mathbf{W}(\mathbf{W}-\bar{\mathbf{W}})\right]$$
$$K = -\mathbf{P}\left[R\bar{R}_\alpha\right] - i\frac{c}{2}\mathbf{P}N$$

represent perturbative terms in the equation.
$M_1 := \mathbf{P}\left[W\bar{Y}\right]_\alpha - \bar{\mathbf{P}}\left[\bar{W}Y\right]_\alpha$

The linearized equation

In linearized var. (w, q): non-diagonal degenerate first order hyperbolic system. Better, use diagonal variables $(w, r) = \mathbf{A}(w, q) = (w, q - Rw)$:

$$\begin{cases} w_t + \mathbf{P}\left[\underline{b} w_\alpha\right] + \mathbf{P}\left[\dfrac{1}{1+\bar{\mathbf{W}}} r_\alpha\right] + \mathbf{P}\left[\dfrac{R_\alpha}{1+\bar{\mathbf{W}}} w\right] = \mathcal{G}(w, r) \\ r_t + \mathbf{P}\left[\underline{b} r_\alpha\right] + icr - i\mathbf{P}\left[\dfrac{g+\underline{a}}{1+\mathbf{W}} w\right] = \mathcal{K}(w, r). \end{cases}$$

Quasilinear energy:

$$E(w, r) = \int_{\mathbb{R}} (g + \underline{a})|w|^2 + \Im(\bar{r} r_\alpha) d\alpha$$

Control norms for purely gravity waves:

$$A := \|\mathbf{W}\|_{L^\infty} + \|Y\|_{L^\infty} + \||D|^{\frac{1}{2}} R\|_{L^\infty} \qquad \text{(scale invariant)}$$

$$B := \||D|^{\frac{1}{2}} \mathbf{W}\|_{BMO} + \|R_\alpha\|_{BMO} \qquad \text{(control. by } \|(\mathbf{W}, R)\|_{\dot{\mathcal{H}}^1}\text{)}$$

$$\|(w, r)\|^2_{\dot{\mathcal{H}}_n} := \sum_{k=0}^{n} \|\partial_\alpha^k (w, r)\|^2_{L^2 \times \dot{H}^{\frac{1}{2}}}, \; n \geq 1.$$

Control norms for lower order terms introduced by ω:

$$A_{-1/2} := \||D|^{\frac{1}{2}} W\|_{L^\infty} + \|R\|_{L^\infty} \quad \text{(controlled by } \dot{\mathcal{H}}_0 \text{ norm of } (\mathbf{W}, R))$$

$$A_{-1} := \|W\|_{L^\infty} \quad \text{(controlled by } \dot{\mathcal{H}}_{\frac{1}{2}} \text{ norm of } (W, Q))$$

Notations:

$$\underline{B} := B + cA + c^2 A_{-1/2}, \qquad \underline{A} := A + cA_{-1/2} + c^2 A_{-1}.$$

Energy estimate:

$$\frac{d}{dt} E(w, r) \lesssim_{\underline{A}} (\underline{B} + c\underline{A}) E(w, r)$$

Local well-posedness:

The translation invariance assures us that the pair $(\mathbf{W}, R)$ solves the linearized equation $\to$ leads to the local well-posedness result.

Normal forms and long time existence

Goal: Find improved lifespan estimates for small data solutions.

(i) Equations with quadratic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\| E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-1}$

(ii) Equations with cubic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\|^2 E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-2}$

Solution 1: Normal form method (Shatah '85): transform an equation with a quadratic nonlinearity into one with a cubic one via a normal form transformation,

$$u \to v = u + B(u, u).$$

Solution 2: Improved normal form choices

Find a cubic and higher correction

$$u \to v = u + B(u,u) + B^{3+}(u), \text{ such that}$$

- the normal form transformation becomes invertible
- the equation in v is a *good* equation to work with

Ways to achieve this:

- S. Wu (2009, gravity water waves): direct (partial) change of coordinates and a secondary normal form transfomation
- Hunter-I. (2011, Burgers-Hilbert equation, and gravity water waves): change of coordinates obtained via a flow:

$$(\star) \quad \frac{d}{d\epsilon} u_\epsilon = B(u_\epsilon, u_\epsilon), \quad u_0 = u, \quad t = 1 \to v = u_1$$

  - Burgers-Hilbert (2010) : eqn. $(\star)$ is Hilbert (Burgers).
  - Gravity water waves (2011): eqn. $(\star)$ is Hilbert (Burgers) in both variables.
    Just "rediscovered" by Craig, Walter in 2016.

Solution 3: Quasilinear modified energy method

The idea is to change the energy rather than change the equations. This was first implemented in for the *Burgers-Hilbert equation* by Hunter-I. Tataru-Wong (2012), and then to various water wave equations:

- *Gravity waves in deep water* by Hunter-I.-Tatatru (2014),
- *Capillary waves in deep water* by I.-Tataru (2014),
- *Constant vorticity gravity waves in deep water* by I.-Tataru (2015),
- *Finite depth gravity waves* by Harrop-Griffiths-I.-Tataru (2016)

The goal is to construct an energy $E^n(u)$ with the properties:

- It is equivalent to the linear energy functional of the problem

$$E^n(u) = (1 + O(\|u\|))\|u\|_{H^n}^2$$

- It has good cubic energy estimates

$$\frac{d}{dt}E^n(u) \lesssim \|u\|^2 E^n(u).$$

Dispersive character ($c \neq 0$ gravity waves)

Linearization around zero:

$$\begin{cases} \partial_t w + q_\alpha = 0 \\ \partial_t q + icq - igw = 0 \end{cases}$$

Energy:

$$\mathcal{E} = \int_{\mathbf{R}} |w|^2 - iq\bar{q}_\alpha \, d\alpha = \|(w,q)\|^2_{\dot{\mathcal{H}}^0} = \|w\|^2_{L^2} + \|q\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Dispersion relation:

$$\tau^2 + c\tau + g\xi = 0, \qquad \xi \leq 0$$

Group velocity of waves:

$$v = \pm \frac{g}{\sqrt{c^2 - 4g\xi}}$$

Quasilinear model:

$$\begin{cases} (\partial_t + b\partial_\alpha) w + q_\alpha = 0 \\ (\partial_t + b\partial_\alpha) q + icq - i(g + a) w = 0 \end{cases}$$

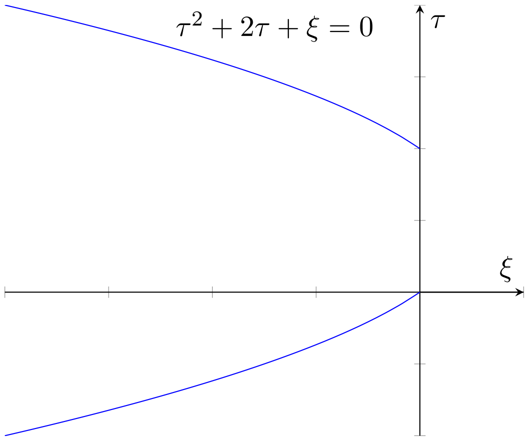

Figure : Dispersion relation (v)

Normal forms for gravity water waves

Existence of a normal form transformation is related to the absence of resonant bilinear interactions. For 2-d gravity waves in holomorphic coordinates, such a normal form transformation exists and is given by

$$\tilde{W} = W + \mathbf{P}W^{[2]}$$
$$\tilde{Q} = Q + \mathbf{P}Q^{[2]}$$

The normal variables solve an equation of the form

$$\begin{cases} \partial_t \tilde{W} + \tilde{Q}_\alpha = \text{ cubic and higher} \\ \partial_t \tilde{Q} - i\tilde{W} = \text{ cubic and higher} \end{cases}$$

However, the cubic and higher nonlinearities also contain higher derivatives, so one cannot close the energy estimates. This is related to the fact that the normal form transformation is not invertible, and further to the fact that the water wave equation is quasilinear, rather than semilinear.

The expressions for $(W^{[2]}, Q^{[2]})$:

$$
\begin{aligned}
W^{[2]} = & -(W+\bar{W})W_\alpha - \frac{c}{2g}\left[(Q+\bar{Q})W_\alpha + (W+\bar{W})Q_\alpha\right] \\
& + \frac{ic^2}{2g}\left[(\partial^{-1}W - \partial^{-1}\bar{W})W_\alpha + W^2 + \frac{1}{2}|W|^2\right] - \frac{c^2}{4g^2}(Q+\bar{Q})Q_\alpha \\
& + \frac{ic^3}{4g^2}\left[(Q+\bar{Q})W + (\partial^{-1}W - \partial^{-1}\bar{W})Q_\alpha\right] \\
& + \frac{c^4}{4g^2}(\partial^{-1}W - \partial^{-1}\bar{W})W, \\
Q^{[2]} = & -(W+\bar{W})Q_\alpha - \frac{c}{2g}(Q+\bar{Q})Q_\alpha + \frac{ic}{4}(W^2 + 2|W|^2) \\
& + \frac{ic^2}{2g}\left[(\partial^{-1}W - \partial^{-1}\bar{W})Q_\alpha + \frac{1}{2}(Q+\bar{Q})W\right] \\
& + \frac{c^3}{4g}(\partial^{-1}W - \partial^{-1}\bar{W})W.
\end{aligned}
$$

The modified energy method

Idea: Modify the energy rather than the equation in order to get cubic energy estimates.

Step 1: Construct a cubic normal form energy

$$E^n_{NF}(W,Q) = (quadratic + cubic)(\|\tilde{W}^{(n)}\|^2_{L^2} + \|\tilde{Q}^{(n)}\|^2_{\dot{H}^{\frac{1}{2}}})$$

Then

$$\frac{d}{dt}E^n_{NF}(W,Q) = quartic + higher$$

Here higher derivatives arise on the right, making it impossible to close.

Step 2: Switch $E^n_{NF}(W,Q)$ to diagonal variables $E^n_{NF}(\mathbf{W},R)$.

Step 3: To account for the fact that the equation is quasilinear, replace the leading order terms in $E^n_{NF}(\mathbf{W},R)$ with their natural quasilinear counterparts to obtain a good cubic quasilinear energy $E^n(\mathbf{W},R)$.

Cubic estimates, I:

Observation

- We do not have cubic energy estimates for the linearized equation as in the irrotational case!
- We do have cubic energy estimates for the diff. eqs in $(\mathbf{W}, R)$ etc.

Toy model:

$$\begin{cases} w_t + \mathbf{P}\left[\underline{b} w_\alpha\right] + \mathbf{P}\left[\frac{1}{1+\bar{\mathbf{W}}} r_\alpha\right] + \mathbf{P}\left[\frac{R_\alpha}{1+\bar{\mathbf{W}}} w\right] = -\mathbf{P}\left[\mathbf{W}\bar{r}_\alpha\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathbf{P}\left[R\bar{w}_\alpha\right] + G \\ r_t + \mathbf{P}\left[\underline{b} r_\alpha\right] + icr - i\mathbf{P}\left[\frac{g+\underline{a}}{1+\mathbf{W}} w\right] = -\mathbf{P}[R\bar{r}_\alpha] + K. \end{cases}$$

Modified energy:

$$E^3(w,r) := \int_{\mathbb{R}} (g+\underline{a})|w|^2 + \Im(r\bar{r}_\alpha) + 2\Im(\bar{R} w r_\alpha) - 2\Re(\bar{\mathbf{W}} w^2)\, d\alpha.$$

Cubic estimates, II:

- Equivalence:

$$E^3(w,r) \approx E^2(w,r).$$

- Estimate:

$$\begin{aligned}\frac{d}{dt}E^3(w,r) = 2\Re \int_{\mathbf{R}} & \left[(g+\underline{a})\bar{w} - i\bar{R}_\alpha r_\alpha - 2\bar{\mathbf{W}} w\right] G \\ & - i\left[\bar{r}_\alpha - (\bar{R}w)_\alpha\right] K \, d\alpha \\ & + c^2 \Im R|w|^2 \, d\alpha + O_A(\underline{A}\,\underline{B})E^2(w,r).\end{aligned}$$

($\dot{\mathcal{H}}^0$ bound for linearized eqn $(w,r) = (w, q - Rw)$)

Energy estimates for constant vorticity gravity waves

Applies directly to $(\mathbf{W}, R)$. Higher order counterpart holds.

Energy estimates:

We can construct energy functionals $E^n(\mathbf{W}, R)$ with these properties:

(i) Energy equivalence:

$$E^{n,(3)}(\mathbf{W}, R) = (1+O(\underline{A}))\mathcal{E}(\partial^n \mathbf{W}, \partial^n R)+O(c^4\,\underline{A})\mathcal{E}(\partial^{n-1}\mathbf{W}, \partial^{n-1}R)$$

(ii) Cubic energy estimate:

$$\frac{d}{dt}E^{n,(3)}(\mathbf{W}, R) \lesssim_{\underline{A}} \underline{B}\,\underline{A}\,\left(\mathcal{E}(\partial^n \mathbf{W}, \partial^n R) + c^4\mathcal{E}(\partial^{n-1}\mathbf{W}, \partial^{n-1}R)\right).$$

Cubic lifespan bounds

Theorem

Let (W, Q) be a solution for the system system whose initial data satisfies

$$\|(W_0, Q_0)\|_{\dot{\mathcal{H}}_0} + \|(\mathbf{W}_0, R_0)\|_{\dot{\mathcal{H}}_1} \leq \epsilon \ll 1.$$

Then the solution exists for a time $T_\epsilon \approx \epsilon^{-2}$, with bounds

$$\|(W, Q)(t)\|_{\dot{\mathcal{H}}_0} + \|(\mathbf{W}, R)(t)\|_{\dot{\mathcal{H}}_1} \lesssim \epsilon, \qquad |t| < T_\epsilon.$$

Further, higher regularity is also preserved,

$$\|(\mathbf{W}, R)(t)\|_{\dot{\mathcal{H}}_n} \lesssim \|(\mathbf{W}, R)(0)\|_{\dot{\mathcal{H}}_n} \qquad |t| < T_\epsilon,$$

whenever the norm on the right is finite.

- Proof idea: *quasilinear modified energy method*
- Bounds for these and higher norms propagate on same timescale.

Later similar result obtained by Bieri, Miao, Shahsahani, Wu.

Thank you !