Stability of capillary waves on fluid sheets

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Theoretical and Computational Aspects of Nonlinear Surface Waves

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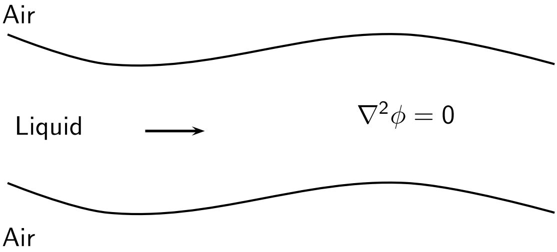
Acknowledgment

Joint work with Mark Blyth (UEA)

Problem sketch

Waves on a fluid sheet



Assume inviscid, incompressible and irrotational flow

Steady capillary waves

- Crapper (1957) Exact solution. Limiting profile (trapped bubbles)
- Kinnersley (1976) Finite depth - symmetric and antisymmetric waves
- Crowdy (1999) Simplified version of Kinnersley symmetric waves
- Blyth & Vanden-Broeck (2004) Bifurcations from symmetric branch

Stability of deep-water capillary waves

- Hogan (1988) superharmonic perturbations
- Chen & Saffman (1985) subharmonic perturbations
- Tiron & Choi (2012) super/subharmonic perturbations

The problem is

$$\nabla^2 \phi = 0$$

with

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + (\gamma/\rho)\kappa = 0, \quad \bar{y}_t + \bar{y}_X \phi_X = \phi_Y$$

on $Y = \bar{y}(X, t)$

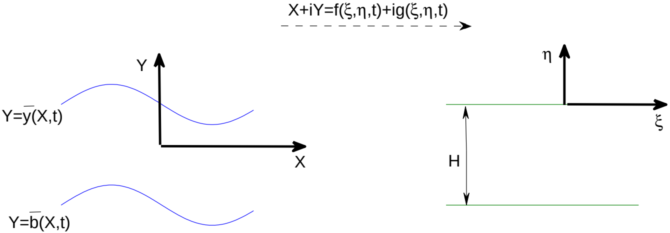
and

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 - (\gamma/\rho)\hat{\kappa} = 0, \quad \bar{b}_t + \bar{b}_X \phi_X = \phi_Y$$

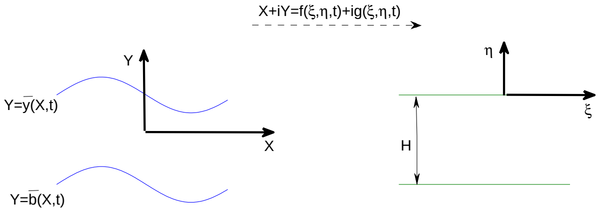
on $Y = \bar{b}(X, t)$

Aim: Rewrite in terms of surface variables

Conformal mapping (Dyachenko *et al.* 1996; Choi & Camassa 1999)



$$X + iY = f(\xi, \eta, t) + i\, g(\xi, \eta, t)$$



$f + ig$ analytic in the strip. Conformal map:

$$g_{\xi\xi} + g_{\eta\eta} = 0 \qquad \text{for } -H \leq \eta \leq 0$$

with $g = y(\xi, t)$ at $\eta = 0$ and $g = \hat{y}(\xi, t)$, where

$$\bar{y}(X(\xi, 0, t), t) = y(\xi, t) \qquad \bar{b}(X(\xi, -H, t), t) = \hat{y}(\xi, t)$$

Write

$$y(\xi, t) = \sum_{n=-\infty}^{\infty} a_n(t) \mathrm{e}^{\mathrm{i}n\xi}, \qquad \hat{y}(\xi, t) = \sum_{n=-\infty}^{\infty} b_n(t) \mathrm{e}^{\mathrm{i}n\xi}$$

Write

$$y(\xi, t) = \sum_{n=-\infty}^{\infty} a_n(t) e^{in\xi}, \qquad \hat{y}(\xi, t) = \sum_{n=-\infty}^{\infty} b_n(t) e^{in\xi}$$

Solution

$$g = a_0 + (a_0 - b_0)\eta/H + \sum{}' \left[a_n \frac{\sinh(n[\eta + H])}{\sinh(nH)} - b_n \frac{\sinh(n\eta)}{\sinh(nH)} \right] e^{in\xi},$$

Write

$$y(\xi,t)=\sum_{n=-\infty}^{\infty}a_n(t)\mathrm{e}^{\mathrm{i}n\xi},\qquad \hat{y}(\xi,t)=\sum_{n=-\infty}^{\infty}b_n(t)\mathrm{e}^{\mathrm{i}n\xi}$$

Solution

$$g=a_0+(a_0-b_0)\eta/H+\sum{}' \left[a_n\,\frac{\sinh(n[\eta+H])}{\sinh(nH)}-b_n\,\frac{\sinh(n\eta)}{\sinh(nH)}\right]\mathrm{e}^{\mathrm{i}n\xi},$$

Cauchy-Riemann: $f_\xi=g_\eta,\ f_\eta=-g_\xi \quad \left[X+iY=f(\xi,\eta,t)+i\,g(\xi,\eta,t)\right]$

Match periods: $H=a_0-b_0$

We have

$$x_\xi = 1 - T\left(y_\xi\right) + S\left(\hat{y}_\xi\right), \qquad \hat{x}_\xi = 1 - S\left(y_\xi\right) + T\left(\hat{y}_\xi\right),$$

Non-local operators

$$T(f(\xi)) = \frac{1}{2H} \int_{-\infty}^{\infty} f(\xi') \coth\left[\frac{\pi}{2H}(\xi' - \xi)\right] \, \mathrm{d}\xi'$$

$$S(f(\xi)) = \frac{1}{2H} \int_{-\infty}^{\infty} f(\xi') \tanh\left[\frac{\pi}{2H}(\xi' - \xi)\right] \, \mathrm{d}\xi'$$

Working similarly, for the fluid problem we obtain

$$\psi_\xi = T(\phi_\xi) - S(\hat{\phi}_\xi), \qquad \hat{\psi}_\xi = S(\phi_\xi) - T(\hat{\phi}_\xi)$$

where $\phi(\xi, t)$, $\psi(\xi, t)$ etc. are surface values.

Kinematic conditions

Upper wave

$$y_t = y_\xi \left[T\left(\frac{\psi_\xi}{J}\right) - S\left(\frac{\hat{\psi}_\xi}{\hat{J}}\right) \right] - \frac{x_\xi \psi_\xi}{J}.$$

Lower wave

$$\hat{y}_t = \hat{y}_\xi \left[S(\psi_\xi / J) - T(\hat{\psi}_\xi / \hat{J}) \right] - \hat{x}_\xi \hat{\psi}_\xi / \hat{J}.$$

$$J = x_\xi^2 + y_\xi^2, \qquad \hat{J} = \hat{x}_\xi^2 + \hat{y}_\xi^2$$

- Upper surface (x, y) equations

$$y_t = y_\xi \Big[T(\psi_\xi/J) - S(\hat{\psi}_\xi/\hat{J}) \Big] - x_\xi \psi_\xi/J,$$

$$\phi_t + \Big[S(\hat{\psi}_\xi/\hat{J}) - T(\psi_\xi/J) \Big] \phi_\xi + \frac{1}{2J} \left(\phi_\xi^2 - \psi_\xi^2 \right) + (\gamma/\rho)\kappa = 0$$

$$\psi_\xi = T(\phi_\xi) - S(\hat{\phi}_\xi), \qquad \hat{\psi}_\xi = S(\phi_\xi) - T(\hat{\phi}_\xi)$$

$$\kappa = \frac{y_\xi x_{\xi\xi} - x_\xi y_{\xi\xi}}{J^{3/2}}$$

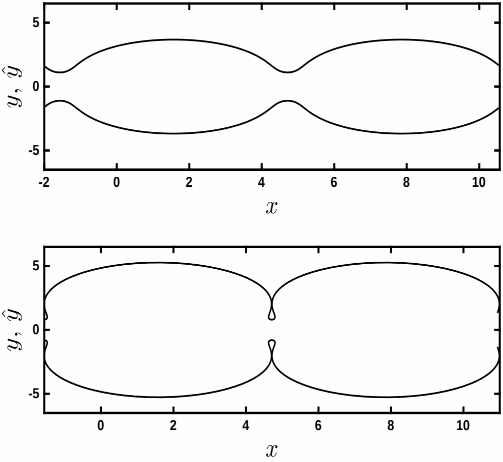
Similar equations for the lower wave

Solution parameters: (H, c)

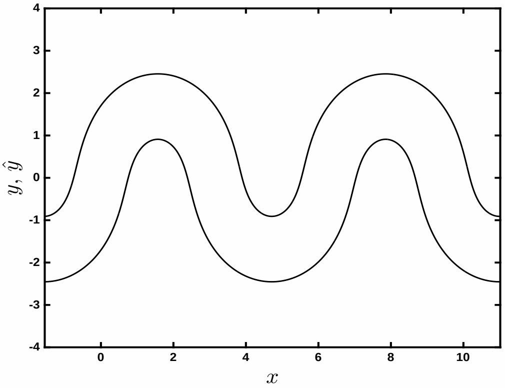
Physical sheet thickness

$$\mathcal{H} = \int_0^1 (y x_\xi - \hat{y}\hat{x}_\xi)\,\mathrm{d}\xi.$$

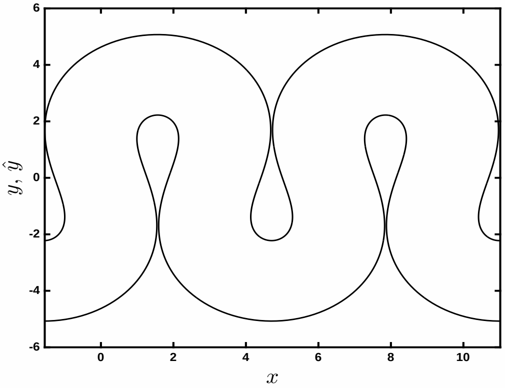
Steady travelling waves



$H = 4$ and two different c



$H = 1$



$H = 1$

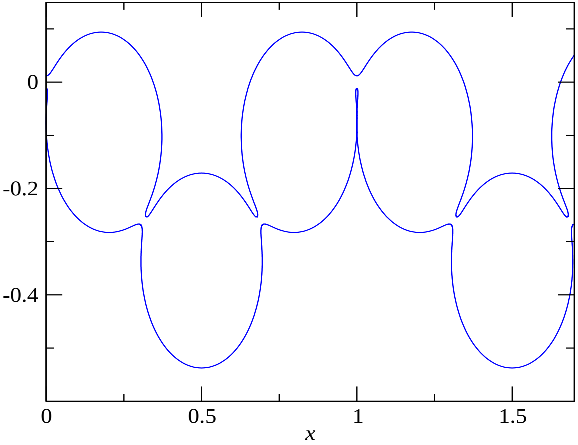
For fixed H:

$c = c_s$ or c_a small amplitude waves

$c = c^*$ trapped bubbles

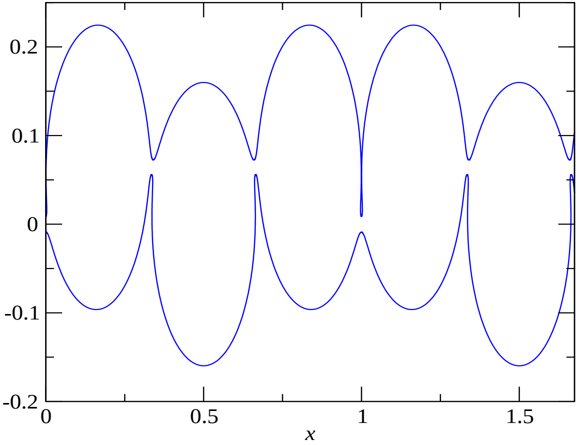
$c_s = \sqrt{\tanh(H/2)}$ (symmetric base wave)

$c_a = \sqrt{\coth(H/2)}$ (antisymmetric base wave)

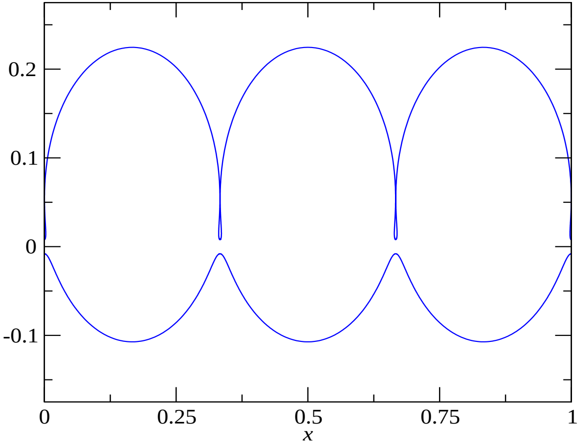


$H = 0.628$

Steady travelling waves



$H = 0.628$



$H = 0.628$

Perturb base waves:

$$x = X(\xi) + \tilde{x}(\xi, t), \quad y = Y(\xi) + \tilde{y}(\xi, t), \quad \phi = \xi + \tilde{\phi}(\xi, t), \quad \psi = \tilde{\psi}(\xi, t)$$

$$\hat{x} = \hat{X}(\xi) + \tilde{\chi}(\xi, t), \quad \hat{y} = \hat{Y}(\xi) + \tilde{b}(\xi, t), \quad \hat{\phi} = \xi + \tilde{\Phi}(\xi, t), \quad \hat{\psi} = \tilde{\Psi}(\xi, t)$$

Using Floquet-Bloch theory, we write

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = e^{\sigma t} e^{ip\xi} \sum_{n=-\infty}^{\infty} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} e^{in\xi}, \qquad \begin{pmatrix} \tilde{\chi} \\ \tilde{b} \\ \tilde{\psi} \\ \tilde{\Psi} \end{pmatrix} = e^{\sigma t} e^{ip\xi} \sum_{n=-\infty}^{\infty} \begin{pmatrix} \hat{a}_n \\ \hat{b}_n \\ \hat{c}_n \\ \hat{d}_n \end{pmatrix} e^{in\xi},$$

$$p \in [0, 1)$$

Small amplitude base waves

σ purely imaginary

$$\sigma = \pm \mathrm{i}\left[p'^3 \tanh(p'H/2)\right]^{1/2} - \mathrm{i}p'c, \quad \sigma = \pm \mathrm{i}\left[p'^3 \coth(p'H/2)\right]^{1/2} - \mathrm{i}p'c$$

where $p' = p + m$ for integer m.

Small amplitude base waves

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Superharmonic perturbations: $p = 0$

Subharmonic perturbations: $0 < p < 1$

Small amplitude base waves

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where $p' = p + m$ for integer m.

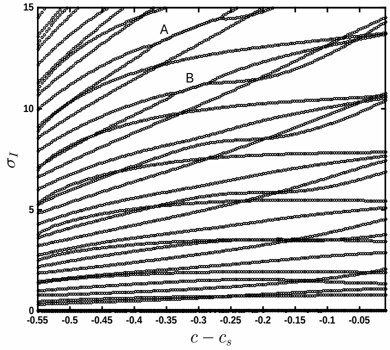
Superharmonic perturbations: $p = 0$

Subharmonic perturbations: $0 < p < 1$

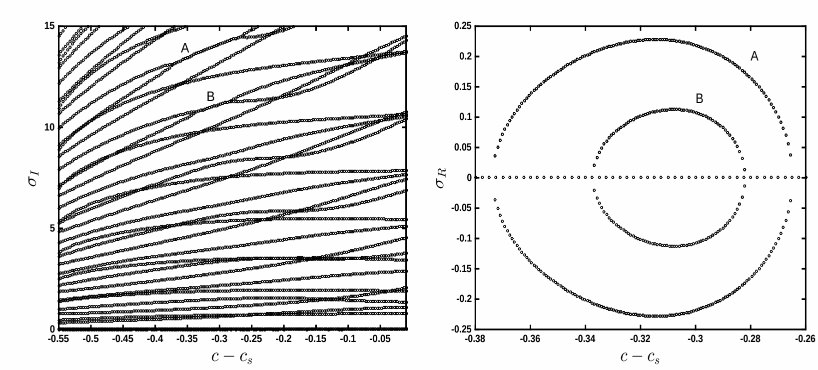
Infinite depth ($H \to \infty$): (Tiron & Choi 2012)

- Superharmonically stable
- Subharmonically unstable

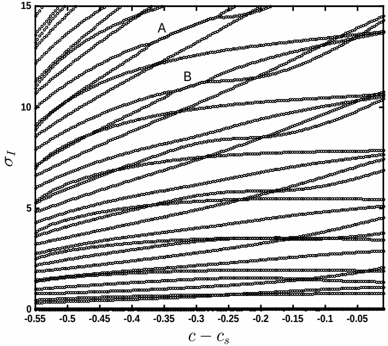
Symmetric base, $p = 0$, $H = 1$

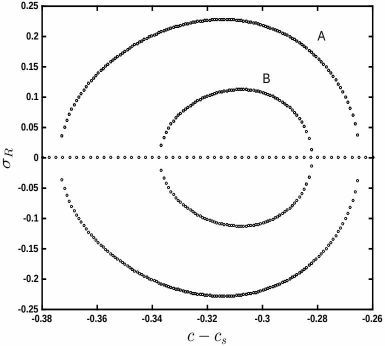


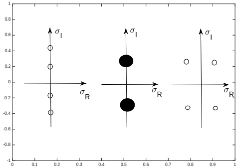
Symmetric base, $p = 0$, $H = 1$



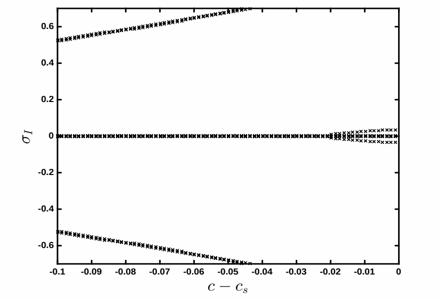
Symmetric base, $p = 0$, $H = 1$



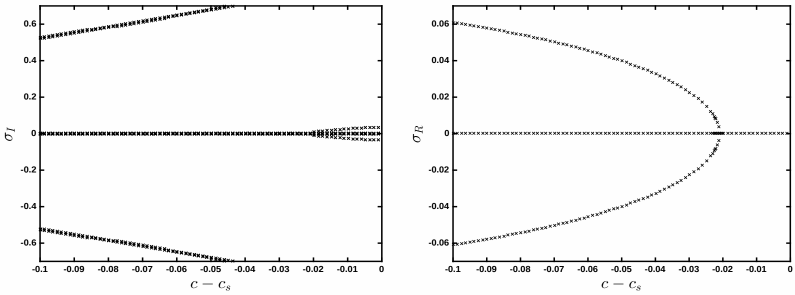




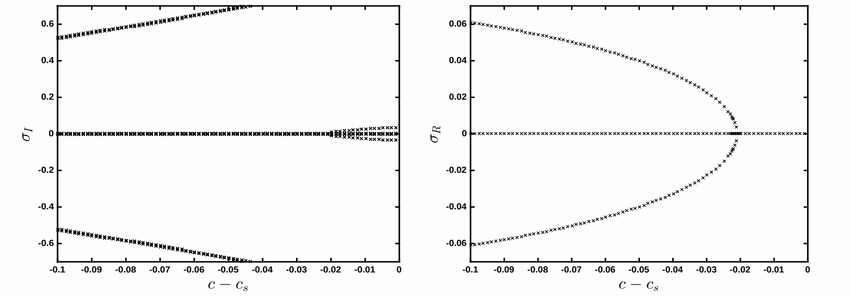
Symmetric base, $p = 0$, $H = 4$

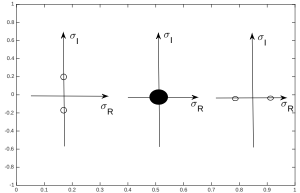


Symmetric base, $p = 0$, $H = 4$

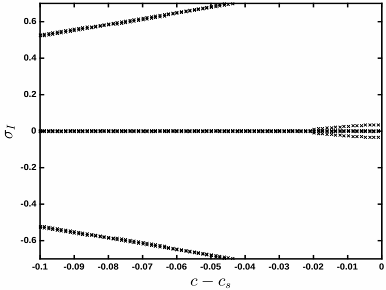


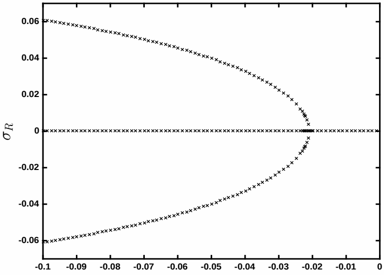
Symmetric base, $p = 0$, $H = 4$

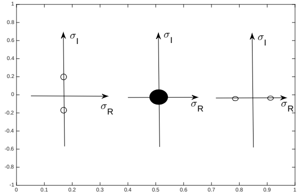


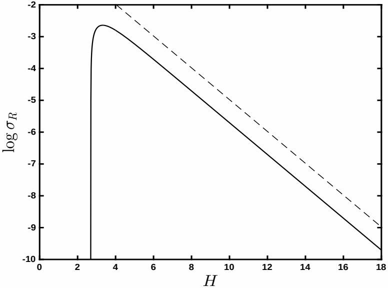


Symmetric base, $p = 0$, $H = 4$









Time-dependent calculations

We confirm the instability via time-dependent calculations

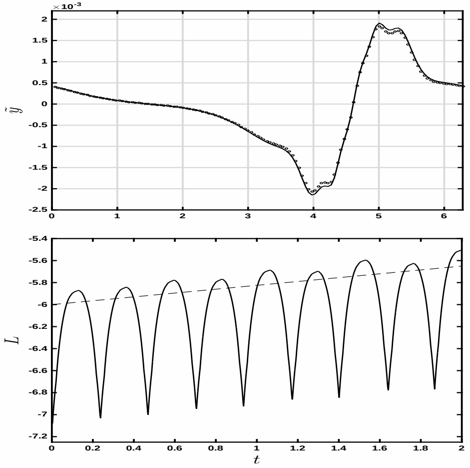
$$y_t = y_\xi \Big[T(\psi_\xi / J) - S(\hat{\psi}_\xi / \hat{J}) \Big] - x_\xi \psi_\xi / J,$$

$$\phi_t + \Big[S(\hat{\psi}_\xi / \hat{J}) - T(\psi_\xi / J) \Big] \phi_\xi + \frac{1}{2J} \left(\phi_\xi^2 - \psi_\xi^2 \right) + (\gamma/\rho)\kappa = 0$$

$$\psi_\xi = T(\phi_\xi) - S(\hat{\phi}_\xi), \qquad \hat{\psi}_\xi = S(\phi_\xi) - T(\hat{\phi}_\xi)$$

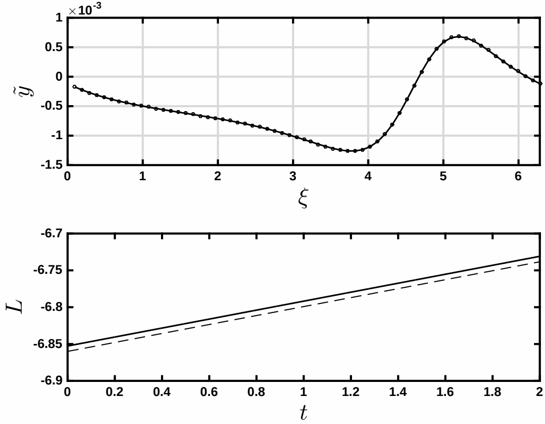
- Spatial derivatives computed spectrally in Fourier space
- 4th order Runge-Kutta integration in time

Time-dependent calculations



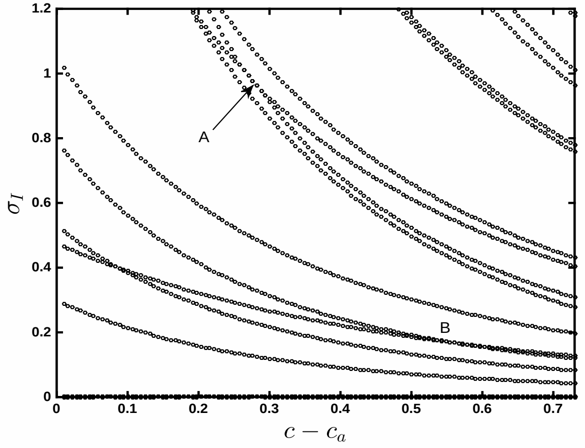
$H = 1, \quad c - c_s = -0.35, \quad$ symmetric base, $\sigma = 0.1758 + 13.45i$

Time-dependent calculations



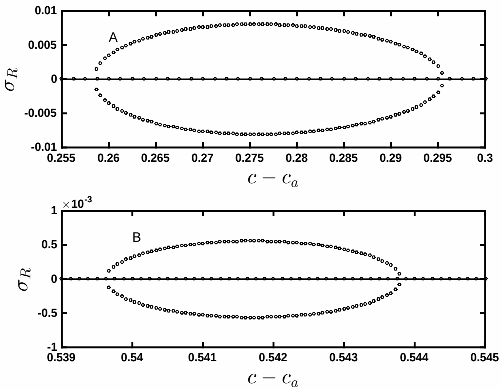
$H = 4$, $c - c_s = -0.1$, symmetric base, $\sigma = 0.061$.

Antisymmetric base, $p = 0$



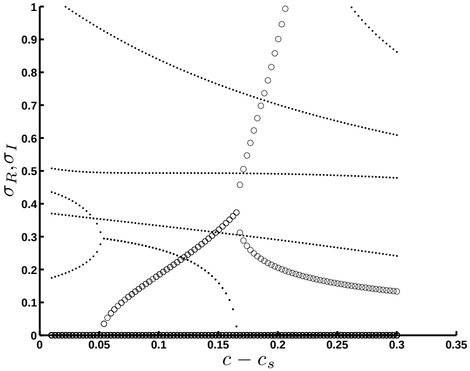
$H = 1$

Antisymmetric base, $p = 0$



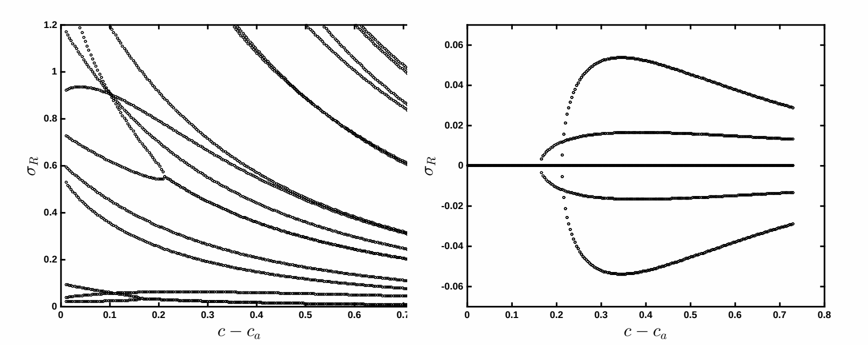
$H = 1$

Subharmonic perturbations, symmetric-base ($p = 1/2$)



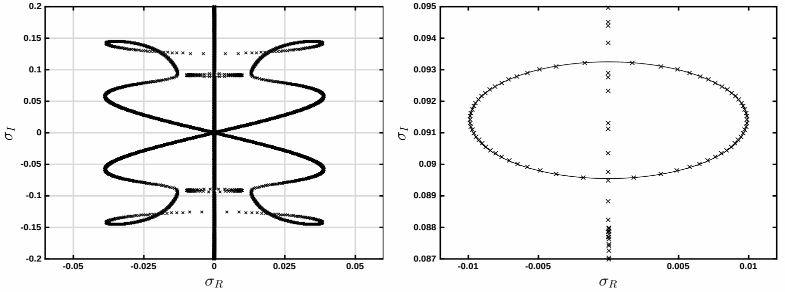
- Real part σ_R (circles) and imaginary part σ_I (dots) versus $c - c_s$ at $H = 1$

Subharmonic perturbations, antisymmetric-base ($p = 1/2$)



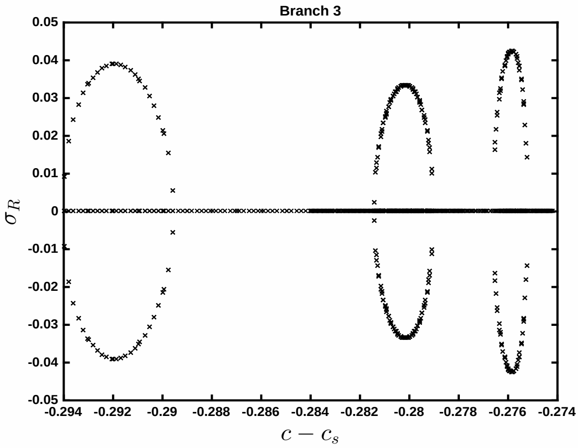
- Imaginary part, σ_I (left) and real part σ_R (right)versus relative wave speed $c - c_a$ at $H = 1$

Antisymmetric base,eigenvalue spectrum for $0 < p < 1$



$H = 2, \quad c - c_a = 0.04$

Bifurcation branch 3, $p = 0$



Krein signature of a disturbance

Following McKay & Saffman (1986) we examine the energy

$$E = K + V - c_f P$$

where

$$K = \int_{\mathcal{A}} \frac{1}{2}\rho|\nabla\phi|^2 \, \mathrm{d}A, \quad P = \int_{\mathcal{A}} \rho\phi_x \, \mathrm{d}A,$$

$$V = \int_{\mathcal{S}_U} \gamma \left[(1+\eta_x^2)^{1/2} - 1\right] \mathrm{d}l + \int_{\mathcal{S}_L} \gamma \left[(1+\zeta_x^2)^{1/2} - 1\right] \mathrm{d}l,$$

Krein signature of a disturbance

$$E = K + V - c_f P$$

The Krein signature is positive if $E > 0$ and negative if $E < 0$

E is easy to compute for small amplitude disturbances

Krein signature

For symmetric or antisymmetric linear waves:

$$E = K + V - c_f P \quad \propto \quad \frac{(\hat{c} - c_f)}{\hat{c}}$$

$\hat{c} = \hat{\omega}/\hat{k}$.

Symmetric waves : $\hat{\omega}^2 = (\gamma/\rho)\hat{k}^3 \tanh(\hat{k}H/2)$

Antisymmetric waves : $\hat{\omega}^2 = (\gamma/\rho)\hat{k}^3 \coth(\hat{k}H/2)$

For symmetric perturbations:

$$s_K = \operatorname{sgn}\left([\hat{k}^3 \tanh(H\hat{k}/2)]^{1/2} - \nu \hat{k} c_f \right)$$

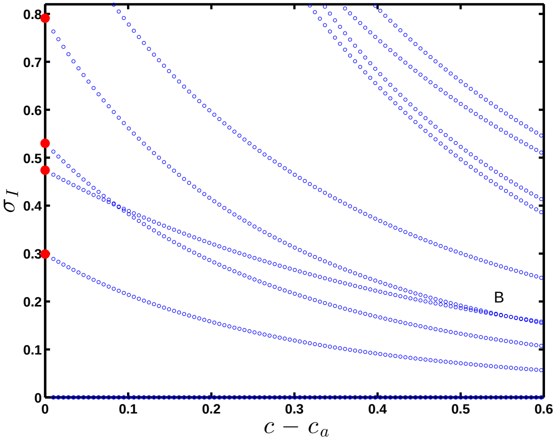
For antisymmetric perturbations:

$$s_K = \operatorname{sgn}\left[[\hat{k}^3 \coth(H\hat{k}/2)]^{1/2} - \nu \hat{k} c_f \right]$$

$$\nu = \pm 1 \qquad \hat{k} = p + m \qquad p \in [0, 1) \qquad m \text{ an integer}$$

$$c_f = c_s \quad \text{or} \quad c_a$$

Krein signature: antisymmetric base, $p = 0$



$H = 1$

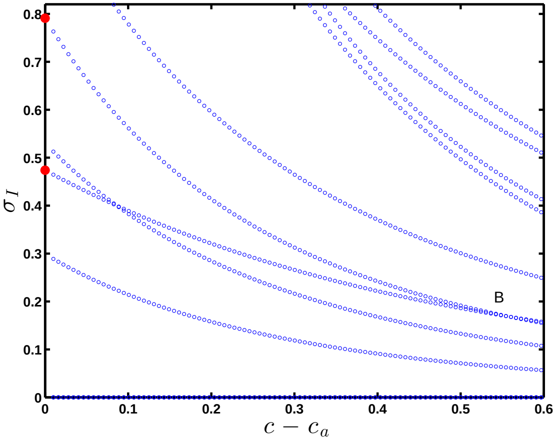
$\sigma^{\nu}_{s,m}$	$\sigma^{\nu}_{a,m}$	ν	m	s_K
–	0.299	1	2	1
0.474	–	-1	-2	-1
0.530	–	1	3	1
0.791	–	-1	-1	-1
–	1.049	1	3	1

$c_f = c_a = 1.471$

$\sigma^{\nu}_{s,m}$	$\sigma^{\nu}_{a,m}$	ν	m	s_K
–	0.299	1	2	1
0.474	–	-1	-2	-1
0.530	–	1	3	1
0.791	–	-1	-1	-1
–	1.049	1	3	1

$c_f = c_a = 1.471$

Krein signature: antisymmetric base, $p = 0$



$H = 1$

Summary

- PDE formulation in terms of surface variables
- Recover steady Kinnersley waves
- Kinnersley waves are superharmonically unstable
- Instability through collision of same signed eigenvalues?

IMA Conference Nonlinearity and Coherent Structures

19-21 June 2017, UEA, Norwich