Computing Flexural-Gravity Waves in Three Dimensions

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Acknowledgements

This is joint work with

- **Jean-Marc Vanden-Broeck** at University College London
- **Emilian Părău** at UEA
- **Paul Milewski** at University of Bath

Outline

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Goal

Compute solutions to Euler's equations as efficiently and as accurately as possible.

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Model for Water Waves

For an inviscid, incompressible fluid with velocity potential $\phi(x, y, z, t)$, the forced Euler's equations are given by

$$\begin{cases} \triangle\phi = 0, & (x, y, z) \in \Omega, \\ \phi_z = 0, & z = -h, \\ \eta_t + \eta_x\phi_x + \eta_y\phi_y = \phi_z, & z = \eta(x, y, t), \\ \phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{F^2}\eta + P(x, y, t) = -D\frac{\delta H}{\delta\eta}, & z = \eta(x, y, t), \end{cases}$$

where

h: depth

F: Froude number

D: flexural rigidity

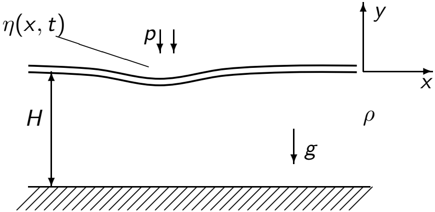
$\eta(x, y, t)$: variable surface

$P(x, y, t)$: external pressure distribution

$\frac{\delta H}{\delta\eta}$: condition at the interface.

$\Omega = \{-\infty < x < \infty, -\infty < y < \infty, -\infty < z < \eta(x, y, t)\}$

Models For a Thin Sheet of Ice



We consider two models

- Biharmonic (linear) model

$$H_L = D\frac{1}{2}\int (\triangle\eta)^2 dA$$

- Cosserat (nonlinear) model

$$H_N = D\frac{1}{2}\int (\kappa_1+\kappa_2)^2 dS \quad \text{with } \kappa_1, \kappa_2 \text{ principle curvatures}$$

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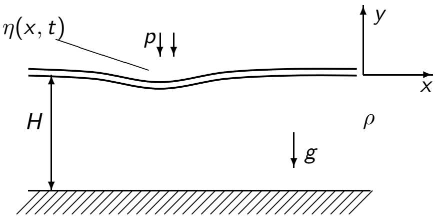

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Reformulation: Bernoulli Equation

- Switch into surface variables via the velocity potential at the surface

$$q(x, y, t) = \phi(x, y, z = \eta, t)$$

- Go into a moving frame of reference
- Combine the dynamic and kinematic boundary conditions

Then the steady-state Bernoulli equation becomes

$$\frac{1}{2}\frac{(1+\eta_x^2)q_y^2 + (1+\eta_y^2)q_x^2 - 2\eta_x\eta_y q_x q_y}{1+\eta_x^2+\eta_y^2} + \frac{\eta}{F^2} + P + D\frac{\delta H}{\delta \eta} = \frac{1}{2}$$

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Models for Ice

The two different models are considered

- Biharmonic (linear) model

$$\frac{\delta H}{\delta \eta} = \nabla^4 \eta$$

- Cosserat (nonlinear) model

$$\frac{\delta H}{\delta \eta} = \frac{2}{\sqrt{a}} \left[\partial_x \left(\frac{1+\eta_y^2}{\sqrt{a}} \partial_x H \right) - \partial_x \left(\frac{\eta_x \eta_y}{\sqrt{a}} \partial_y H \right) - \partial_y \left(\frac{\eta_x \eta_y}{\sqrt{a}} \partial_x H \right) + \partial_y \left(\frac{1+\eta_x^2}{\sqrt{a}} \partial_y H \right) \right] + 4H^3 - 4KH$$

where

$$a = 1 + \eta_x^2 + \eta_y^2$$

$$H = \frac{1}{2} a^{3/2} \left[(1+\eta_y^2)\eta_{xx} - 2\eta_{xy}\eta_x\eta_y + (1+\eta_x^2)\eta_{yy} \right]$$

$$K = \frac{1}{a^2} \left[\eta_{xx}\eta_{yy} - \eta_{xy}^2 \right]$$

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Reformulation: Boundary Integral Method

Following the formulation by Forbes (1989), use Green's second identity

$$\int_V (\alpha \Delta \beta - \beta \Delta \alpha) dV = \oint_{S(V)} \left(\alpha \frac{\partial \beta}{\partial n} - \beta \frac{\partial \alpha}{\partial n} \right) dS$$

where in three dimensions, β is the fundamental solution given by the Green's function

$$\frac{1}{4\pi} \frac{1}{((x - x^*)^2 + (y - y^*)^2 + (z - z^*)^2)^{1/2}}$$

and $\alpha = \phi - x$, which satisfies Laplace's equation.

System of Equations

The final form of equations to solve for flexural-gravity waves in infinite depth is

$$\frac{1}{2}\frac{(1+\eta_x^2)q_y^2+(1+\eta_y^2)q_x^2-2\eta_x\eta_y q_x q_y}{1+\eta_x^2+\eta_y^2}+\frac{\eta}{F^2}+P+D\frac{\delta H}{\delta\eta}=\frac{1}{2}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(q-q^*-x+x^*)K_1+\eta_x K_2\right]dxdy=2\pi(q^*-x^*)$$

where

$$K_1=\frac{1}{d^{3/2}}(\eta-\eta^*-(x-x^*)^2\eta_x-(y-y^*)^2\eta_y)$$

$$K_2=\frac{1}{d^{1/2}}$$

with

$$d(x,y,x^*,y^*,\eta)=(x-x^*)^2+(y-y^*)^2+(\eta-\eta^*)^2$$

.

Symmetry

Symmetry in y direction

$$\eta(x,y) = \eta(x,-y)$$

and

$$q(x,y) = q(x,-y)$$

implies additional terms

$$\frac{1}{2}\frac{(1+\eta_x^2)q_y^2+(1+\eta_y^2)q_x^2 - 2\eta_x\eta_y q_x q_y}{1+\eta_x^2+\eta_y^2}+\frac{\eta}{F}-\frac{1}{2} = F(\eta)$$

$$\int_0^\infty \int_{-\infty}^\infty \left[(q-q^*-x+x^*)\tilde{K}_1 + \eta_x \tilde{K}_2\right] dxdy = 2\pi(q^*-x^*)$$

where

$$\tilde{K}_1 = \bar{K}_1(x,y,\eta,x^*,y^*,\eta^*) + \bar{K}_1(x,-y,\eta,x^*,y^*,\eta^*)$$
$$\tilde{K}_2 = \bar{K}_2(x,y,\eta,x^*,y^*,\eta^*) + \bar{K}_2(x,-y,\eta,x^*,y^*,\eta^*)$$

Outline

Discretisation

- Let x_i and y_j be equally spaced points such that $i = 1, \ldots, N$ and $j = 1, \ldots, M$.
- Let the vector of unknowns be $q_{x(i,j)}$ and $\eta_{x(i,j)}$ such that

$$u = \left[q_{x(1,1)}, \cdots, q_{x(N,1)}, \cdots, q_{x(N,M)}, \eta_{x(1,1)}, \cdots, \eta_{x(N,M)}\right]^T$$

- Use finite differences to discretise the derivatives
- Obtain 2NM equations

$$G(u) = 0$$

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Numerical Approach

To solve the system

1. Set up an initial guess u^0
2. Until convergence
   2.1 Solve $J(u^n)\delta^n = -G(u^n)$
   2.2 Set $u^{n+1} = u^n + \lambda\delta^n$, $0 < \lambda < 1$
   2.3 Test for convergence

This method relies on an initial guess u^0 and the Jacobian J.

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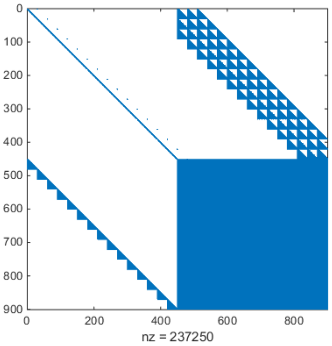
Jacobian

The sparsity of the linearised Jacobian for flexural-gravity waves



Solving the System of Equations

We consider two ways of solving the system of equations

1. Inexact Newton Method: (direct method) uses an inexact Jacobian (not computed at each step).
2. Modified Newton Method: (iterative method) using a preconditioned Krylov method to construct the solution, not keeping the full Jacobian matrix.
    - Preconditioner is constructed as shown in Pethiyagoda *et al* (2014)
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Initial Condition

Newton's method is very sensitive to initial conditions. In order to compute different wave amplitudes, generate a bifurcation diagram

- Guess a small amplitude solution
- Use this guess in Newton's method to compute the true solution.
- Scale the previous solution to get a guess for a larger amplitude solution
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Can use the Jacobians from previous steps in the bifurcation branch as preconditioners.

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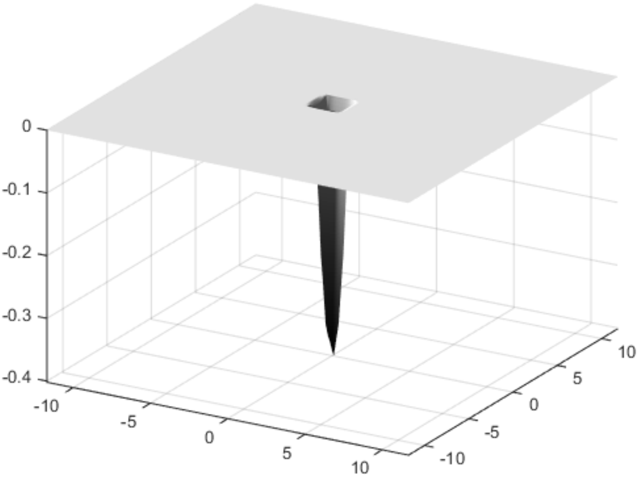
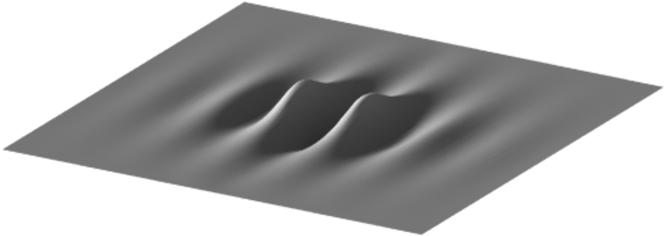
Forcing Term

We use the following pressure as a forcing for depression waves



Sample Solutions

Solutions for forced waves underneath an ice sheet



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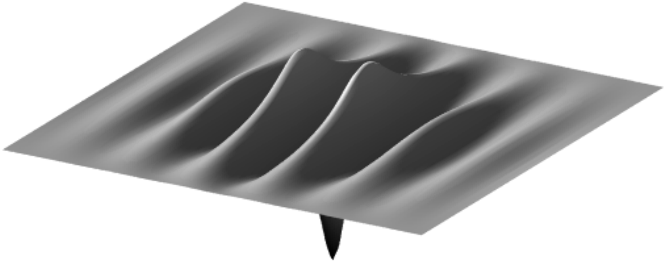

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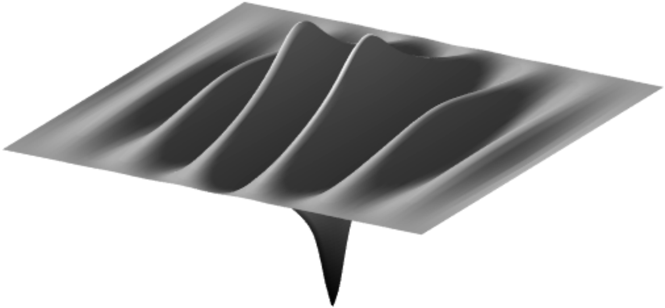
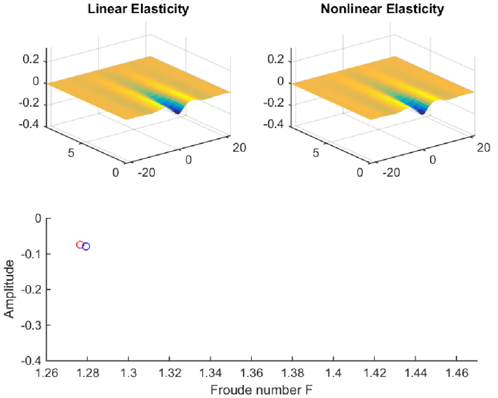


Bifurcation Branch

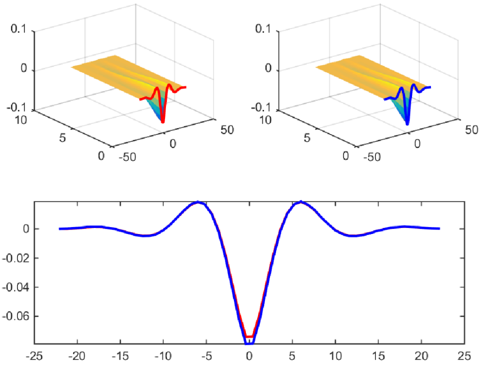
Comparison of the bifurcation branches for flexural-gravity waves with the linear and the nonlinear elasticity models



Note: both models give the same wave amplitude, but different Froude numbers

Flexural-Gravity Wave Profiles

Comparison of the solution profiles for linear elasticity model and the nonlinear elasticity model.



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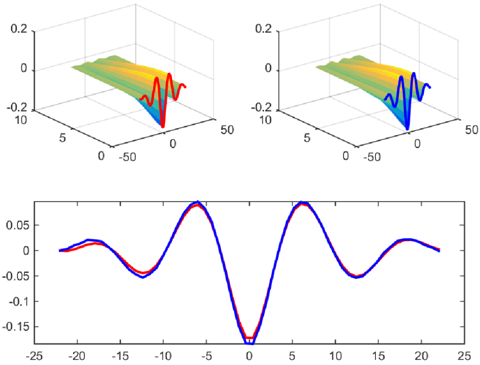

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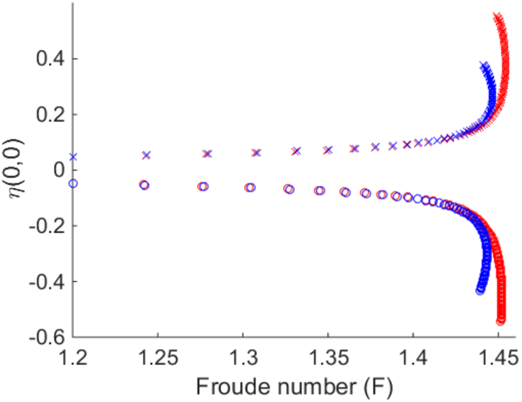

Elevation waves are represented as crosses and depression waves as circles.

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- Both models for produce similar shaped profiles, but at different Froude numbers
- The code is easy to use and easy to modify
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Thank you for your attention