Variational modelling of nonlinear water wave and ship dynamics: continuum and finite element modelling

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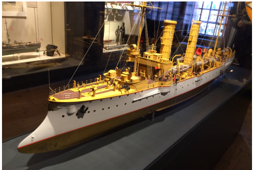
School of Mathematics, University of Leeds
Theoretical and computational aspects of surface waves, Banff, 2016
EPSRC & EU European Industry Doctorate funding (MARIN)



1. Introduction

- According to the latest IPCC report "It is virtually certain that there has been an increase in the frequency and intensity of the strongest tropical cyclones . . . in the North Atlantic since the 1970s . . . there is low confidence regarding regional changes of intensity of extratropical cyclones ".
- Wind and water-wave impact on offshore structures and ships at sea can thus be expected to intensify.

Introduction

I will sketch how we started to model:

- time evolution of oblique rogue waves (Kalogirou, Gidel),
- a rogue wave-energy device (Kalogirou, Zweers) and
- wave impact on ships (Kalogirou).

Rogue Waves

Rogue waves are anomalously high waves defined relative to a significant wave height H_s.

- Index (Kharif et al. '09, Dysthe et al. '08):

$$AI = H_{rw}/H_s > 2 \qquad \text{or} \qquad AI = \eta_{rw}/H_s > 1.25 \qquad (1)$$

- Relevance in maritime & coastal engineering —ship design & safety offshore structures
- Pyramidal rogue wave (Faulkner 2001):

Fig.1. Pyramidal wave off south Japan

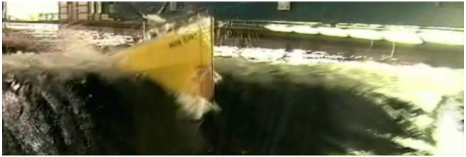
Rogue Waves

There are many causes of rogue waves, e.g., Kharif et al. (2009) & Faulkner (2001, 2003):

- ...
- spatial wave focussing due to coastal or submarine convergences
- episodic waves generated elsewhere
- crossing seas, nearly standing waves with pyramidal waves.
- Man-made analog: $AI = \frac{H_{rw}}{H_s} = \frac{3.5}{0.35} \approx 10$: *bore soliton splash*

Shipping

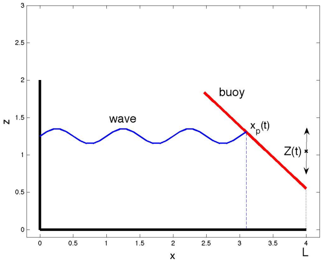
- Shipping at sea is the "invisible industry that brings you 90% of everything" (Rose George)
- Wavetank experiments MARIN of fast axe-bowed vessel:



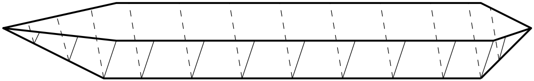
- *Axe-bowed vessel experiments*. Huijsmans TU Delft.
- On faster & larger ships safe seakeeping becomes more important, due to the larger accelerations involved, causing potential damage.
- Mathematical modelling of fast ships in heavy seas aims to improve hull design.

Introduction: Wave Energy Buoys & Ships

- Buoy in vertical cross-section



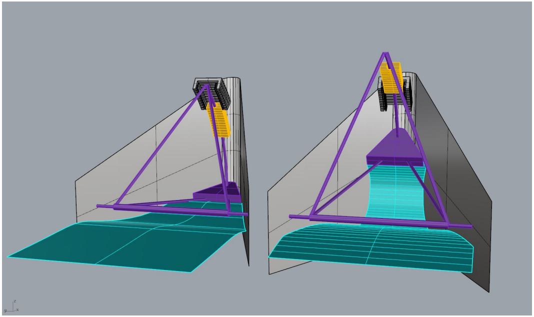
- Buoy/wave-energy device in 3D
- Simple model with V–shaped ship cross-sections:



Introduction: Wave Energy Buoy

Design by B. & Zweers (2013 trial proof-of-principle):

- Sketch 2nd version wave energy device (robustness):



- *Proof-of-principle* 1$^{\mathrm{st}}$ version.

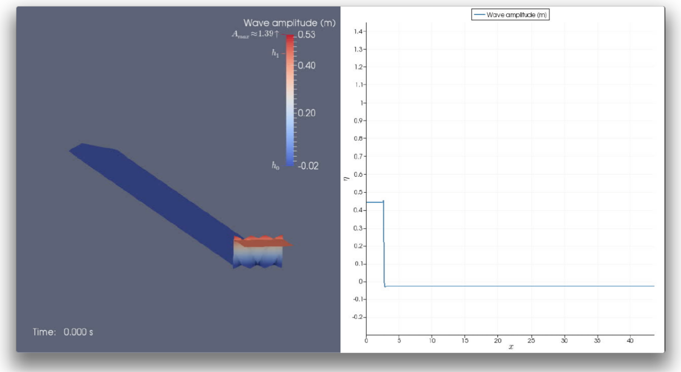
2. (Variational) Principles

Challenge will be addressed using following 3 principles:

- Our first principle is that, even when damping and wave breaking are absent, the appropriate coupled models should contain a conservative limit.
- Our second principle is that the conservative, coupled wave-energy-device and wave-ship systems should "simply" consist of the sum of the variational principles of the separate systems.
- Our third principle is that we "simply" discretise these (nonlinear and coupled) systems consistently in space and time, to obtain a space-time discrete algebraic variational (finite element) system. Its variation than "semi-automatically" yields a stable numerical scheme.

3. Rogue Waves: Soliton Splash

- Nonlinear *Benney-Luke model* (Pego & Quintero 1999, B. & Kalogirou 2016).
- *Soliton Splash*, `www.firedrakeproject.org` example:



Rogue Waves: Soliton Splash Event

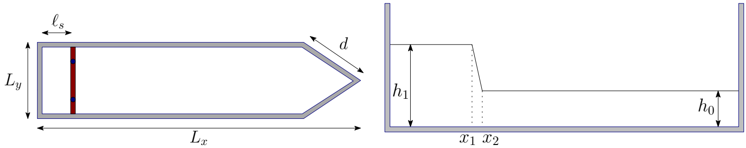


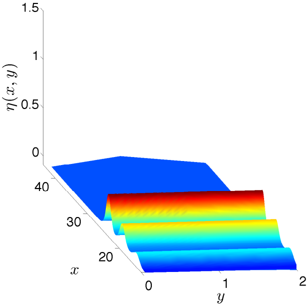
Figure : Sketch of the wave channel set-up: top (left panel) and side views (right panel).

Validation: Soliton Splash Event

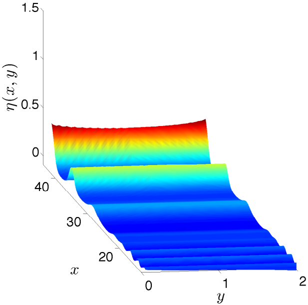
Table : Details about the soliton splash experiment, including wavetank dimensions.

Wavetank length	$L_x = 43.63 \pm 0.1$ m
Wavetank width	$L_y = 2$ m
Wavetank height	$L_z = 1.2$ m
Contraction length	$d = 2.7$ m
Location of sluice-gate	$\ell_s = 2.63$ m
Rest water level (hight)	$h_1 = 0.9$ m
Rest water level (low)	$h_0 = 0.43$ m
Sluice-gate release speed	$V_g \approx 2.5$ m/s
Sluice-gate removal time	$T_s = h_1/V_g \approx 0.36$ s
$\mu = 0.04$, $\epsilon = 0.55$	

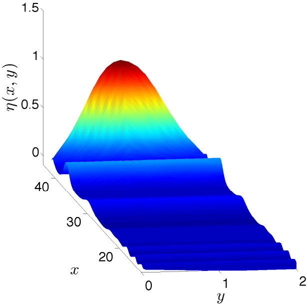
Rogue Waves: Soliton Splash Event



Rogue Waves: Soliton Splash Event

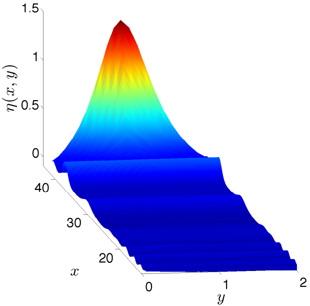


Rogue Waves: Soliton Splash Event



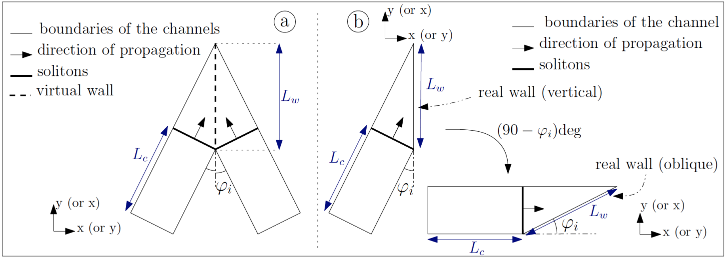
Rogue Waves: Soliton Splash Event

Time evolution:



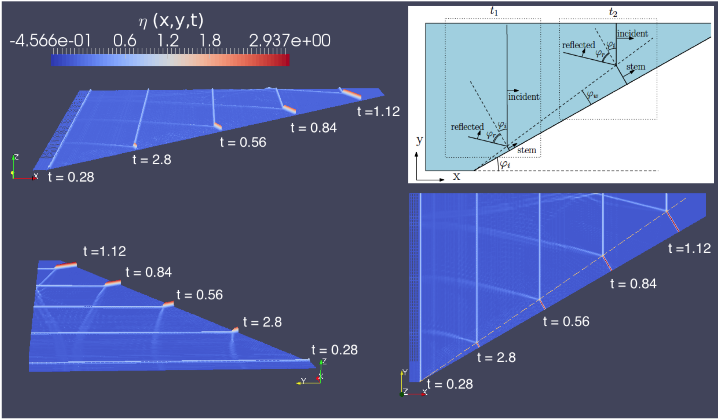
Rogue Waves: oblique soliton

Floriane Gidel & O.B., space-time geometric simulations of Benney-Luke system in closed channel:



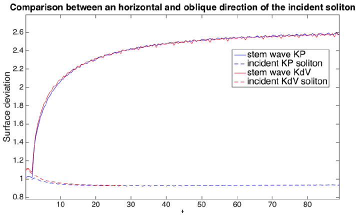
Rogue Waves: oblique soliton

Floriane Gidel & O.B.

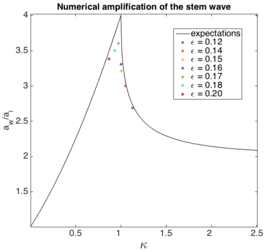


Rogue Waves: oblique soliton

Gidel & O.B. amplification 3.6 (Ablowitz & Curtis: 3.9):



Comparison between an horizontal and oblique direction of the incident soliton



Numerical amplification of the stem wave

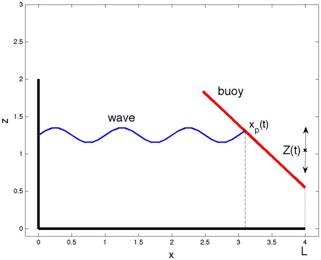
Relevance to (sinking) ships:
`https://www.youtube.com/watch?v=72k9JR9otSg?`

4. Shallow Water Wave Impact on a Buoy

Consider buoy motion in shallow water waves in a plane:

- Shallow water depth $h(x,t)$, velocity $u(x,t) = \partial_x \phi(x,t)$, buoy keel $z = Z(t) - H_k$, buoy position $Z(t)$ and shape:

$$h_b(x, Z(t)) = Z(t) - H_k - \tan\alpha(x - L)$$



- Water line point at x_p defined by $h(x_p, t) = h_b(x_p, Z)$.

Shallow Water Wave Impact on a Buoy

- By introducing constraint $\underline{h - h_b + \mu^2 = 0}$ with global Lagrange multiplier λ we impose non-negative nature of $h_b - h$ as (in)equality on the VP

$$
\begin{aligned}
0 = & \delta \int_0^T \int_0^L -\rho h \partial_t \phi \\
& \quad - \frac{1}{2}\rho h (\partial_x \phi)^2 - \frac{1}{2}\rho g h^2 + \rho g h H \\
& \quad + \underline{\rho \lambda (h - h_b + \mu^2)} \, \mathrm{d}x \\
& \quad - M Z \dot{W} - \frac{1}{2} M W^2 - M g Z \, \mathrm{d}t
\end{aligned}
$$

Shallow Water Wave Impact on a Buoy

- The resulting equations of motion are:

$$
\begin{aligned}
\delta h:&\quad \partial_t\phi+\frac{1}{2}(\partial_x\phi)^2+g(h-H)-\lambda=0\\
\delta\phi:&\quad \partial_t h+\partial_x(h\partial_x\phi)=0\\
\delta\lambda:&\quad h-h_b+\mu^2=0\\
\delta Z:&\quad M\dot{W}+Mg+\rho\int_0^L\lambda\frac{\partial h_b}{\partial Z}\,\mathrm{d}x=0\\
\delta W:&\quad M\dot{Z}=MW\\
\delta\mu:&\quad \lambda\,\mu=0
\end{aligned}
$$

- So either $\lambda=0$ and $\mu^2>0$ where $h_b-h=\mu^2>0$, or
- $\mu=0$ with $\lambda>0$ under the buoy where $h-h_b=0$
- Why $\lambda=0$ at the waterline *. . . Cotter & B, 2010, JEM*?

Shallow Water Wave Impact on a Buoy

- Introduce steady rest state $\phi = 0, W = 0, Z = \bar{Z}$

$$\begin{aligned} 0 < x < L_p: \quad & h = H(x) = H_0, \lambda = \Lambda(x) = 0,\\ & \mu = \bar{\mu}(x) = \sqrt{h_b(x,\bar{Z}) - H_0}, h_b = H_b(x,\bar{Z}),\\ L_p \le x < L: \quad & h = H(x) = H_b(x,\bar{Z}), \mu = \bar{\mu}(x) = 0,\\ & \lambda = \Lambda(x) = g(h_b(x,\bar{Z}) - H_0), h_b = H_b(x,\bar{Z}) \end{aligned}$$

- with rest waterline point at $x = L_p$, and linearise

$$\begin{aligned} \phi = \tilde{\phi}, h = H(x) + \eta, h_b(x,Z) = H_b(x,\bar{Z}) + \tilde{Z},\\ \lambda = \Lambda(x) + \tilde{\lambda}, \mu = \bar{\mu} + \tilde{\mu}, W = \tilde{W}, Z = \bar{Z} + \tilde{Z}. \end{aligned}$$

Shallow Water Wave Impact on a Buoy

- VP linear system:

$$0 = \delta \int_0^T \int_0^L -\rho\eta\partial_t\tilde{\phi} - \frac{1}{2}\rho H(x)(\partial_x\tilde{\phi})^2 - \frac{1}{2}\rho g\eta^2 + \underline{\rho\tilde{\lambda}(\eta - \tilde{Z} + 2\bar{\mu}\tilde{\mu}) + \rho\Lambda\tilde{\mu}^2}\,\mathrm{d}x - M\tilde{Z}\dot{\tilde{W}} - \frac{1}{2}M\tilde{W}^2\,\mathrm{d}t$$

- Resulting equations of motion:

$$\begin{gathered}\delta\eta : \partial_t\tilde{\phi} + g\eta - \tilde{\lambda} = 0, \quad \delta\phi : \partial_t\eta + \partial_x(H(x)\partial_x\tilde{\phi}) = 0\\ \delta\tilde{\lambda} : \eta - \tilde{Z} + 2\bar{\mu}\tilde{\mu} = 0, \quad \delta\tilde{\mu} : \Lambda\tilde{\mu} + \bar{\mu}\tilde{\lambda} = 0\\ \delta\tilde{Z} : M\dot{\tilde{W}} + \rho\int_0^L \tilde{\lambda}\,\mathrm{d}x = 0, \quad \delta\tilde{W} : \dot{\tilde{Z}} - \tilde{W} = 0.\end{gathered}$$

Shallow Water Wave Impact on a Buoy

- Variational/symplectic C/DGFEM time discretization (RATTLE Cotter et al., 2004, B. & Kalogirou 2016ab, Firedrake):

$$
\begin{aligned}
\int_0^L \delta\eta\Big(\tilde\phi^{n+1/2} - \tilde\phi^n + \frac{\Delta t}{2} g\eta^n - \frac{\Delta t}{2}\tilde\lambda^{n+1/2}\Big)\,\mathrm{d}x &= 0\\
M\tilde W^{n+1/2} - M\tilde W^n + \frac{\Delta t}{2}\int_0^L \tilde\lambda^{n+1/2}\,\mathrm{d}x &= 0\\
\int_0^L \delta\phi(\eta^{n+1} - \eta^n) - \Delta t H(x)\partial_x(\delta\phi)\partial_x\tilde\phi^{n+1/2}\,\mathrm{d}x &= 0\\
\tilde Z^{n+1} - \tilde Z^n - \Delta t\tilde W^{n+1/2} &= 0\\
\int_0^L \delta\tilde\lambda\Big(\eta^{n+1} - \tilde Z^{n+1} + 2\bar\mu\tilde\mu^{n+1/2}\Big)\,\mathrm{d}x &= 0\\
\int_0^L \delta\tilde\mu\Big(\Lambda\tilde\mu^{n+1/2} + \bar\mu\tilde\lambda^{n+1/2}\Big)\,\mathrm{d}x &= 0, \dots
\end{aligned}
$$

Wave-Energy Buoy Coupled to Induction Motor

- Linear induction motor with iron core/magnet, induction $L\,(Z(t))$ (measure), output voltage $v(t)$, current $i(t) = \mathrm{d}q/\mathrm{d}t$, coupled to mast on top of wave buoy:

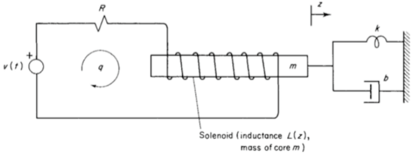


Figure 7.25

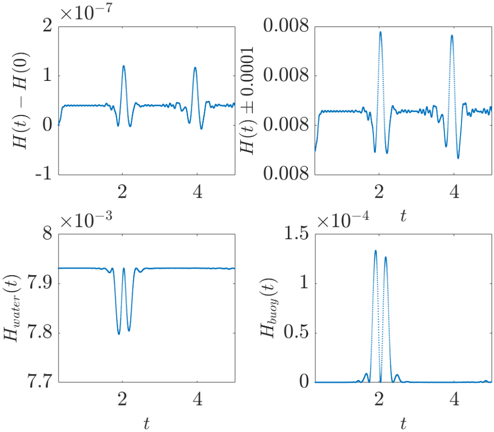
- Port-Hamiltonian (including damping/resistor and output port) or Lagrangian formulation (Wellstead 2000)

$$0 = \delta \int_0^T \cdots - MZ\dot{W} - \frac{1}{2}MW^2 - MgZ + L(Z)i\dot{q} - \frac{1}{2}L_k(Z)i^2 + qv\,\mathrm{d}t. \qquad (2)$$

Water Wave Impact on Buoys

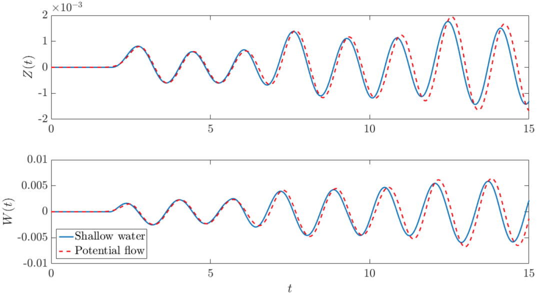
Likewise for a 3D buoy and 3D ship; numerical results:

- *Buoy wave maker case* & *Sluice gate case*
  driven waves & *3D buoy*



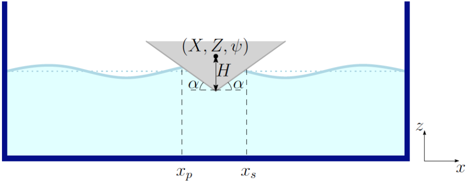
Water Wave Impact on Buoys

- Comparison shallow water & potential flow models:



Water Wave Impact on 2D Ships

- Model: $h_s(x;t) = Z(t) - H + \tan(\alpha \pm \psi(t))|x - X(t)|$.



- Variational principle

$$0 = \delta \int_0^T \Big\{ \int_0^L \rho\Big(\eta\partial_t\phi + \frac{1}{2}H(x)(\partial_x\phi)^2 + \frac{1}{2}g\eta^2\Big)\,\mathrm{d}x$$

$$- \int_{L_p}^{L_s} \rho\lambda\Big(\eta + \tan\alpha\,\mathrm{sign}(x-\bar{X})X - Z + \sec^2\alpha\,(x-\bar{X})\psi\Big)\,\mathrm{d}x$$

$$+M(X\dot{U} + Z\dot{W}) + \psi\dot{p}_\psi + \frac{1}{2}M(U^2+W^2) + \frac{1}{2}\frac{p_\psi^2}{I_1}\Big\}\,\mathrm{d}t.$$

Water Wave Impact on 2D Ships

Evolution equations

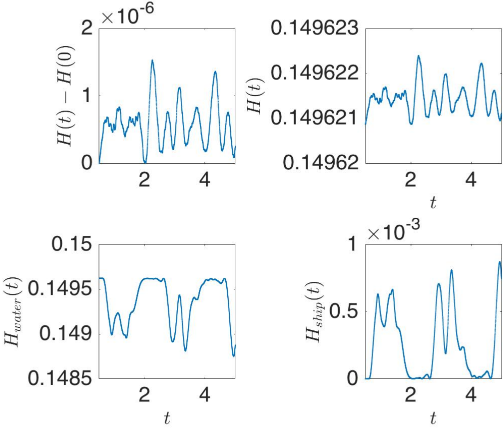
$$
\begin{aligned}
&\partial_t \eta + \partial_x\big(H(x)\partial_x \phi\big) = 0 && \partial_t \phi + g\eta - \lambda\Theta(L_p < x < L_s) = 0\\
&\dot{X} = U && M\dot{U} - \rho \tan\alpha \int_{L_p}^{L_s} \mathrm{sign}(x - \bar{X})\lambda \,\mathrm{d}x = 0\\
&\dot{Z} = W && M\dot{W} + \rho \int_{L_p}^{L_s} \lambda \,\mathrm{d}x = 0\\
&\dot{\psi} = \frac{p_\psi}{I_x} && \dot{p}_\psi - \rho \sec^2\alpha \int_{L_p}^{L_s} (x - \bar{X})\lambda \,\mathrm{d}x = 0
\end{aligned}
$$

Constrained equation

$$
\eta = -\tan\alpha\, \mathrm{sign}(x - \bar{X})X + Z - \sec^2\alpha\,(x - \bar{X})\psi
$$

Water Wave Impact on 2D Ships

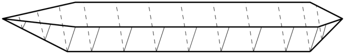
- *2D Ship sluice gate* & *2D Ship wave maker*



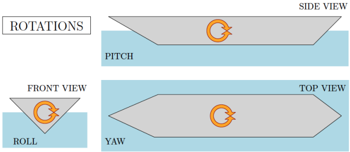
- Extends to 3D linear potential flow (Ambati, *YouTube*)
- *Wave maker case*, *sluice gate*.

Water Wave Impact on 3D Ships

- Simple model of a ship with V-shaped cross-sections:



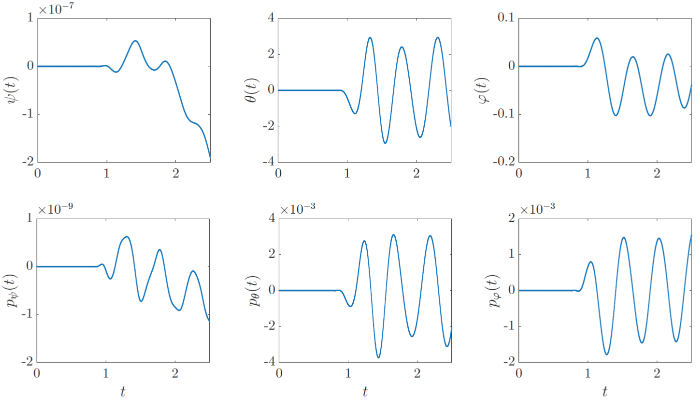
- Dynamics of ship described by: [Marsden & Ratiu, 1994]
  1. Centre of mass $\mathbf{X} = (X, Y, Z)$
  2. Angles $\boldsymbol{\theta} = (\theta, \psi, \varphi)$ pitch-roll-yaw rotations
  3. Velocity centre of mass $\mathbf{U} = (U, V, W)$
  4. Angular momenta $\mathbf{p}_{\boldsymbol{\theta}} = (p_\theta, p_\psi, p_\varphi)$.



- Ship: mass M, moments of inertia $\mathbf{I} = (I_1, I_2, I_3)$ & angular velocities $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$.

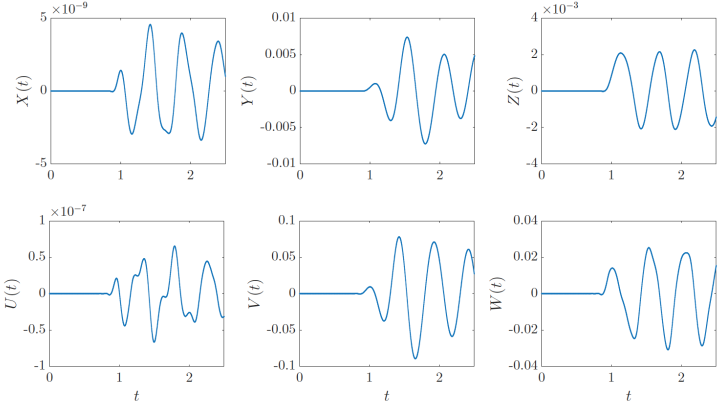
Water Wave Impact on 3D Ships

3D wave tank with ship in centre: ship response – rotations



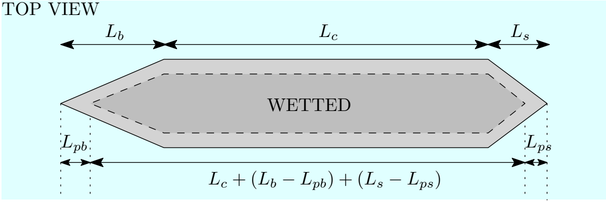
Water Wave Impact on 3D Ships

3D wave tank with ship in centre: ship response – translations



Water Wave Impact on 3D Ships

- *3D Ship with wave maker*



6. Outlook

- Modelled nonlinear rogue waves.
- Modelled (linear) wave-energy device.
- Modelled (linear) wave-ship dynamics.
- Next step: nonlinear coupling.
- Related: fully two-way variational wave-turbine-mast coupling (Salwa, Kelmanson, B.).

- B. & Kalogirou '16a: *Lect. on . . . Water Waves.*
- Gidel & B. '16: . . . extreme waves . . . oblique . . . solitary waves. Nonl. Proc. Geophys.
- Kalogirou & B. '16b: Math. . . . modelling of wave impact on wave-energy buoys. OMAE2016.
- `www.facebook.com/resurging.flows` & *Wetropolis*
- Salwa, B. Kelmanson, '16ab: Int. Conf. Ocean, Offshore Arctic Eng. & IWWWFB
- Gagarina et al. '16: J. Comp. Phys.

Outlook

- Next step: breaking wave impact on structures. *modelling of breaking waves* via mixture theory or NS-Cahn-Hilliard model/stratified (potential flow) model?
- Idea: combine robustness of SPH/VOF with accuracy of potential flow.

Questions?

X-bowed vessel experiments Courtesy: Ulstein Sea-of-Solutions.